QUADRATIC APPROXIMATION OF ADMITTANCE DIAGRAMS FOR THE THEORETICAL EXAMINATION OF TURBO-GENERATORS IN ASYNCHRONOUS OPERATION

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As has been seen in the chapters of the previous two parts [1, 2] approximating each of the direct-axis and quadrature-axis admittance diagrams by one straight line, or some straight sections, from the condition $g =$ const. the relation $s(\delta)$ may be determined and its reciprocal function integrated. After having discussed the linear approximation in details, the present third part is devoted to the examination of the quadratic approximation.

6. The primitive parabolic approximation

Instead of the more general case of the conductance and also the susceptance being a general quadratic expression of the slip — as the relations are very complicated — the following discussion starts from the most simple parabolic approximation, when the conductance consists of a term merely containing the first power of the slip, while the susceptance is composed of two members containing only the zero and second power of the slip. (This approximation is suggested, however, by expression (2—33) of the admittances [1], too. Decomposing, namely, each admittance to a real and an imaginary part and expanding both components in series with respect to the *small* slips, it may be observed that the real part contains the 1st, 3rd, 5th, etc., while the imaginary one the 0, 2nd, 4th, etc. powers of the slip.) The above-mentioned approximation will be called in the following primitive parabolic approximation.

6.1. Slip-angle relation in case of a primitive parabolic approximation

If the direct-axis and quadrature-axis admittance diagrams, respectively, are approximated by a primitive parabola, then

$$\bar{y}_q = s\, g_{q1} - j\,(b_{q0} + s^2\, b_{q2})$$

$$\bar{y}_d = s\, g_{d1} - j\,(b_{d0} + s^2\, b_{d2}). \qquad (6-1)$$

Consequently, each parabola is symmetric with respect to the negative imaginary axis. Now

$$g_q(s) = s\,g_{q1} \qquad b_q(s) = b_{q0} + s^2\,b_{q2}$$
$$g_d(s) = s\,g_{d1} \qquad b_d(s) = b_{d0} + s^2\,b_{d2}. \tag{6—2}$$

Substituting the relations (6—2) into the fundamental equation (3—1) (*see* [1]):

$$g = s\,g_{S1} + s\,g_{D1}\cos 2\delta - b_{D0}\sin 2\delta - s^2\,b_{D2}\sin 2\delta\,, \tag{6—3}$$

where

$$g_{S1} = \frac{1}{2}(g_{q1} + g_{d1}) \qquad g_{D1} = \frac{1}{2}(g_{q1} - g_{d1})$$
$$b_{D0} = \frac{1}{2}(b_{q0} - b_{d0}) \qquad b_{D2} = \frac{1}{2}(b_{q2} - b_{d2})\,.$$

Reducing the relation (6—3) to zero, the following two equations may be obtained for the slip and its reciprocal, respectively:

$$s^2\,b_{D2}\sin 2\delta - s\,(g_{S1} + g_{D1}\cos 2\delta) + (g + b_{D0}\sin 2\delta) = 0 \tag{6—4}$$

$$\left(\frac{1}{s}\right)^2 (g + b_{D0}\sin 2\delta) - \left(\frac{1}{s}\right)(g_{S1} + g_{D1}\cos 2\delta) + b_{D2}\sin 2\delta = 0\,. \tag{6—5}$$

Solving the two quadratic equations (6—4) and (6—5), we obtain the slip-angle relation and its reciprocal, respectively.

To simplify the calculations, that is, to reduce the number of the figuring parameters, introduction of some relative quantities is again advisable. Starting from Fig. 6—1, for the circumstances outlined there

$$g_{S1} > 0\,; \quad b_{D0} > 0\,; \quad g_{D1} < 0\,; \quad b_{D2} < 0 \tag{6—6}$$

further, in asynchronous generator operation

$$g < 0.$$

Let it be again

$$-\frac{b_{D0}}{g} = \beta_0\,; \quad -\frac{g_{D1}}{g_{S1}} = \varkappa_1\,; \quad \frac{g}{g_{S1}} = s_0\,, \tag{6—7}$$

where (for the case of Fig. 6—1) $\beta_0 > 0$, $\varkappa_1 > 0$, while $s_0 < 0$.

Let it further be

$$-\frac{b_{D2}}{g_{S1}} = \lambda_2\,, \tag{6—8}$$

where (for the case of Fig. 6—1) $\lambda_2 > 0$.

By the aid of the relative quantities introduced here, the quadratic equations (6—4) and (6—5), respectively, may assume the following form:

$$\left(\frac{s}{s_0}\right)^2 (-\lambda_2 s_0) \sin 2\delta - \left(\frac{s}{s_0}\right) (1 - \varkappa_1 \cos 2\delta) + (1 - \beta_0 \sin 2\delta) = 0 \qquad (6-9)$$

and

$$\left(\frac{s_0}{s}\right)^2 (1 - \beta_0 \sin 2\delta) - \left(\frac{s_0}{s}\right) (1 - \varkappa_1 \cos 2\delta) - \lambda_2 s_0 \sin 2\delta = 0 \,. \qquad (6-10)$$

Solution of the quadratic equation (6—10) yields

$$\frac{s_0}{s(\delta)} = \frac{1}{2} \frac{1 - \varkappa_1 \cos 2\delta}{1 - \beta_0 \sin 2\delta} + \sqrt{\left(\frac{1}{2} \frac{1 - \varkappa_1 \cos 2\delta}{1 - \beta_0 \sin 2\delta}\right)^2 + \frac{\lambda_2 s_0 \sin 2\delta}{1 - \beta_0 \sin 2\delta}} \,. \qquad (6-11)$$

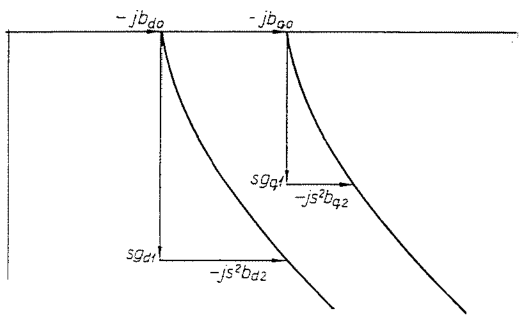


Fig. 6—1

It must be noted that in the solution (6—11) the square root may be taken only with positive sign, as with substitution of lim $\lambda_2 = 0$ the relation (4—26), valid for the primitive linear approximation must be regained.

Similarly, solution of the quadratic equation (6—9) is:

$$\frac{s(\delta)}{s_0} = \frac{1 - \varkappa_1 \cos 2\delta}{-2\lambda_2 s_0 \sin 2\delta} \pm \sqrt{\left(\frac{1 - \varkappa_1 \cos 2\delta}{-2\lambda_2 s_0 \sin 2\delta}\right)^2 - \frac{1 - \beta_0 \sin 2\delta}{-\lambda_2 s_0 \sin 2\delta}} \,. \qquad (6-12)$$

In order (6—12) should correctly express the reciprocal of (6—11), in the former the square root must be taken alternately with $\pm$ sign, according to

$$\frac{1 - \beta_0 \sin 2\delta}{-\lambda_2 s_0 \sin 2\delta} \lessgtr 0 \,.$$

The sign problems can be avoided, if the slip is calculated from the reciprocal of (6—11) as follows:

$$\frac{s(\delta)}{s_0} = \frac{1}{\dfrac{1}{2}\dfrac{1-\varkappa_1\cos 2\delta}{1-\beta_0\sin 2\delta} + \sqrt{\left(\dfrac{1}{2}\dfrac{1-\varkappa_1\cos 2\delta}{1-\beta_0\sin 2\delta}\right)^2 - \dfrac{\lambda_2 s_0 \sin 2\delta}{1-\beta_0\sin 2\delta}}} \,. \qquad (6-13)$$

It must be noted, that all of the expressions (6—11)...(6—13) are valid if, and only if the inequality

$$(1-\varkappa_1 \cos 2\delta)^2 \geqq 4(1-\beta_0 \sin 2\delta)\,(-\lambda_2 s_0) \sin 2\delta \qquad (6-14)$$

is valid, as physically only the real slip has a meaning. On the other hand, always $s/s_0 > 0$, as if $s \gtreqless 0$, then $s_0 \gtreqless 0$.

From the afore-said it becomes clear, that the first step of the method suggested in *chapter 3* of [1] may be realized, that is, the slip and its reciprocal, respectively, may be expressed as an explicite function of the angle.

6.2. Determination of the time-angle relation

To find the time-angle function, as the second step of the method suggested, the expression (6—11) should be integrated with respect to the angle. As in the expression in question only the trigonometric functions of the double angle 2δ figure, instead of the full rotation 2π it is sufficient to restrict ourselves to the half-rotation π and to integrate only within the range $-\pi/2 \leqq \delta \leqq \pi/2$. Choosing the initial value $\delta = -\pi/2$ means, that at the initial moment $t = 0$, the voltage vector leads by 90° with respect to its synchronous no-load position determined by the quadrature axis q, *i. e.*, the quadrature axis q and the flux-linkage vector coincide with each other, while the direct-axis d lags by 90° with respect to the revolving field.

The integral of expression (6—11) may be divided into two parts:

$$-s_0\,\omega_0\,t = I_1 + I_2, \qquad (6-15)$$

where the first is an elementary integral

$$I_1 = \frac{1}{2}\int\limits_{-\frac{\pi}{2}}^{\delta} \frac{1-\varkappa_1\cos 2\delta}{1-\beta_0\sin 2\delta}\, d\delta$$

solution of which *e. g.* according to Eq. (4—17) from Eq. (4—28) (*see* [2]):

$$I_1 = \frac{1}{2}\frac{1}{\sqrt{1-\beta_0^2}}\left\{\text{arc tg}\left[\sqrt{\frac{1+\beta_0}{1-\beta_0}}\,\text{tg}\left(\delta-\frac{\pi}{4}\right)\right] - \text{arc tg}\sqrt{\frac{1+\beta_0}{1-\beta_0}}\right\} +$$

$$+\frac{1}{2}\frac{\varkappa_1}{2\beta_0}\ln(1-\beta_0\sin 2\delta). \qquad (6-16)$$

The second integral

$$I_2=\frac{1}{2}\int\limits_{-\frac{\pi}{2}}^{\delta}\frac{\sqrt{(1-\varkappa_1\cos 2\delta)^2+4\lambda_2 s_0\sin 2\delta\,(1-\beta_0\sin 2\delta)}}{1-\beta_0\sin 2\delta} \qquad (6-17)$$

cannot be expressed by elementary functions, being, however, an elliptic integral. For the evaluation, the integral I_2 in question must be expressed by the basic elliptic integrals of first, second and third kind,

$$F(\varphi, k),\quad E(\varphi, k)\quad \text{and}\quad \Pi(\varphi, a^2, k)$$

as for the evaluation of the latter ones tables and manuals are available (*e.g.*[3]).

For this purpose, first of all the root expression must be brought to the denominator:

$$I_2=\frac{1}{2}\int\limits_{-\frac{\pi}{2}}^{\delta}\frac{4\lambda_2 s_0\sin 2\delta+\dfrac{1-2\varkappa_1\cos 2\delta+\varkappa_1^2\cos^2 2\delta}{1-\beta_0\sin 2\delta}}{\sqrt{1-2\varkappa_1\cos 2\delta+\varkappa_1^2\cos^2 2\delta+4\lambda_2 s_0\sin 2\delta-4\lambda_2 s_0\beta_0\sin^2 2\delta}}\,d\delta. \qquad (6-18)$$

Then by the well-known substitution $x=\operatorname{tg}\delta$, $\delta=\operatorname{arctg}x$ applying relations

$$\sin 2\delta=\frac{2x}{1+x^2};\qquad \cos 2\delta=\frac{1-x^2}{1+x^2};\qquad d\delta=\frac{dx}{1+x^2}$$

the respective integral may assume the following form

$$I_2=\frac{1}{2}\int\limits_{-\frac{\pi}{2}}^{\delta}\frac{\dfrac{8\lambda_2 s_0 x}{x^2+1}+\dfrac{(1+\varkappa_1)^2\left(x^2+\dfrac{1-\varkappa_1}{1+\varkappa_1}\right)^2}{(x^2-2\beta_0 x+1)(x^2+1)}}{\sqrt{(1+\varkappa_1)^2x^4+8\lambda_2 s_0x^3+2(1-\varkappa_1^2-8\lambda_2 s_0\beta_0)x^2+8\lambda_2 s_0 x+(1-\varkappa_1)^2}}\,dx. \qquad (6-19)$$

Let us factorize the coefficient belonging to the term of the highest degree in the denominator, divide with its root the numerator, then let us expand the numerator into partial fractions! After all the integral may be written as follows:

$$I_2=\int\limits_{\infty}^{y}\frac{\operatorname{Re}\left\{N_0+\dfrac{N_j}{x-j}+\dfrac{\overline{N}_n}{x-\overline{n}}\right\}}{\sqrt{P(x)}}\,dx, \qquad (6-20)$$

where

$$N_0 = \frac{1 + \varkappa_1}{2} \tag{6—21}$$

$$N_j = \frac{1}{1 + \varkappa_1} \left(4\lambda_2 s_0 + \frac{\varkappa_1^2}{\beta_0}\right) \tag{6—22}$$

$$\bar{N}_n = \frac{1 + \varkappa_1}{2} \frac{\left[(\beta_0 + j\sqrt{1-\beta_0^2})^2 + \frac{1-\varkappa_1}{1+\varkappa_1}\right]^2}{j\sqrt{1-\beta_0^2}\,[(\beta_0 + j\sqrt{1+\beta_0^2})^2 + 1]} =$$

$$= (1 + \varkappa_1) \frac{\left[\beta_0\,(\beta_0 + j\sqrt{1-\beta_0^2})^2 - \frac{\varkappa_1}{1+\varkappa_1}\right]^2}{j\beta_0\sqrt{1-\beta_0^2}\,(\beta_0 + j\sqrt{1-\beta_0^2})} \tag{6—23}$$

and

$$n = \beta_0 + j\sqrt{1 - \beta_0^2} \tag{6—24}$$

finally

$$P(x) = x^4 + \frac{8\lambda_2 s_0}{(1+\varkappa_1)^2} x^3 + 2\,\frac{1 - \varkappa_1^2 - 8\lambda_2 s_0 \beta_0}{(1+\varkappa_1)^2} x^2 + \frac{8\lambda_2 s_0}{(1+\varkappa_1)^2} x + \left(\frac{1-\varkappa_1}{1+\varkappa_1}\right)^2 . \tag{6—25}$$

From condition (6—14) substituting $x = \text{tg}\,\delta$ it follows that $P(x) \geqq 0$.

The equation $P(x) = 0$ of fourth degree has four different roots and these may form only *two conjugate complex pairs of roots.* (Otherwise there existed two, or four different real roots, consequently, in the interval

$$-\infty < x < +\infty$$

the function $P(x)$ became zero for two, or four real values of x and $P(x)$ changed sign in the vicinity of these points. Accordingly, in certain intervals determined by the roots, the condition $P(x) < 0$ would prevail, engendering a contradiction to condition $P(x) \geqq 0$ and besides $\sqrt{P(x)}$ would become imaginary.)

It must be noted the roots not being too far from the values given by the following expression

$$x_{1,2,3,4.} = \frac{-2\lambda_2 s_0}{(1+\varkappa_1)^2} \pm \sqrt{\left(\frac{-2\lambda_2 s_0}{(1+\varkappa_1)^2}\right)^2 + \frac{4\lambda_2 s_0 \beta_0}{(1+\varkappa_1)^2}} \pm$$

$$\pm j \sqrt{\left(\frac{1-\varkappa_1}{1+\varkappa_1}\right) - \left(\frac{-2\lambda_2 s_0}{(1+\varkappa_1)^2} \pm \sqrt{\left(\frac{-2\lambda_2 s_0}{(1+\varkappa_1)^2}\right)^2 + \frac{4\lambda_2 s_0 \beta_0}{(1+\varkappa_1)^2}}\right)^2} . \tag{6—26}$$

Namely, the equation of fourth degree, the roots of which are expressions (6—26), has the same coefficients as Eq. (6—25), merely instead of

$$\frac{8\lambda_2 s_0}{(1+\varkappa_1)^2}$$

the coefficient belonging to the term of first degree is

$$\frac{1-\varkappa_1}{1+\varkappa_1}\,\frac{8\lambda_2 s_0}{(1+\varkappa_1)^2}\,.$$

6.3. Evaluation of the elliptic integral

To evaluate the elliptic integral (6—20), let the two complex pairs of roots be

$$\bar{a} = a_r + j\,a_j \qquad \hat{a} = a_r - j\,a_j$$

and (6—27)

$$\bar{c} = c_r + j\,c_j \qquad \hat{c} = c_r - j\,c_j\,.$$

Evidently,

$$P(x) = (x-\bar{a})\,(x-\hat{a})\,(x-\bar{c})\,(x-\hat{c}) = [(x-a_r)^2 + a_j^2]\,[(x-c_r)^2 + c_j]. \quad (6\text{—}28)$$

Following [3], but with some other notations, let us further denote

$$\begin{aligned} A^2 &= (a_r - c_r)^2 + (a_j + c_j)^2 \\ B^2 &= (a_r - c_r)^2 + (a_j - c_j)^2 \end{aligned} \quad (6\text{—}29)$$

$$k^2 = \frac{4\,A\,B}{(A+B)^2} \quad (6\text{—}30)$$

$$\gamma = \frac{2}{A+B} \quad (6\text{—}31)$$

$$\alpha^2 = \frac{4a_j^2 - (A-B)^2}{(A+B)^2 - 4a_j^2}\,. \quad (6\text{—}32)$$

Let us introduce a new variable:

$$\operatorname{tg}\vartheta = \frac{x - a_r + a_j\,\alpha}{-\alpha x + a_r\,\alpha + a_j}\,, \quad (6\text{—}33)$$

i. e.

$$x = \frac{a_r - a_j\,\alpha + (a_r\,\alpha + a_j)\operatorname{tg}\vartheta}{1 + \alpha\operatorname{tg}\vartheta} \quad (6\text{—}34)$$

and so the new limit of integration is:

$$\varphi = \operatorname{arc\,tg}\frac{y - a_r + a_j\,\alpha}{-\alpha y + a_r\,\alpha + a_j}\,. \quad (6\text{—}35)$$

Some coherent values of the limits of integration are

$$
\begin{aligned}
&y = -\infty && \varphi_{-\infty} = \operatorname{arc\,tg} \frac{1}{-a} \\
&y = 0 && \varphi_0 = \operatorname{arc\,tg} \frac{-a_r + a_j a}{a_r a + a_j} \\
&y = y_0 = a_r - a_j a && \varphi = 0 \\
&y = y_R = \frac{a_r a + a_j}{a} && \varphi = \frac{\pi}{2} \\
&y = +\infty && \varphi = \varphi_{-\infty} + \pi .
\end{aligned}
\tag{6—36}
$$

On the basis of the above relations (6—27)...(6—36), it may first of all be shown [3] that

$$
\int_{y_0}^{y} \frac{dx}{\sqrt{P(x)}} = \gamma \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} = \gamma F(\varphi, k) \tag{6—37}
$$

i. e. the latter integral may be reduced to the basic elliptic integral of first kind $F(\varphi, k)$.

Secondly it may be proved [3, 4] that

$$
\int_{y_0}^{y} \frac{dx}{(x-n)\sqrt{P(x)}} =
$$

$$
= \frac{\gamma}{y_0 - n} \left[\int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} + \frac{a_n - a}{a} \int_0^{\varphi} \frac{d\vartheta}{(1 + a_n \operatorname{tg} \vartheta)\sqrt{1 - k^2 \sin^2 \vartheta}} \right] =
$$

$$
= \frac{\gamma}{y_0 - n} \frac{1 + a a_n}{1 + a_n^2} \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}} + \tag{6—38}
$$

$$
+ \frac{\gamma}{y_0 - n} a_n \frac{a_n - a}{1 + a_n^2} \int_0^{\varphi} \frac{d\vartheta}{[1 + (1 + a_n^2)\sin^2 \vartheta]\sqrt{1 - k^2 \sin^2 \vartheta}} -
$$

$$
- \frac{y}{y_0 - n} (a_n - a) \int_\theta^{\varphi} \frac{\cos \vartheta \sin \vartheta}{[1 - (1 + a_n^2)\sin^2 \vartheta]\sqrt{1 - k^2 \sin^2 \vartheta}} d\vartheta,
$$

where

$$
a_n = \frac{a_r a + a_j - a n}{a_r - a_j a - n} = a \frac{y_R - n}{y_0 - n} . \tag{6—38'}
$$

The integral figuring in the last term of Eq. (6—38) may be expressed in a closed form by the elementary function:

$$f(\varphi, 1+a_n^2, k) = -\int_0^{\varphi} \frac{\cos\vartheta \sin\vartheta}{[1-(1+a_n^2)\sin^2\vartheta]\sqrt{1-k^2\sin^2\vartheta}}\, d\vartheta = \tag{6—39}$$

$$= -\frac{1}{2\sqrt{(1+a_n^2)(k'^2+a_n^2)}} \ln \frac{\dfrac{\sqrt{1+a_n^2}\sqrt{1-k^2\sin^2\varphi}+\sqrt{k'^2+a_n^2}}{\sqrt{1+a_n^2}+\sqrt{k'^2+a_n^2}}}{\dfrac{\sqrt{1+a_n^2}\sqrt{1-k^2\sin^2\varphi}-\sqrt{k'^2+a_n^2}}{\sqrt{1+a_n^2}-\sqrt{k'^2+a_n^2}}},$$

where $k' = \sqrt{1-k^2}$ is the so-called complementary modulus.

Again considering the definitions of the basic elliptic integrals, as well as relation (6—39), the integral (6—38) may be reduced according to

$$\int_{y_0}^{y} \frac{dx}{(x-n)\sqrt{P(x)}} = \frac{\gamma}{y_0-n}\,\frac{1+a\,a_n}{1+a_n^2}\,F(\varphi,k) +$$

$$+\frac{\gamma}{y_0-n}\,a_n\,\frac{a_n-a}{1+a_n^2}\,\Pi(\varphi, 1+a_n^2, k) + \frac{\gamma}{y_0-n}(a_n-a) f(\varphi, 1+a_n^2, k) \tag{6—40}$$

onto the basic elliptic integrals of first and third kind,

$$F(\varphi, k) \text{ and } \Pi(\varphi, (\sqrt{1+a_n^2})^2, k).$$

Knowing integrals (6—37) and (6—40), establishment of the primitive function of integral (6—20) involves no difficulty (the limits of integration, however, are not yet substituted):

$$(I_2) = \mathrm{Re}\left\{\left(N_0 + \frac{N_j}{y_0-j}\,\frac{1+a\bar{a}_j}{1+\bar{a}_j^2} + \frac{\bar{N}_n}{y_0-\bar{n}}\,\frac{1+a\bar{a}_n^2}{1+\bar{a}_n^2}\right)\gamma F(\varphi,k) + \right.$$

$$+\frac{N_j}{y_0-j}\,\bar{a}_j\,\frac{\bar{a}_j-a}{1+\bar{a}_j^2}\,\gamma\,\bar{\Pi}(\varphi, 1+\bar{a}_j^2, k) + \frac{\bar{N}_n}{y_0-\bar{n}}\,\bar{a}_n\,\frac{\bar{a}_n-a}{1+\bar{a}_n^2}\,\gamma\,\bar{\Pi}(\varphi, 1+\bar{a}_n^2, k) +$$

$$\left. +\frac{N_j}{y_0-j}(\bar{a}_j-a)\,\gamma\bar{f}(\varphi, 1+\bar{a}_j^2, k) + \frac{\bar{N}_n}{y_0-\bar{n}}(\bar{a}_n-a)\,\gamma\bar{f}(\varphi, 1+\bar{a}_n^2, k)\right\}, \tag{6—41}$$

where

$$\bar{a}_j = (a_n)_{n=j} \qquad \bar{a}_n = (a_n)_{n=\bar{n}}$$

consequently, in Eq. (6—38′) $n = j$ and $n = \bar{n}$, respectively, are to be substituted.

Finally, substituting the limits of integration (6—36), the desired solution is:

$$\begin{aligned} I_2 = \mathrm{Re}\{&\bar{M}_F[F(\varphi,k) - F(\varphi_{-\infty},k)] + \\ &+ \bar{M}_{\Pi j}[\bar{\Pi}(\varphi, 1+\bar{a}_j^2, k) - \bar{\Pi}(\varphi_{-\infty}, 1+\bar{a}_j^2, k)] + \\ &+ \bar{M}_{\Pi n}[\bar{\Pi}(\varphi, 1+\bar{a}_n^2, k) - \bar{\Pi}(\varphi_{-\infty}, 1+\bar{a}_n^2, k)] + \\ &+ \bar{M}_{fj}[\bar{f}(\varphi, 1+\bar{a}_j^2, k) \quad -\bar{f}(\varphi_{-\infty}, 1+\bar{a}_j^2, k)] + \\ &+ \bar{M}_{fn}[\bar{f}(\varphi, 1+\bar{a}_n^2, k) \quad -\bar{f}(\varphi_{-\infty}, 1+a_n^2, k)]\}, \end{aligned} \tag{6—42}$$

where the values of the complex coefficients are:

$$\bar{M}_F = \left(N_0 + \frac{N_j}{y_0 - j}\,\frac{1 + a\bar{a}_j}{1 + \bar{a}_j^2} + \frac{\bar{N}_n}{y_0 - \bar{n}}\,\frac{1 + a\bar{a}_n}{1 + \bar{a}_n^2}\right)\gamma$$

$$\bar{M}_{\Pi j} = \frac{N_j}{y_0 - j}\,\bar{a}_j\,\frac{\bar{a}_j - a}{1 + \bar{a}_j^2}\,\gamma$$

$$\bar{M}_{\Pi n} = \frac{\bar{N}_n}{y_0 - \bar{n}}\,\bar{a}_n\,\frac{\bar{a}_n - a}{1 + \bar{a}_n^2}\,\gamma$$

$$\bar{M}_{fj} = \frac{N_j}{y_0 - j}\,(\bar{a}_j - a)\,\gamma$$

$$\bar{M}_{fn} = \frac{\bar{N}_n}{y_0 - \bar{n}}\,(\bar{a}_n - a)\,\gamma\,.$$

6.4. Some remarks concerning determination of the time-angle function

In knowledge of relations (6—42) and (6—16) the desired time-angle function (6—15) may now be calculated.

The most expedient solution is to choose round figures for φ and to calculate the integral I_2 for these values. Then considering Eq. (6—34) and $\delta = \operatorname{arc\,tg} x$, on the basis of relation

$$\delta = \operatorname{arc\,tg} \frac{a_r - a_j a + (a_j + a_r a)\operatorname{tg}\varphi}{1 + a \operatorname{tg}\varphi}$$

for the chosen values φ the angle δ may be determined. By the aid of the latter, the integral I_1 may be calculated. Finally, the values of I_2 and I_1 in this way evaluated are to be summarized.

While φ is changing within the limits $\varphi_{-\infty}$ and $\varphi_{-\infty} + \pi = \varphi_{+\infty}$ (*i. e.* y is within the limits $-\infty$ and $+\infty$) δ is changing within the limits $-\pi/2$ and $+\pi/2$.

To evaluate the incomplete basic elliptic integrals of first and second kind, in the range of $0 \leqq \varphi \leqq \pi/2$ tables are available (*e. g.* [3, 5]).

Determination of the incomplete elliptic integral of third kind is somewhat more complicated. For its evaluation, besides the elementary functions, the tabulated *Heuman's Lambda* functions $\Lambda_0(\varphi, k)$ and the tabulated function $KZ(\varphi, k)$ — *i. e.* the K-times value of *Jacobi's Zeta* function, where K is the complete elliptic integral of first kind — also *Theta* functions are needed, necessitating an evaluation by infinite series.

In the mathematical books (*e. g.* [3]) formulas are available for reducing the functions $\Pi(\varphi, \bar{a}^2, k)$ with *complex parameters* and the expressions

$$\operatorname{Re} \bar{M}\, \bar{\Pi}(\varphi, \bar{a}^2, k) = \frac{1}{2} \bar{M}\, \bar{\Pi}(\varphi, \bar{a}^2, k) + \frac{1}{2} \hat{M}\, \hat{\Pi}(\varphi, \hat{a}^2, k)$$

respectively, to elliptic integrals of third kind with *real parameters.*

Evaluation of the elementary functions with complex parameters theoretically involves no difficulties.

As demonstrated in the afore-said, the time-angle function may be expressed in a not too complicated form by elliptic and elementary functions. Nevertheless, numerical evaluation is a quite lengthy and troublesome procedure.

Naturally, in possession of the time-angle function, determination of the slip-time, current-time, apparent power-time, reactive power-time, etc. functions may be effected according to the procedure already known [1].

6.5. Determination of the period

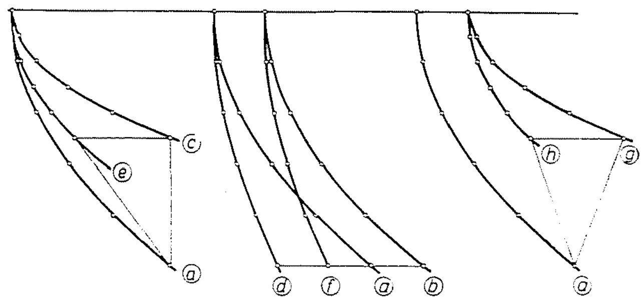
Choosing the value $\delta = \pi/2$ for the upper limit of integral I_1, the value $\varphi = -\pi/2$ ($\gamma = \gamma_R - \pi$; $\delta = \delta_R - \pi$) for the lower limit of integral I_2 and for its upper limit the value $\varphi = +\pi/2$ ($\gamma = \gamma_R$; $\delta = \delta_R$), finally, multiplying both integrals by 2, for the whole period the following relation may be obtained:

$$\begin{aligned} - s_0 \omega_0 T = {} & \frac{\pi}{\sqrt{1-\beta^2}} + \qquad (6\text{—}43) \\ & + 4 \operatorname{Re} \left\{ \bar{M}_F K + \bar{M}_{\Pi j} \bar{\Pi}\left(\frac{\pi}{2}, 1 + \bar{a}_j^2, k\right) + M_{\Pi n} \bar{\Pi}\left(\frac{\pi}{2}, 1 + \bar{a}_n^2, k\right) + \right. \\ & \left. + \bar{M}_{fj} \bar{f}\left(\frac{\pi}{2}, 1 + \bar{a}_j^2, k\right) + \bar{M}_{fn} \bar{f}\left(\frac{\pi}{2}, 1 + \bar{a}_n^2, k\right) \right\}, \end{aligned}$$

where K is the complete elliptic integral of first kind: $K = F\left(\frac{\pi}{2}, k\right)$.

6.6. Special cases. The possibility of generalization.

The primitive parabolic approximation may have seven special cases, when among the three parameters β_0, $\varkappa_1$, λ_2 one, or more, become zero. These possibilities are illustrated in Fig. 6—2. In all cases the direct-axis admittance diagram $\overline{y}_d$ is the same (curve a), and only the quadrature-axis admittance diagram $\overline{y}_q$ is changed. The points belonging to identical slips are marked by crosses on the respective curves and some of them are also linked by a thin line for the sake of a better illustration. It must be noted, that in Fig. 6—2 all para-



β	0	$\beta>0$	0	0	0	$\beta>0$	$\beta>0$	$\beta>0$
$\varkappa$	0	0	$\varkappa>0$	0	$\varkappa>0$	0	$\varkappa>0$	$\varkappa>0$
λ	0	0	0	$\lambda>0$	$\lambda>0$	$\lambda>0$	0	$\lambda>0$
$\bar{y}_d$	a	a	a	a	a	a	a	a
$\bar{y}_q$	a	b	c	d	e	f	g	h

Fig. 6—2

meters are positive, or zero. (Naturally, some parameters may be negative too, and also in this case suitable curves may be plotted, these, however, are not discussed here.)

The respective special cases are as follows:

a) All of three parameters are zero. The curves $\overline{y}_d$ and $\overline{y}_q$ are coincident (the rotor is symmetric), to the different torques different slips, but constant slips are belonging. Function $s(\delta)$ is now again given by (4—5), while function $t(\delta)$ by Eqs. (4—6) and (4—7), respectively.

b) Solely $\beta \neq 0$ (while the other two parameters equal zero). Curves $\overline{y}_d$ and $\overline{y}_q$ are running parallelly. Though both curves are parabolas, the function $s(\delta)$ and $t(\delta)$ may now again be calculated by formula (4—22).

c) Exclusively $\varkappa_1 \neq 0$. The abscissae of the points belonging to the same slip at the two admittance curves are of the same magnitude. For calculating functions $s(\delta)$ and $t(\delta)$ now again formulae (4—11) and (4—13), respectively, may be adopted.

d) Only $\lambda_2 \neq 0$. On curves $\bar{y}_d$ and $\bar{y}_q$ the ordinates of the points belonging to the same slip are of the same value. Now formula (6—11) becomes considerably more simple

$$\frac{s_0}{s(\delta)} = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\lambda_2 s_0 \sin 2\delta}. \qquad (6-44)$$

$$(1 > |4\lambda_2 s_0|).$$

As cos 2δ is not present, but only sin 2δ is figuring, the slip curve

$$\frac{s(\delta)}{s_0} = \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\lambda_2 s_0 \sin 2\delta}} \qquad (6-45)$$

may be divided into four sections during a complete relative rotor rotation, while angle δ changes in the range 2π. The duration of each section equals $\pi/2$. In the odd sections the slip curves are of the same course, similarly the even sections are also congruent, while in any section the course of the slip is axially symmetrical with respect to the slip curve of the contiguous section, that is, each of the two adjacent sections may be regarded as reflected images.

By the proper choice of the integration interval, the calculation of integral I_2 necessary for determining the time-angle function becomes considerably more simple. If *e. g.* $\lambda_2 > 0$ (and $s_0 < 0$), then the most practicable solution is to integrate the reciprocal slip function $-1/s(\delta)$ in the domain $-\pi/4 < \delta \leqq \pi/4$:

$$I_2' = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\delta} \sqrt{1 - (-4\lambda_2 s_0)\sin 2\delta}\, d\delta . \qquad (6-46)$$

The latter integral is again an elliptic one. To calculate it, the following substitution is realized:

$$\sin\vartheta = \sqrt{\frac{1 + \sin 2\delta}{2}}; \qquad \delta = \frac{1}{2}\arcsin(2\sin^2\vartheta - 1); \qquad d\delta = d\vartheta .$$

Thereby

$$I_2' = \frac{1}{2}\int_0^{\varphi} \sqrt{1 - 4\lambda_2 s_0 - (-8\lambda_2 s_0)\sin^2\vartheta}\quad d\vartheta$$

that is

$$I_2' = M_E\int_0^{\varphi} \sqrt{1 - k^2\sin^2\vartheta}\, d\vartheta = M_E E(\varphi, k), \qquad (6-47)$$

where

$$M_E = \frac{1}{2}\sqrt{1 - 4\lambda_2 s_0} > \frac{1}{2}$$

$$k^2 = \frac{-8\lambda_2 s_0}{1 - 4\lambda_2 s_0} < 1 .$$

Consequently, the integral I_2' may be expressed by the basic elliptic integral of second kind. As at the same time

$$I_1' = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\delta} d\delta = \frac{1}{2}\left(\delta + \frac{\pi}{4}\right) \tag{6—48}$$

thus:

$$-s_0\,\omega_0\,t = \frac{1}{2}\left(\delta(\varphi) + \frac{\pi}{4}\right) + M_E\,E\,(\varphi, k) \tag{6—49}$$

where $\delta(\varphi)$ denotes, that for values φ adopted in the course of the evaluation the values of δ must be calculated by the following formula:

$$\delta = \frac{1}{2}\arcsin\left(2\sin^2\varphi - 1\right).$$

The relation (6—49) provides the function $t(\delta)$ merely for $-\pi/4 < \delta \leqq \pi/4$ *i. e.* in an interval of magnitude $\pi/2$, yet we need it, at least in an interval of magnitude π. Instead of further calculations, it is sufficient to consider only the enumerated symmetry conditions, as on the basis of these, *e. g.* in the interval $-3\pi/4 < \delta \leqq -\pi/4$, curve $t(\delta)$ has to be with respect to point $t(-\pi/4) = 0$ in center symmetry, as compared with the curve given by formula (6—49).

For calculating the period of the complete rotation ($\delta = 2\pi$), in the right side of Eq. (6—49) the value $\varphi = \pi/2$ has to be substituted, then multiplying by 4, results:

$$\omega_0 T = -\frac{\pi + 4\,M_E\,E}{s_0}. \tag{6—50}$$

This case is worthy of special attention, because it may be regarded as the most simple parabolic approximation. The quadratic approximation leads accordingly, even in its most simple form to an elliptic integral, whose evaluation is, however, now very easy.

Finally, if $\lambda_2 < 0$ (*i. e.* curve $\bar{y}_q$ is more deflected than curve y_d) the most simple solution is to integrate between the limits $\pi/4 < \delta \leqq 3\pi/4$. With substitution of $\delta' = \delta - \pi/2$ this case may be reduced to case $\lambda_2 > 0$ and for the evaluation of the elliptic integral the above formulas (6—46)...(6—49) may directly be adopted, merely writing instead of $-4\,\lambda_2 s_0 > 0$ the value $4\,\lambda_2 s_0 > 0$ everywhere.

e) Solely $\beta_0 = 0$, while $\varkappa_1 \neq 0$, $\lambda_2 \neq 0$. The curves $\bar{y}_d$ and $\bar{y}_q$ start from a common point, but the ordinates, as well as the abscissae of the points belonging to the same slip, are different. This case does not involve a considerable simplification with respect to $\beta_0 \neq 0$. Since the numerator of integral I_2 is now

$$\mathrm{Re}\left\{ N_0 + \frac{N_{1j}}{x-j} + \frac{N_{2j}}{(x-j)^2} \right\}$$

not only the integrals

$$\int\limits_{y_0}^{y} \frac{dx}{\sqrt{P(x)}} \qquad \text{and} \qquad \int\limits_{y_0}^{y} \frac{dx}{(x-j)\sqrt{P(x)}}$$

but also the integral of form

$$\int\limits_{y_0}^{y} \frac{dx}{(x-j)^2\sqrt{P(x)}}$$

must be evaluated. Consequently, this case is even more complicated. In addition to the basic elliptic integrals of first and third kind, now also the integral of second kind plays a part. (Not speaking of the two elementary functions.)

f) Exclusively $\varkappa_1 = 0$, while $\beta_0 \neq 0$, $\lambda_2 \neq 0$. Curves $\bar{y}_q$ and $\bar{y}_d$ start from different points, the ordinates of the points belonging to the same slip are of the same magnitude. At that time only $\sin 2\delta$ figures in the formula of the slip ($\cos 2\delta$ is absent), therefore similarly to case d) it is sufficient also now to restrict ourselves to the integration interval $-\pi/4 < \delta \leq \pi/4$ and substitute $x = \sin 2\delta$. In the integral I_2 besides an elementary function only the basic elliptic integrals of first and third kind are figuring.

If $-\lambda_2 s_0 = \beta_0$ and $\varkappa_1 = 0$ then from formula (6—11) $s = s_0$, that is, the slip is constant. This occurs if the straight line $g = \text{const.}$ passes through the point of intersection of the two curves $\bar{y}_d$ and $\bar{y}_q$.

g) Merely $\lambda_2 = 0$, while $\beta_0 \neq 0$ and $\varkappa_1 \neq 0$. The difference between the abscissas of the suitable points lying on curves $\bar{y}_d$ and $\bar{y}_q$ is always equal. At such times function $s(\delta)$ may be determined through formula (4—26), while function $t(\delta)$ from relation (4—27), or (4—28). Consequently, the general formulas of the primitive linear approximation may directly be adopted for calculating the primitive parabolic approximation of this type.

h) In the most general case of the primitive parabolic approximation, when none of the three parameters equals zero, for calculating the slip-angle function $s(\delta)$ formula (6—13), while for determining the time-angle function $t(\delta)$ formulas (6—15), (6—16) and (6—42) are to be applied.

It is worth-while mentioning here, that instead of the primitive parabolic approximation also a more general quadratic approximation may be adopted. In the most general case both $g(s)$ and $b(s)$ are general quadratic expressions. The condition $g = \text{const.}$ now also leads to an equation of second degree in s. Accordingly, $s(\delta)$ may be expressed in an explicite form. When integrating its reciprocal function, merely elliptic integrals are again needed for the evaluation.

Summary

Adjoining the two preceding articles [1, 2], the present paper gives a survey of the results obtained by substituting each of the direct-axis and quadrature-axis admittance diagrams by single parabolas, that is, through quadratic approximation, in order to examine the asynchronous operation of turbogenerators and to determine the slip variating with the time.

As regards the method previously suggested, the slip-time function may be expressed in an explicite form and its reciprocal function may be integrated in a closed form now also, nevertheless, there is a need for the evaluation of elliptic integrals of first, second, moreover of third kind.

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