DEFINITION AND CALCULATION METHODS OF THE AMPLIFICATION FACTOR OF NEUTRON AMPLIFIERS

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1. The neutron amplifier

The neutron amplifier is a subcritical nuclear reactor containing a neutron source. Neutrons emitted from this source are partially absorbed in the amplifier, and partially escape from the system through the boundary surfaces. A part of the neutrons absorbed in fissionable material causes fissions, thereby new neutrons are released. Some of these again cause new fissions, *i. e.* a chain reaction is brought about. As the system is subcritical (*i. e.* $k_{\text{eff}} < 1$), this chain reaction is not divergent, not even self-sustaining. In the stationary condition, therefore, neutrons can only be in the amplifier, if they are uniformly emitted by the source.

The system may be subcritical because its dimensions are smaller than the critical value pertaining to the given material composition. In this case the multiplication factor of a system with infinite dimensions is greater than one ($k > 1$). The effective multiplication factor is less than one because of escaping neutrons. If the material composition is such that even the "infinite" multiplication factor is less than one, than the system cannot be critical at any dimensions. (This happens *e. g.* in case of a homogenous mixture of natural uranium and of graphite or of common water.) Such a system may be called definitely subcritical, but such types of amplifiers are not customary because of very great dimensions.

Basically, neutron amplifiers can be used for two purposes. On the one hand, it can serve as a reactor model and can be used for the experimental determination of critical dimensions, of various material characteristics, etc. (Lately, the application of impulse-operation neutron amplifiers has arisen. In this place however only amplifiers with a stationary neutron source are discussed.) On the other hand, the neutron amplifier may serve as neutron source, that means it may supply neutrons for neutron-technical experiments, for isotope production, etc. It may even function, at least in theory, as an energy source in case of satisfactory neutron flux and volume.

The concept of the amplification factor may come to the fore, especially in connection with the above-mentioned second field of application. The definition of the amplification factor, however, is by no means unequivocal.

Therefore our aim will be to make a survey on some fundamental questions in connection with the definition of the amplification factor. These are:

a) What is the practical definition of the amplification factor?

b) How can various amplifiers be compared?

c) How the expression of the amplification factor is influenced by the applied calculation method?

2. Processes in the neutron amplifier

Our following tests will be limited to thermal neutron amplifiers, in which a great part of fast neutron, both those emitted by the source and those released by nuclear fission, are slowed down to thermal energy. This also means that the amplifier contains beyond the fissionable material, some moderator, too. In a thermal neutron amplifier of this kind the same processes take place as in a thermal reactor. The freed fast neutrons are slowed down: a part of the slowed down neutrons escape from the system, others are absorbed, while a third part of neutrons becomes thermal. Diffusing neutrons with thermal energy, partially escape from the system and are partially absorbed. A part of the absorbed neutrons causes fission, consequently fast fission neutrons are also produced beyond source neutrons.

For the sake of simplifying calculations some approximate suppositions are made. First of all let us suppose that slowing-down neutrons are not absorbed by the moderator, which is a very good approximation. Let us further suppose that the U^{238} isotope alone is responsible for the absorption of slowing-down neutrons, consequently the probability of slowing down is equal to resonance escape probability. This assumption means that the neutron absorption of the U^{235} isotope in the range of slowing-down neutrons is neglected, consequently fissions caused by slowing-down neutrons are also neglected. In other words, we apply the usual approximation, that only thermal neutrons cause nuclear fission in a thermal reactor. Finally our examinations are limited to the first period of amplifier operation, this means that neither changes in fissionable material quantity (burn-up and breeding), nor the accumulation of fission products (poisoning and contaminating isotopes) are taken into account during the operation of the amplifier.

It should be noted from the point of view of calculation technique, that there is an essential difference between neutron amplifiers and reactors in critical conditions. In the neutron amplifier, namely, fast neutrons are produced not only by fission, but fast neutrons are emitted by the source, too. In the near surrounding of the source this latter effect is overwhelming. As fission neutron density is proportional to thermal neutron density, these two density values can be regarded proportional at any point of the reactors.

This supposition is not valid for neutron amplifiers, consequently this fact should also be taken into account during calculations.

In the course of our calculations only the mean value of the neutron flux stabilized in time is taken into account. This means, on one hand, the supposition that neutrons are emitted uniformly by the source, and on the other hand, the statistic fluctuation of neutron density is not taken into account, *i. e.* the variation of values is neglected beside the average. This fact is worth mentioning because relative scatter is small in generally used critical reactors, in subcritical systems, however, it may reach considerable values. This is, however, of no greater importance, except when the neutron amplifier is used, not as a neutron source, but as a measuring instrument. In the latter case natural scatter should be taken into account when evaluating the results.

3. Definition of the amplification factor

Subcritical reactors used as neutron amplifiers can be characterized by various data. One of the data groups comprises specifications of the material composition, and nuclear characteristics of the amplifier. Another group of data specifies the form and dimensions of the system, as well as the critical dimensions pertaining to the given material composition and form. Costs of the amplifier also belong under this heading, as being determined by the used fissionable material and moderator, consequently calculatable from previous data. The third group of data furnishes information on neutron distribution as brought about by the source, consequently neutron distribution should be described in function of position and of energy.

After all, from the point of view of application three data are essential. The first is the necessary investment cost. The second is the degree of criticality according to definition, which gives information to the extent to which the amplifier approximates the critical condition. The third essential data is a suitably defined amplification factor, by which the basic reservation is made, that it should be a dimensionless number, in accordance with general usage. The amplification factor should be defined in such a way that the quantity essential from the point of view of application could, by its aid, be easily calculated. (This point will be discussed later.) Beyond this, amplification factor is to be defined in such a way, that both its calculation and the checking of calculation results by measurement, should be as simple as possible.

As neutron amplifiers can be used for various purposes, no such definition of the amplification factor can be set up, which would unequivocally characterize the situation in every case. If radioactive isotopes are to be produced in the amplifier, the evident aim is to absorb as many neutrons in the material to be activated, as possible. By presuming, in accordance with practice, that the quantity of the material to be activated is much less than

the quantity of the fertile medium, then conditions are not greatly influenced by its presence. Hence the number of thermal neutrons absorbed in the material to be activated during the unit of time is proportional to the number of thermal neutrons absorbed in the amplifier in the unit of time. This latter value, however, can be expressed in the following form:

$$Q_T = \int_V \Sigma_a(\mathbf{r})\, \Phi(\mathbf{r})\, dV, \tag{1}$$

where Σ_a is the thermal absorption cross section, Φ is the thermal neutron flux and V is the volume in which the material to be activated is arranged. In the followings, for the sake of simplicity, the supposition is that V is the total volume of the active zone (core) of the amplifier.

Now, an obvious definition of the amplification factor is the following: The quotient of thermal neutrons absorbed in the system in the unit of time (Q_T) and of the number of fast neutrons emitted by the source in the unit of time is the amplification factor, which will be named *thermal amplification factor* in the following:

$$K_T = K_1 = \frac{Q_T}{Q} = \frac{1}{Q} \int_V \Sigma_a(\mathbf{r})\, \Phi(\mathbf{r})\, dV \tag{2}$$

It should be noted that Q_T would be equal to the number of neutrons becoming thermal during the unit of time only in that case if the system should be infinite. (This condition can be approximated by a definite subcritical system.) Namely a part of the neutrons which become thermal escape through the boundary surfaces. From the points of view of practical application, of calculation and of measurement, the Q_T value as written in expression (1) is more important and more accessible.

Hence the number of thermal neutrons absorbed in the unit of time can easily be calculated by the equation

$$Q_T = K_T Q \tag{3}$$

if the thermal amplification factor K_T, as defined under (2), is known. It should be noted, that although the concept of the thermal amplification factor was arrived at by way of isotope production, nevertheless, the factor K_T supplies some information on the thermal neutron distribution of the amplifier in general. Consequently this is a general characteristic of neutron amplifiers.

For whichever purpose thermal neutrons are needed, they can be produced by a source of fast neutrons without fissionable material, that is, in a more simple way. Let us surround the neutron source with a zone consisting of pure moderator containing no fissionable material. Be Φ_0 the thermal

neutron flux in a system which only differs from the examined amplifier by not containing fissionable material (distribution of moderator and dimensions being equal). The number of thermal neutrons absorbed in the unit of time is, in this case:

$$Q_{T0} = \int_V \Sigma_{a0}(r)\, \Phi_0(r)\, dV. \tag{4}$$

Evidently the ratio Q_T/Q_{T0} also characterizes the effectiveness of the amplifier, therefore, the following definition of the amplification factor is also generally used:

$$K_{T0} = K_2 = \frac{Q_T}{Q_{T0}} = \frac{\int \Sigma_a \Phi\, dV}{\int \Sigma_{a0} \Phi_0\, dV}. \tag{5}$$

The amplification factor so defined supplies information about the "worthiness" of putting fissionable material into the system, but the number of neutrons becoming thermal cannot be directly calculated by its aid.

If the suberitical system is used to infer from measurements, carried out in it, on a critical system of similar material composition and dimensions, then first of all the value of the neutron flux should be known. Neutron flux in any place of the sysem is proportional to the count as recorded in the unit of time by the neutron counter arranged at that place. Neutron flux, *i. e.* the count proportional to it, depends, beyond the material composition and geometry of the system, also on the intensity and position of the source. Source intensity can be eliminated if the neutron count is divided by the number of neutrons emitted by the source in the unit of time. In this way the following definition of the amplification factor is obtained:

$$K_3 = \frac{C}{Q}. \tag{6}$$

where C is the count measured in the unit of time. In practice, it is more convenient not to divide by source intensity, but by a measured value proportional to it. Let C_0 be the counting rate measured at a given distance from the source arranged in the moderator, free of fissionable material. In this case the amplification factor can also be defined in the form

$$K_3' = \frac{C}{C_0}. \tag{7}$$

If measurements were carried out by the same counter, or if the two counters

are calibrated to each other, then K_3' is simultaneously equal to the proportion of neutron fluxes as measured at the suitable place:

$$K_\Phi' = K_3' = \frac{\Phi(\mathbf{r}_1)}{\Phi_0(\mathbf{r}_0)} \tag{8}$$

where $\Phi(\mathbf{r}_1)$ is the flux in the system also containing fissionable material, at the point determined by position vector $\mathbf{r}_1$. $\Phi_0(\mathbf{r}_0)$ the flux in the system containing no fissionable material, at point $\mathbf{r}_0$ taken arbitrarily. (Source is naturally supposed as being fixed *e. g.* in point $\mathbf{r} = 0$.) As a final result this amplification factor determines neutron flux in a selected point of the subcritical system.

$\Phi_0(\mathbf{r})$ "base level" is sometimes not regarded as a universal value related to the whole system, but the flux $\Phi_0(\mathbf{r})$ of the system free of fissionable material is measured at every point, one after the other. In this way, the following amplification factor can be defined in point $\mathbf{r}_1$:

$$K_\Phi'' = K_3'' = \frac{\Phi(r_1)}{\Phi_0(\mathbf{r}_1)} \tag{9}$$

where $\Phi(\mathbf{r}_1)$ and $\Phi_0(\mathbf{r}_1)$ is the neutron flux (or the count proportional to it) at the same points of the systems containing and not containing fissionable material, respectively. This factor, however, is greatly influenced by the relative position of the source, consequently it does not characterize flux $\Phi(\mathbf{r}_1)$ as unequivocally as does factor K_Φ' as defined under (8). The amplification factor can also be formulated for fast neutrons originating in the unit of time. Let Σ_f be the cross section of fission and ν_0 the neutron produced by fission (the average number of neutrons released by one fission). In this case the number of fast (fission) neutrons released in the unit of time in the amplifier is:

$$Q_F = \int_V \nu_0 \Sigma_f(\mathbf{r}) \Phi(\mathbf{r}) \, dV . \tag{10}$$

Consequently another definition of the amplification factor is:

$$K_F = K_4 = \frac{Q_F}{Q} = \frac{1}{Q} \int_V \nu_0 \Sigma_f(\mathbf{r}) \Phi(\mathbf{r}) \, dV . \tag{11}$$

If ν_0, Σ_f and Σ_a are independent of position, then

$$K_F = \nu_0 \frac{\Sigma_f}{\Sigma_a} K_T . \tag{12}$$

Amplification factor K_F simultaneously characterizes the produced heat power, too. The number of fissions occurring in the unit of time in the amplifier is Q_f/ν_0. If W_F is the average energy released at every fission then the thermal power output of the amplifier is:

$$P = \frac{Q_F}{\nu_0} W_F = K_F \frac{QW_F}{\nu_0} . \tag{13}$$

If the fissionable material is U^{235}, than $\nu_0 = 2{,}46$ and $W_F = 194$ MeV $= 31 \cdot 10^{-12}$ Ws. Source intensity Q is generally measured, in place of neutron/sec, in curie values. A Po-Be neutron source of 1 curie intensity supplies about $2{,}1 \cdot 10^6$ neutrons in a second. Hence

$$P^{\text{watt}} \approx 2{,}65 \cdot 10^{-6} K_F Q^{\text{curie}} . \tag{14}$$

As a reasonable limit of source intensity is around 10 curie, to produce outputs in the order of magnitude of one watt, amplification factors of $K_F \approx 10^5$ are necessary. As will be shown later by numerical data, this is practically impossible.

Neutron amplifiers can finally be regarded as a neutron source of large extension. The only important thing is the number of neutrons flowing out across the surface. If $\mathbf{J}$ is the neutron current density on the surface, so the total neutron current is:

$$I = \int_A \mathbf{J}\, d\mathbf{A} . \tag{15}$$

Along the previous lines the following new possibilities of defining the amplification factor arise:

$$K_5 = \frac{I}{Q} , \tag{16}$$

or for the system free of fissionable material:

$$K_6 = \frac{I}{I_0} . \tag{17}$$

Amplification factors K_5 and K_6 can be interpreted both for thermal neutron and for neutrons, the energy of which falls within a determined energy interval.

If the reactor is in a critical condition, there are neutrons in the system even without extraneous source (*i. e.* in case of $Q = 0$). If the reactor contains an extraneous source too, so the neutron level becomes infinitely high, consequently any one of the amplification factors, as defined above, also become

infinite. The critical dimensions of the amplifier can just be determined from this condition. The dimensions of the system (*e. g.* edge length of a prism) be H_1, H_2, H_3 ... The value of the amplification factor evidently depends on these dimensions. We have therefore to determine those H_{10}, H_{20}, H_{30} ... dimensions for which

$$\lim_{H_i \to H_{i0}} K(H_1, H_2, H_3 \ldots) = \infty \tag{18}$$

These Hi_0 dimensions will just be the critical dimensions of the amplifier.

4. Calculation methods

Several methods for technical reactor calculations are known. Or these only two methods can be carried out without a large calculating machine equipment. One method is the model of continuous slowing-down (calculation by the Fermi age), the other is the group method. However, in any case these methods only lead to mathematical problems analytically solvable in case of simple arrangements. The application of the group method should also be limited to a small number of groups, otherwise calculation work will be excessively great and results will become confused.

According to the model of continuous slowing-down, thermal flux Φ and slowing-down density q in the stationary condition can be obtained by solving the following equations:

$$\Delta\Phi - \frac{1}{L^2}\Phi + \frac{p}{D} q(\tau_T) = 0, \tag{1}$$

$$\Delta q = \frac{\partial q}{\partial \tau}. \tag{2}$$

where L is the diffusion length, p the slowing-down probability, D the diffusion coefficient, τ and τ_T the age and its value taken at thermal energy, respectively, and $\Delta =$ div grad is the Laplace operator. On the extrapolated boundary surface of the reactor, functions Φ and q should have a value of zero and source density should fulfil the following "initial" condition:

$$q(\mathbf{r}, 0) = \frac{k}{p} \Sigma_a \Phi(\mathbf{r}) + q_0(\mathbf{r}), \tag{3}$$

where q_0 is the density of extraneous sources. Let it be supposed here that the energy and age of neutrons emitted by the sources corresponds to the

energy of fission neutrons and to age $\tau = 0$, respectively. This method is difficult to apply in a case when material characteristics are only within part volume constant (*e. g.* when using a reflector), because on the boundary surface of two mediums the boundary conditions for slowing-down density are difficult to fulfil.

When using the group method, the supposition is made that neutrons, the energy of which falls within a determined interval, can be characterized by a single position function (neutron density or neutron flux), further, that processes within single groups take place according to the elementary diffusion theory. This means that the flux of every group fulfils a diffusion equation. For informative calculations in reactor technics, the simplest method, the one-group approximation is often used. The corrected one-group diffusion equation for the flux is:

$$\Delta\Phi + \frac{k-1}{M^2}\Phi = \frac{p}{D} q_0 . \tag{4}$$

The applied correction consists, on one hand, of calculating, in place of the formula $D/\Sigma_a = L^2$, by the formula

$$\frac{D}{\Sigma_a} = M^2 = L^2 + \tau \tag{5}$$

where M is the migration distance and $\tau = \tau_T$ is the age pertaining to the thermal energy. The correction, on the other hand, consists of multiplying the source member by the p slowing-down probability, taking, thereby, the absorption of slowing-down neutrons approximately into account.

The lawfulness of the application of one-group approximation should be separately examined in the case of neutron amplifiers. In neutron amplifiers, namely, the correlation between thermal and fast flux is not so exact as in critical reactors because of the presence of the extraneous fast source, as mentioned above. The two-group equations for the thermal neutron flux Φ and the fast neutron flux Ψ are the followings:

$$\Delta\Phi - \frac{1}{L^2}\Phi - p\frac{D_F}{D_T}\frac{1}{\tau}\Psi = 0 . \tag{6}$$

$$\Delta\Psi - \frac{1}{\tau}\Psi + \frac{k}{p}\frac{D_T}{D_F}\frac{1}{L^2}\Phi = \frac{1}{D_F} q_0 . \tag{7}$$

Here D_F and D_T is the diffusion coefficient in the range of fast (fission) and thermal neutrons, respectively.

By increasing the number of groups, the accuracy of computation also increases, but simultaneously the necessary labor also increases and the perspicuity of results steadily decreases. Boundary conditions to be fulfilled are the following: Every neutron flux should disappear at the (extrapolated) boundary surface of the system; on the boundary surface of two mediums every neutron flux and the normal components of every neutron current density should be continuous.

The solution of the equation system (9—10) is generally looked for in the following form:

$$\Delta\Phi + \chi^2 \Phi = 0, \quad \Delta\Psi + \chi^2 \Psi = 0. \tag{8}$$

When resubstituting into homogeneous equations, a biquadratic equation for χ originates. The roots of the equation are $\chi_{12} = \pm \mu$ and $\chi_{34} = \pm j\nu$ where

$$\mu^2 = \frac{M^2}{2L^2\tau}\left[\sqrt{1 + 4(k-1)\frac{L^2\tau}{M^4}} - 1\right]. \tag{9}$$

$$\nu^2 = \frac{M^2}{2L^2\tau}\left[\sqrt{1 + 4(k-1)\frac{L^2\tau}{M^4}} + 1\right]. \tag{10}$$

If $(k-1) \ll 1$, or if values L^2 and τ differ greatly, then the following approximative equations can be arrived at by series of development:

$$\mu^2 \approx \frac{k-1}{M^2}\left[1 - (k-1)\frac{L^2\tau}{M^4}\right] \approx \frac{k-1}{M^2}. \tag{11}$$

$$\nu^2 \approx \frac{M^2}{L^2\tau} - \mu^2 \approx \frac{M^2}{L^2\tau}. \tag{12}$$

The correction member in the expression for μ^2 is the highest in the case of $L^2 = \tau$, having a value of $(k-1)/4$, *i. e.* in respect to μ the correction member may be $(k-1)/8$ at the maximum. It is worth-while noting, that

$$\frac{\mu^2}{\nu^2} \approx (k-1)\frac{L^2\tau}{M^4} \leq \frac{k-1}{4}. \tag{13}$$

These inequations supply information regarding the order of magnitude of the error caused by a one-group approximation in respect to a two-group approximation. According to equation (4) namely, the parameter of the one-group approximation is $(k-1)/M^2$ that is just equal to the roughest approximation of parameter μ^2 under (11).

Functions Φ and Ψ pertaining to parameters μ and ν respectively are not independent of each other, because

$$\left(\frac{\Psi}{\Phi}\right)_{\chi^2=\mu^2} = \alpha = \frac{\zeta\tau}{p}\left(\mu^2 - \frac{1}{L^2}\right) \tag{14}$$

$$-\left(\frac{\Psi}{\Phi}\right)_{\chi^2=\nu^2} = \beta = \frac{\zeta\tau}{p}\left(\nu^2 - \frac{1}{L^2}\right); \quad \zeta = \frac{D_T}{D_F} \tag{15}$$

The method of solution is not discussed in detail here. The aim of the above discussion was to clarify used symbols.

5. The degree of criticality

The neutron amplifier has to perform a given task. *E. g.* a determined number of fast neutrons should be released in the system, according to the definition of the amplification factor under (3.8). This task can be solved in various ways, as at present we may freely choose the following parameters:

geometrical form of the amplifier,

dimensions of the amplifier (it should, however, be smaller than the critical one),

source intensitiy,

degree of concentration of the uranium used,

material of the moderator used,

ratio of the fissionable material and of the moderator and their respective position.

Basically, various systems can only be compared if they are equally far from the critical condition. It is evident, namely, that the nearer the critical condition is approximated, the greater is the danger of the amplifier becoming critical. Maybe the only advantage of the neutron amplifier in comparison to reactors is, that just because of the system being subcritical, one has not to fear its runaway — consequently no complicated protection devices are necessary and the whole equipment is completely safe as regards the danger of explosion. If dimensions of the amplifier approximate the critical ones very much, so the danger exists, that in consequence of some fluctuation (*e. g.* temperature change), the system becomes critical, or even supercritical, *i. e.* runs away and explodes.

The previous note on the non-criticality of the system is evident, nevertheless it is very difficult to find a simple mathematical value characterizing the degree of criticality of the system. In principle the following procedure would be necessary. All the possible effects able to make the amplifier critical should be examined. Such as *e. g.* Because of evaporation or leakage of the

moderator fluid, the proportion of fissionable material is increased; fuel solute is diluted *e. g.* because of moderator medium getting into the active zone from the reflector (of these two effects only one is dangerous by a given material composition); temperature is rapidly increased or decreased, depending on the sign of the temperature coefficient of the amplifier, etc. Now, those systems can be similarly named non-critical, by which an extraneous effect occurring with identical probability just makes the system critical. Non-critical to a higher degree is that system, by which the probability of its growing critical is smaller (roughly speaking: which needs a more violent extraneous effect to grow critical).

Such an examination lasts very long, it contains several complicated parameters and it produces such a complicated result that comparison cannot be carried out in practice. Therefore, it is practical to search for a new criterion, not so general but more simple, or for a new definition for the degree of criticality.

The following reasoning referred to below is self evident. Though the neutron amplifier has no protective devices, neutron flux, however, is continuously checked. Now if the system grows supercritical, neutron flux starts growing. If this growth is slow enough, the operator has time to intervene and to make the amplifier subcritical in some way (by draining fuel, by inserting a control rod or other absorbent, etc.). The approximate expression of the so-called reactor period is known from reactor technics:

$$T \approx \frac{\beta - \varrho_1}{\varrho_1 \lambda}, \tag{1}$$

where β and λ are given values (the yield of delayed neutrons and the avarage decay constant, respectively), while ϱ is the reactivity, which depends on the effective multiplication factor according to the following expression:

$$\varrho = \frac{k_{\text{eff}} - 1}{k_{\text{eff}}} = 1 - \frac{1}{k_{\text{eff}}}. \tag{2}$$

The longer the reactor period is, *i. e.* the slower the processes are, the smaller reactivity ϱ_1 is. In subcritical state $k_{\text{eff}} < 1$, consequently $\varrho_0 < 0$. In case of a given $\Delta\varrho$ change of reactivity, the expression for the reactivity ϱ_1 coming into being is:

$$\varrho_1 = \Delta\varrho + \varrho_0 = \Delta\varrho - (-\varrho_0). \tag{3}$$

Accordingly, reactor period is the longer, the greater $(-\varrho_0)$ is, *i. e.* the initial negative reactivity.

Hence, negative reactivity can be taken as a simple measure of non-criticality. Those systems are said to be of similar non-criticality which have

identical negative reactivity. The greater negative reactivity is, the less critical is the amplifier, because the greater change in reactivity is necessary to make the amplifier critical, and in the supercritical condition variations will be correspondingly slower.

Against this method of comparison the objection may be raised that reactivity change $\Delta\varrho$ cannot be regarded as a given value for different arrangements. Effects acting with similar probability may cause different changes of reactivity in various amplifiers. If the temperature factor of the amplifier is negative, the system is selfregulating, in a manner known from reactor technics, consequently running away should be feared only in cases of extreme and practically negligible temperature changes. The first cause of growing critical, therefore, is a change in material composition of the amplifier. Naturally it is difficult to find a general characteristic for this case. Nevertheless, it can be roughly said that the ratio of fissionable material in the amplifier and of the critical fissionable material quantity pertaining to the given material composition is characteristic of how near the amplifier has approximated the critical condition. If the quotient of fissionable material and of moderator is changed, namely, the critical quantity of fissionable material changes, too, and thereby (in an unfavourable case) the quotient gets nearer to one.

The quantity of fissionable material is proportional to the volume of the active zone in case of a given material composition. Therefore this same quantity, that will be denominated the *degree of criticality* in the following, can also be written as the quotient of the V actual volume of the active zone, and of the V_0 critical volume:

$$s^3 = \frac{V}{V_0} . \tag{4}$$

Quotient V/V_0 has the notation s^3 (in place of simply s), because in this way linear dimensions can be expressed by its aid. *E. g.* in case of a sphere $s = R/R_0$.

Summarizing, it can be stated that basically only such amplifiers can be compared for which the degree of criticality s, as defined under (4) is the same.

Naturally the question may arise as to how great a negative reactivity is meant by the prescription of the degree of criticality. If the B geometrical buckling is known (*e. g.* for a bare sphere $B = \pi/R$), then the expression of the effective multiplication factor *e. g.* in two-group approximation:

$$k_{\text{eff}} = \frac{k}{(1 + B^2 L^2)\,(1 + B^2 \tau)} \tag{5}$$

By substituting $k_{\text{eff}} = 1$ for the buckling B_0 ensuring critical condition:

$$(1 + B_0^2 L^2)\,(1 + B_0^2 \tau) = k . \tag{6}$$

The expression for negative reactivity on the basis of (2) is:

$$-\varrho = \frac{(1 + B^2 L^2)(1 + B^2 \tau)}{k} - 1 = \frac{(B^2 - B_0^2) M^2 + (B^4 - B_0^4) L^2 \tau}{k} \quad (7)$$

By the given geometry, values $(B^2 - B_0^2)$ and $(B^4 - B_0^4)$ can be determined.

For the sake of information, the member of the fourth order should be neglected (which corresponds to a corrected one-group approximation) and let us supposed that $B = B_0 s$, what is evidently true in case of a bare sphere. In this case:

$$-\varrho = \frac{B_0^2 M^2}{k} \left(\frac{1}{s^2} - 1 \right) = \frac{k - 1}{k} \cdot \frac{1 - s^2}{s^2} . \quad (8)$$

where the effective multiplication factor in the one-group approximation is:

$$k_{\text{eff}} = \frac{k}{1 + M^2 B^2} ; \quad 1 = \frac{k}{1 + M^2 B_0^2} \quad (9)$$

Hence, reactivity is not unequivocally defined by the degree of criticality even in case of simple geometrical forms, because it is also influenced by the expression of the infinite multiplication factor.

In the followings systems with an identical degree of criticality, will be regarded as equivalent from the point of view of subcriticality. Nevertheless, in single cases, the consequences of the prescription of negative reactivity will also be examined. It should be noted that the practical advantage of comparison on the basis of the degree of criticality lies in the fact that actual dimensions pertaining to a given degree of criticality can be determined from the critical dimension. At the same time the effective multiplication factor or reactor parameter, necessary for computing reactivity, can only be determined for simple geometric arrangements. Thus *e. g.* for systems with reflector, their exact calculation is not known, consequently the correlation between negative reactivity and actual dimensions cannot be expressed exactly, not even theoretically.

6. Optimum arrangement

As mentioned at the beginning of the previous chapter, tasks connected with the amplifier can be solved in various ways, as various parameters can be chosen. Nevertheless, a restriction is that the degree of criticality as defined previously is regarded as prescribed. In practice there are further restrictions. The degree of enrichment of the used uranium and the material of the moderator is determined by given circumstances, consequently in the course of design-

ing these data can generally be regarded as given values. The arrangement of the amplifier is generally determined by technological aspects, *e. g.* the optimum spherical form is applicable in the case of a homogeneous system, however in the case of heterogeneous arrangement not. Source intensity has a reasonable upper limit. this is about 10 curie in case of a Po-Be source, or 0.5 curie for a Ra-Be source. After all, only the ratio of fuel and of moderator can freely be determined, and eventually some geometric characteristics (*e. g.* ratio of the radius and of height of the cylinder).

From the amplifiers fulfilling the given task evidently that one can be regarded as optimum, which costs the least. Costs consist of the following components: costs of fissionable material, of moderator and of the source, and various other costs (tank, sheathing, auxiliary equipment). These last named various costs are on the one hand smaller than the former ones, and on the other hand they can be regarded as nearly constants, consequently they can be neglected in the course of comparison. Our task is to determine that arrangement which produces a determined number of neutrons by prescribed degree of criticality and the costs of which are the smallest. This principle is a natural guide in choosing uranium enrichment, moderator and geometry.

It appears from the detailed examination that costs of the source are much less than costs of fissionable material and of moderator. Our task is therefore simplified, a minimum is to be found for costs

$$Z = Z_f + Z_m \tag{1}$$

where Z_f is cost of the fuel and Z_m that of the moderator. In a general case, cost of the source Z_Q can also be taken into account, that is roughly proportional to source intensity:

$$Z_Q = a + bQ . \tag{2}$$

If the amplification factor is defined in the form

$$Q_0 = KQ \tag{3}$$

where Q_0 is the number of neutrons (fast or thermal) to be released in the unit of time, then total costs are:

$$Z_0 = Z_f + Z_m + a + b\frac{Q_0}{K} . \tag{4}$$

The minimum of this function is to be found if a, b, and Q_0 are given values, while Z_f, Z_m and K are functions of any parameter.

If the changing parameter is *e. g.* material composition, so the extreme value cannot be analytically determined in practice, consequently, in general,

graphic methods are to be relied on. In the case of some tasks, however, Z_0 may be a simple function of the changing parameter, hence the minimum can be determined by conventional extreme value calculation.

7. Bare homogeneous spherical system

Our above results are applied to a single system. Fuel and moderator form a homogenous mixture, the amplifier is spherical and bare, *i. e.* there is no reflector. The point-like neutron source is in the middle of the sphere. Material composition is characterized by two data. The first one is the dilution:

$$g = \frac{N_0}{N_U}, \tag{1}$$

while the second is the degree of enrichment

$$c = \frac{N_{235}}{N_U} = \frac{N_{235}}{N_{235} + N_{238}}, \tag{2}$$

where N is the number of molecules in 1 cubic centimeter of the moderator (N_0), of the uranium (N_U) and of its two isotopes (N_{235}, N_{238}), respectively. In the function of these parameters the infinite multiplication factor, the diffusion length, the cross sections etc. can be calculated by making use of equations well-known from literature.

Neutron flux distribution can be determined the most simply by the one-group approximation, *i. e* by solving equation (4.4). Taking into account the boundary condition too, the following formula is arrived at:

$$\Phi(r) = \frac{pQ}{4\pi\Sigma_a M} \frac{\sin \varkappa (R - r)}{r \sin \varkappa R}, \quad \varkappa^2 = \frac{k - 1}{M^2}. \tag{3}$$

It should be noted that R means, both in this equation and it the following ones, not the effective radius but the so-called extrapolated radius which is an actual radius R^* plus d extrapolation distance:

$$R^* = R - d, \quad d \approx 0{,}71\,\lambda_{tr} \tag{4}$$

where λ_{tr} is the transport mean free path. Critical dimension are reached at the $R = R_0$ minimum radius, by which case flux becomes infinite, *i. e.* the denominator of Φ is zero. From equation (3) we obtain:

$$\varkappa R_0 = \pi, \quad R_0 = \pi \frac{M}{\sqrt{k - 1}}. \tag{5}$$

This is a well-known formula in elementary reactor technics.

A more exact formula for flux distribution can be arrived at by the two-group method, *i. e.* by solving equation systems (4.6—7). With the symbols of chapter 4. the following equations are obtained for thermal flux Φ and fast flux Ψ respectively:

$$\Phi(r) = \frac{pQ}{4\pi\Sigma_a L^2 \tau (\mu^2 + \nu^2)} \left[\frac{\sin \mu (R-r)}{r \sin \mu R} - \frac{\operatorname{sh} \nu (R-r)}{r \operatorname{sh} \nu R} \right], \tag{6}$$

$$\Psi(r) = \frac{Q(\mu^2 L^2 + 1)}{4\pi D_F L^2 (\mu^2 + \nu^2)} \left[\frac{\sin \mu (R-r)}{r \sin \mu R} + \right.$$
$$\left. + \frac{L^2}{\tau} \frac{\mu^2 \tau - 1}{\mu^2 L^2 + 1} \frac{\operatorname{sh} \nu (R-r)}{r \operatorname{sh} \nu R} \right]. \tag{7}$$

The equation determining critical radius is:

$$\mu R_0 = \pi, \quad R_0 = \frac{\pi}{\mu}. \tag{8}$$

Taking into account the approximate expressions for μ^2 and ν^2 under (4.18—19) the following approximate equations for the thermal flux, which is of first importance, and for the obtained critical radius are:

$$\Phi(r) = \frac{pQ}{4\pi\Sigma_a M^2} \left[\frac{\sin \mu (R-r)}{r \sin \mu R} - \frac{\operatorname{sh} \nu (R-r)}{r \operatorname{sh} \nu R} \right], \tag{9}$$

$$R_0 = \pi \frac{M}{\sqrt{k-1}}. \tag{10}$$

Comparing these with equations (3) and (5) arrived at by the one-group approximation, the following observations can be made. From the point of view of the critical radius, the two-group approximation only means a small correction. According to chapter 4. the correction is $[1 + (k-1)/4]$ at the maximum. If the moderator is common water ($L_0^2 = 8{,}3$ cm^2, $\tau = 33$ cm^2) then correction is $[1 + 0{,}08\,(k-1)]$ at the maximum. Regarding flux distribution the situation is somewhat different. While in the one-group approximation thermal flux is infinitely great at the point of the source, in the two-group approximation, however, the following expression is obtained by limit value formation:

$$\Phi(r=0) = \frac{pQ}{4\pi\Sigma_a M^2} \left(- \frac{\mu}{\operatorname{tg} \mu R} - \frac{\nu}{\operatorname{th} \nu R} \right). \tag{11}$$

(If $R \approx R_0$, $\operatorname{tg} \mu R$ is negative.) As generally $\nu R \gg 1$, so hyperbolic functions

can be substituted in equation (9) by exponential functions for places being far enough from the source:

$$\Phi(r) = \frac{pQ}{4\pi\Sigma_a M^2}\left[\frac{\sin\mu(R-r)}{r\sin\mu R} - \frac{e^{-\mu r}}{r}\right]. \tag{12}$$

The second member in brackets is much smaller than the first one, consequently just a distribution (3) obtained by one-group approximation is obtained after having carried out approximations.

When calculating neutron flux distribution with the aid of the model of continuous slowing-down, the following result is produced:

$$\Phi(r) = \frac{pQ}{2\Sigma_a R^2}\sum_{n=1}^{\infty}\frac{n}{(1+\alpha_n^2 L^2)\,e^{\alpha_n^2\tau} - k}\,\frac{\sin\alpha_n r}{r}, \tag{13}$$

$$\alpha_n = n\frac{\pi}{R}, \quad \tau = \tau_T. \tag{14}$$

The equation determining critical dimensions is:

$$(1+\alpha_0^2 L^2)\,e^{\alpha_0^2\tau} - k = 0; \quad \alpha_0 = \frac{\pi}{R_0}. \tag{15}$$

By substituting the exponential function with its Taylor's polynome of second order and by neglecting the fourth and higher powers of α_0, the transcendent equation can be solved approximately:

$$\alpha_0^2 = \left(\frac{\pi}{R_0}\right)^2 = \frac{L^2+\tau}{\tau(\tau+2L^2)}\left[\sqrt{1+2(k-1)\tau\,\frac{\tau+2L^2}{(L^2+\tau)^2}} - 1\right]. \tag{16}$$

By substituting the square root with its Taylor's polynome of the first order (or by already previously neglecting the member of the fourth order) the critical radius will be:

$$R_0 = \pi\sqrt{\frac{L^2+\tau}{k-1}} = \pi\frac{M}{\sqrt{k-1}}. \tag{17}$$

Therefore by this approximation, the same critical radius is the result as in the one group, or approximate two-group calculation. The correction factor of the radius is the highest in the case $L = 0$, its value being $[1 + (k-1)/4]$. If $R \approx R_0$, *i. e.* the system is near the critical condition, hence it is satisfactory to take into consideration only the first member from the expression for the neutron flux under (13), obtaining thereby the following distribution:

$$\Phi(r) = \frac{pQ}{2\Sigma_a R^2}\,\frac{1}{(1+\alpha^2 L^2)\,e^{\alpha^2\tau} - k}\,\frac{\sin\alpha r}{r}, \quad \alpha = \frac{\pi}{R}. \tag{18}$$

At present, this expression greatly differs from that deduced under (3) by the one-group approximation. By subsituting the exponential function with its Taylor's polynome of the first order:

$$\Phi(r) = \frac{pQ}{2\,\Sigma_a R^2} \frac{1}{(1 + a^2 M^2) - k} \frac{\sin a r}{r} . \tag{19}$$

According to our suppositions, the system is in a condition near to critical, hence

$$R = (1 - \varepsilon) R_0 , \tag{20}$$

where $\varepsilon \ll 1$. From this

$$a = \frac{\pi}{R} \approx \frac{1}{1 - \varepsilon} \frac{\sqrt{k - 1}}{M} = \frac{\varkappa}{1 - \varepsilon} , \tag{21}$$

where $\varkappa$ is identical with the parameter introduced at the one-group approximation. By neglecting the higher powers of ε, equation (19) can be written in the following form:

$$\Phi(r) = \frac{pQ}{4\,\Sigma_a \pi^2 M^2 \varepsilon} \frac{\sin(\varepsilon\pi + \varkappa r)}{r} . \tag{22}$$

Near the source, the two fluxes are naturally different. If, however, $r \approx R$, so

$$(1 + \varepsilon)\varkappa r \approx \varkappa r + \varepsilon\varkappa R \approx \varkappa r + \varepsilon\pi . \tag{23}$$

Hence, in first approximation, the slowing-down equation produces the same result as in the one or two-group method. Deviations are, however, considerable near the source. It should be noted that at the centre the same flux is obtained when calculating, either by the two-group method or by the slowing-down equation, if the system is in a near critical condition. According to equations (11) and (22):

$$\Phi(r = 0) = \frac{pQ\varkappa}{4\pi\,\Sigma_a M^2 \varepsilon} , \tag{24}$$

because $\mu = \varkappa$ in first approximation.

The flux distribution of a system containing no fissionable material can easily be determined along the above lines by substituting $k = 0$, and $p = 1$. In the two-group approximation, by rewriting equation (6) we obtain

$$\Phi_0(r) = \frac{Q}{4\pi\,\Sigma_{a0}(\tau_0 - L_0^2)} \left[\frac{\operatorname{sh}(R - r)/\sqrt{\tau_0}}{r \operatorname{sh} R/\sqrt{\tau_0}} - \frac{\operatorname{sh}(R - r)/L_0}{r \operatorname{sh} R/L_0} \right] , \tag{25}$$

where subscript 0 indicates that values are for the pure moderator. At the centre of the system:

$$\Phi_0(r=0) = \frac{Q}{4\pi\Sigma_{a0}(\tau_0 - L_0^2)}\left[\frac{1}{L_0 \operatorname{th} R/L_0} - \frac{1}{\sqrt{\tau_0}\operatorname{th} R/\sqrt{\tau_0}}\right]. \tag{26}$$

If $R/L_0 \gg 1$ and $R/\sqrt{\tau_0} \gg 1$, then we obtain with good approximation:

$$\Phi_0(r=0) = \frac{Q}{4\pi\Sigma_{a0}L_0\sqrt{\tau_0}(L_0 - \sqrt{\tau_0})}. \tag{27}$$

In one-group approximation, by rewriting equation (3):

$$\Phi_0(r) = \frac{Q}{4\pi\Sigma_{a0}M_0^2}\,\frac{\operatorname{sh}(R-r)/M_0}{r\operatorname{sh}R/M_0}. \tag{28}$$

Results obtained by way of the model of continuous slowing-down are not written down here. Under such conditions, namely, the first member of the infinite series has no dominant role, consequently, neglecting the rest of the members would mean a far greater error than in the case of a multiplier system in a near-critical condition.

8. Summary

Subcritical multiplier systems, the neutron amplifiers can be used for various purposes, accordingly the amplification factor can be defined in various practical ways. The most characteristic is the amplification factor which is defined as the ratio of the number of thermal neutrons absorbed in the unit of time, or the number of fast neutrons released in the unit of time on one hand, and of neutron source intensity on the other hand. The expression for the amplification factor depends, to a small extent, on applied calculation method, too, if the amplifier is, however, in a near-critical condition, deviations caused thereby can be neglected.

Basically, only such systems can be compared which are equally far from the critical condition. The critical condition, however, is difficult to be characterized by a single value. According to our considerations, a simple and well characterizing value is the degree of criticality (s^3), the quotient of actual fissionable material quantity and of the critical fissionable material quantity. In the case of a given material composition this is equal to the quotient of the actual and of the critical volume of the active zone. A value which is characterizing only to a lesser extent, but which, nevertheless, cannot be wholly left out of consideration, is the negative reactivity, which depends, beyond the degree of ciriticality, on material composition too.

In the case of a given degree of criticality (or of negative reactivity) an amplification factor can be produced by different systems. Of all the possible solutions, that one is regarded as optimum, which costs the least. Costs are determined, above all, by the fuel and (in case of heavy water) by the moderator. On designing for a given number of neutrons, costs of the source should also be taken into account, this can, however, generally be neglected beside the previous ones.

In connection with a simple example, it was shown that, in a near critical system flux distributions obtained by different approximate calculation methods only vary to a small extent.

In another paper, to be published later, several numerical data for the thermal and fission amplification factor of simple form, homogeneous neutron amplifiers as well as for their costs will be made known.

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