

# THE INTERPRETATION OF CHARACTERISTICS AND OF FUNDAMENTAL EQUATIONS OF THE ELECTROMAGNETIC FIELD

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It is customary to characterize the electromagnetic field by several vectors. In the course of computations, two vectors are used to describe the electric field and two for the magnetic field (vectors  $\mathbf{E}$  and  $\mathbf{D}$ , and  $\mathbf{H}$  and  $\mathbf{B}$ , respectively). There is much discussion about the physical interpretation of these vectors, even in our days. The basic questions are: Can the field be characterized by a single electric and a single magnetic vector? Which pair of vectors can be taken as of fundamental importance in this respect? What physical interpretation can then be given to the other pair of vectors?

The electric field used to be unequivocally characterized by the electric field strength ( $\mathbf{E}$ ), quantity  $\mathbf{D}$  being regarded as secondary. At present, the viewpoint of using the vector of induction ( $\mathbf{B}$ ) for the characterization of the magnetic field is quite wide-spread, as the power effect, further the induced field strength depends on the vector  $\mathbf{B}$  and on the variation of this latter with time, respectively. Hence value  $\mathbf{H}$  is only of secondary importance, the same as value  $\mathbf{D}$ . Their use is only motivated by the ease of computations attainable by their application. In the presence of substance, namely, several equations for  $\mathbf{D}$  and  $\mathbf{H}$  are very simple, while these same equations describing the field become quite complicated for vectors  $\mathbf{E}$  and  $\mathbf{B}$ .

Relying upon these findings the questions may arise whether these auxiliary vectors can be interpreted physically, further whether their value is determined unequivocally, or whether values used by us are taken arbitrarily. In the following it will be shown that vectors  $\mathbf{D}$  and  $\mathbf{H}$  are vector-potential-like values which are not unequivocally determined by the equation of definition. Further limitations can only be set up in accordance with practicability and only their integral values can be measured.

## 1. Basic equations of the electromagnetic field in vacuo

In vacuo, the electromagnetic field is produced by the charges in the last analysis. In a co-ordinate system fixed in relation to the observer, charges are either static, or in moving condition. Current produced by moving charges

should, however, rather be regarded as an independent quantity. The field strength is divided into electric and magnetic components exactly according to the kind of force acting on static charges, and moving charges (or current), respectively. Accordingly two vector quantities are necessary to describe the field. Vector  $\mathbf{E}$  indicating electric field strength supplies the force acting on a point-like charge  $Q$ :

$$\mathbf{F} = Q\mathbf{E}, \quad (1.1)$$

while vector  $\mathbf{B}$  indicating magnetic field strength supplies the turning moment operative on a point-like circuit of  $\mathbf{m} = IAn^{\circ}$  dipole moment, or the force acting on charge  $Q$  moving at rate  $\mathbf{v}$ :

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \quad \mathbf{F} = Q\mathbf{v} \times \mathbf{B} \quad (1.2)$$

So long as phenomena occur in vacuum, these two vectors are perfectly sufficient for the description of the field. On the basis of Maxwell's equations, charge density ( $\rho$ ) and current density ( $\mathbf{J}$ ) being known, these vectors can be determined. Maxwell's equations take the following form in vacuum:

$$\text{rot } \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \quad (\text{Ia})$$

$$\text{rot } \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{IIa}) \quad (1.3)$$

$$\text{div } \mathbf{B} = 0, \quad (\text{IIIa})$$

$$\text{div } \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (\text{IVa})$$

$$\mathbf{J} = \rho \mathbf{v}, \quad (\text{Va})$$

$$w = \frac{1}{2} \epsilon_0 E^2 + \frac{1}{2} \frac{1}{\mu_0} B^2. \quad (\text{VIa})$$

In the equations,  $\epsilon_0$  and  $\mu_0$  are the permittivity and the permeability of the vacuum, respectively. In the rationalised MKSA (or MSVA) system of units used here:

$$\mu_0 = 4\pi \cdot 10^{-7} \frac{\text{Vs}}{\text{Am}}, \quad \epsilon_0 = \frac{1}{\mu_0 c^2} = 8,855 \cdot 10^{-12} \frac{\text{As}}{\text{Vm}}$$

The equation determining the two field vectors needs not to be given here as it can be deduced from equation (VIa) describing energy density  $w$ . It is

interesting that the equation of continuity

$$\operatorname{div} \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (1.4)$$

which is of fundamental importance, needs not to be written separately, as it can be derived as the divergence of equation (I) by making use of equation (IV).

The two universal constants can be eliminated from the equations by introducing the following two new vectors: The vector of displacement as defined by  $\mathbf{D} = \varepsilon_0 \mathbf{E}$  and the vector named (unfortunately) magnetic field strength as defined by equation  $\mathbf{H} = \mathbf{B} / \mu_0$ . After this formal transformation Maxwell's equations take the following shape:

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (\text{Ib})$$

$$\operatorname{rot} \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{IIb})$$

$$\operatorname{div} \mathbf{B} = 0, \quad (\text{IIIb})$$

$$\operatorname{div} \mathbf{D} = \rho \quad (\text{IVb})$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B}, \quad \mathbf{J} = \rho \mathbf{v} \quad (\text{Vb}) \quad (1.5)$$

$$w = \frac{1}{2} \mathbf{E} \mathbf{D} + \frac{1}{2} \mathbf{H} \mathbf{B}. \quad (\text{VIb})$$

With the help of the first two equations in group (Vb), two vectors can be eliminated and the system of equations in this way arrived at can be solved unequivocally,  $\rho$  and  $\mathbf{J}$  being known values.

## 2. Basic equations in the presence of substance

Equations immediately become complicated if a substance is present in the field. The electromagnetic field, namely, is influenced by the substance, consequently Maxwell's equations should also be modified. Generalization can be effected in three different directions.

The first possibility of generalization is to regard the first four equations under (1.5) as of general validity. From the viewpoint of computation technique, this method is the most convenient, and, consequently, the most wide-spread. The physical meaning of vectors  $\mathbf{E}$  and  $\mathbf{B}$  is determined by equations (1.1) and (1.2), but equation (Vb) determining vectors  $\mathbf{D}$  and  $\mathbf{H}$  should be general-

ized. In the general case  $\mathbf{D}$  and  $\mathbf{H}$  can evidently be written as the sum or difference of a component proportional to vectors  $\mathbf{E}$  and  $\mathbf{B}$ , respectively, and of another vector. For the time being we know nothing about the physical meaning of these additive vectors, and of vectors  $\mathbf{D}$  and  $\mathbf{H}$ . As the last result, Maxwell's equations are expressed in the following form :

$$\operatorname{rot} \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (\text{Ic})$$

$$\operatorname{rot} \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}. \quad (\text{IIc})$$

$$\operatorname{div} \mathbf{B} = 0, \quad (\text{IIIc})$$

$$\operatorname{div} \mathbf{D} = \rho \quad (\text{IVc})$$

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}; \quad \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M},$$

$$\mathbf{J} = \gamma (\mathbf{E} + \mathbf{E}_e) + \rho \mathbf{v}, \quad (\text{Vc}) \quad (2.1)$$

$$w = \frac{1}{2} \mathbf{E} \mathbf{D} + \frac{1}{2} \mathbf{H} \mathbf{B}. \quad (\text{VIc})$$

How can now the vectors characterizing the field be measured within the substance? In fluids and gases  $\mathbf{E}$  and  $\mathbf{B}$  can be measured on the basis of equations (1.1) and (1.2). Vectors  $\mathbf{D}$  and  $\mathbf{H}$ , however, can only be measured indirectly. One of the methods of measurement is, namely, to determine the force effect in vacuo (or in the air, what is practically the same) in slits of given shape cut into the substance at given directions. Now a value  $\mathbf{E}_0$  of field strength and a value  $\mathbf{B}_0$  of induction can be determined from equations (1.1) and (1.2). According to equation (1.5) values of  $\mathbf{D}_0$  and  $\mathbf{H}_0$ , respectively, can be determined in the slit. It follows from the laws on vectors  $\mathbf{D}$  and  $\mathbf{H}$  that in a suitably prepared slit the measured values of  $\mathbf{D}_0$  and  $\mathbf{H}_0$  are just equal to values  $\mathbf{D}$  and  $\mathbf{H}$  produced in the substance. Similar, in principle, is the measurement of  $\mathbf{E}$  and  $\mathbf{B}$  in a solid substance. As the last result, always vectors  $\mathbf{E}$  and  $\mathbf{B}$  are measured, and of these values conclusions are drawn as to the value of vectors  $\mathbf{D}$  and  $\mathbf{H}$ , respectively.

Another method of measurement is to use integral equations

$$\oint_A \mathbf{D} \, d\mathbf{A} = Q,$$

$$\oint_s \mathbf{H} \, ds = \theta \equiv \int_A \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) d\mathbf{A} \quad (2.2)$$

which can be deduced from equations (IVb) and (Ib). As charge  $Q$  and excitation (MMF)  $\mathcal{O}$  can be measured, the above integrals are known. If a condition (with the necessary symmetry or other characteristics) can be produced in such a way that the vectors can unequivocally be determined from the integrals, then the measurement has been carried out. A measurement of this type is *e.g.* the determination of charges induced on a metal plate, or the indication of the compensation of the magnetic field by a dipole. Obviously, in the last result, force effects are observed here too. Under completely homogeneous conditions, in case of a high-grade symmetry,  $\mathbf{D}$  and  $\mathbf{H}$  can also be computed. The control of computations by measurement, however, means in the last effect the measuring of vectors  $\mathbf{E}$  and  $\mathbf{B}$ .

After the determination of vectors  $\mathbf{D}$  and  $\mathbf{H}$  by measurement or computation, the value of vectors  $\mathbf{P}$  and  $\mathbf{M}$  can also be determined. According to experiences, the following formulae are valid for a wide group of substances :

$$\mathbf{P} = \varepsilon_0 \kappa_e \mathbf{E}, \quad \mathbf{M} = \frac{1}{\mu_0} \kappa'_m \mathbf{B} = \kappa_m \mathbf{H} \quad (2.3)$$

where  $\kappa$  is the electric or magnetic susceptibility. (In theory it would be more correct to apply  $\kappa'_m$ , but in general the value  $\kappa_m = \kappa'_m/(1-\kappa'_m)$  is given.) Thus with denominations

$$\varepsilon_0 (1 + \kappa_e) = \varepsilon_0 \varepsilon_r = \varepsilon, \quad \mu_0 (1 + \kappa_m) = \mu_0 \mu_r = \mu \quad (2.4)$$

the following formulae are obtained :

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} \quad (2.5)$$

where  $\varepsilon$  and  $\mu$  are scalar values depending on the substance (eventually also on temperature, density and frequency). In crystalline substances  $\varepsilon$  and  $\mu$  can also be interpreted as tensors (the same can be said of conductivity  $\gamma$ ). For many materials (electretes, ferromagnetic substances), however,  $\varepsilon$  and  $\mu$  cannot at all be analytically determined. For such hard magnetic materials the approximation  $\mathbf{M} = \mathbf{M}_0$  is sometimes allowable, where  $\mathbf{M}_0$  is a vector independent of the magnetic field. In general, however, we have to make use of approximation methods in such cases (*e.g.* the application of the magnetization curve, or any analytic approximation of it).

It can be proved by computation that even vectors  $\mathbf{P}$  and  $\mathbf{M}$  can be given a physical meaning. By imagining the material to be built up, from an electric and magnetic point of view, of molecular dipoles, having dipole moments  $\mathbf{p}$  and  $\mathbf{m}$ , respectively, then the specific density of these dipole moments in the field supplies the values of the electric and magnetic polarization in the given point of the field. These are precisely vectors  $\mathbf{P}$  and  $\mathbf{M}$ :

$$\mathbf{P} = \lim_{\Delta V \rightarrow 0} \frac{\sum \mathbf{p}}{\Delta V}; \quad \mathbf{M} = \lim_{\Delta V \rightarrow 0} \frac{\sum \mathbf{m}}{\Delta V}. \quad (2.6)$$

It should be noted that in case of ferromagnetic materials the origin of magnetic dipoles are much more complicated than those of the electric dipoles. From the viewpoint of the phenomenological discussion, however, this quantum-physical background needs be of no concern.

To sum up, the way of generalization is the following: Vectors  $\mathbf{D}$  and  $\mathbf{H}$ , further  $\mathbf{P}$  and  $\mathbf{M}$  are formally introduced, the method of measuring vectors  $\mathbf{D}$  and  $\mathbf{H}$  is determined, empirical relationships are found between vectors  $\mathbf{E}$ ,  $\mathbf{D}$ ,  $\mathbf{P}$  and  $\mathbf{B}$ ,  $\mathbf{H}$ ,  $\mathbf{M}$ , respectively, finally the physical meaning of vectors  $\mathbf{P}$  and  $\mathbf{M}$  can be given.

### 3. Further possibilities of building up the basic equations

Another method of generalization is based on Lorentz' electron theory, according to which the presence of a substance only means that charges and currents coupled to the substance (molecular dipoles, and molecular circuits or magnetic dipoles, respectively) also produce a field, beyond the external charges and currents. Consequently, Maxwell's equations for vacuum are sufficient to describe the field, as the word "substance" only means the appearance of additive charges and currents. The average value (in respect to time and space) of electric and magnetic field strength values computed in the described way, can be measured. This method is exceptionally clear and well-arranged, it corresponds strictly to physical realities. For practical computations, however, just because of the complete generalization, it is, it may be said, unusable.

The third method of building up the basic equations is in essence a middle way between the previously mentioned two ways. The field is described by the well-defined vectors  $\mathbf{E}$  and  $\mathbf{B}$ , and the presence of the substance is accounted for by the polarization vectors as defined by equation (2.6). The basic equations can be formally obtained by eliminating vectors  $\mathbf{D}$  and  $\mathbf{H}$  from the equations given under (2.1). In this way we arrive at the following system of equations:

$$\text{rot } \mathbf{B} = \mu_0 \left( \mathbf{J} + \text{rot } \mathbf{M} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \right), \quad (\text{Id})$$

$$\text{rot } \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{IIId})$$

$$\text{div } \mathbf{B} = 0, \quad (\text{IIIId})$$

$$\text{div } \mathbf{E} = \frac{1}{\varepsilon_0} (\rho - \text{div } \mathbf{P}), \quad (\text{IVd})$$

$$\mathbf{P} = \lim_{\Delta V \rightarrow 0} \frac{\sum \mathbf{p}}{\Delta V}, \quad \mathbf{M} = \lim_{\Delta V \rightarrow 0} \frac{\sum \mathbf{m}}{\Delta V}, \quad (\text{Vd}) \quad (3.1)$$

$$\mathbf{J} = \gamma (\mathbf{E} + \mathbf{E}_e) + \rho \mathbf{v},$$

$$\frac{\partial w}{\partial t} = \left[ \frac{\varepsilon_0}{2} \frac{\partial \mathbf{E}^2}{\partial t} + \varepsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} \right] + \left[ \frac{1}{2\mu_0} \frac{\partial \mathbf{B}^2}{\partial t} - \mathbf{M} \cdot \frac{\partial \mathbf{B}}{\partial t} \right]. \quad (\text{VIId})$$

Equation systems (2.1) and (3.1) are evidently equivalent in essence, from the mathematical point of view. In the first case vectors  $\mathbf{D}$ ,  $\mathbf{H}$ ,  $\mathbf{P}$  and  $\mathbf{M}$  have only a formal role, the principle of their measuring and their physical meaning will be shown only later. In equation system (3.1) every quantity has a physical interpretation. The difficulty lies in finding the relationship between the polarization vectors and the field intensity vectors. It now stands to reason to declare that polarization is simply proportional to field intensity in general, and equations (2.3), or in single cases the polarization, are values independent of field intensity. The most general linear formulae for vectors  $\mathbf{P}$  and  $\mathbf{M}$  are

$$\mathbf{P} = \varepsilon_0 \mathbf{E}_s + \varepsilon_0 \varkappa_e \mathbf{E} + \mathbf{P}_0, \quad \mathbf{M} = \frac{1}{\mu_0} \mathbf{B} + \frac{\varkappa'_m}{\mu_0} \mathbf{B} + \mathbf{M}. \quad (3.2)$$

The very complicated relationships of ferromagnetic materials cannot naturally be followed up here either, as the difference between the two methods of discussion lies not in the content, but in the way it is comprehended. The equations presented here are somewhat more complicated than usual, but every quantity has a direct physical meaning.

It should be noted that the equation of continuity (1.4) can be deduced from the first four equations here too.

Equation VI is presented in a somewhat different form, as in place of energy density its variation with respect to time has been given. We shall revert to the discussion of this equation later.

#### 4. The analogy of electric and magnetic characteristics

It is evident from the preceding discussion that, at building up the equation system, vectors  $\mathbf{E}$  and  $\mathbf{B}$  were regarded as values determining field strength, vectors  $\mathbf{D}$  and  $\mathbf{H}$  having only the role of auxiliary quantities for computation purposes. Experimental results proving equations (1.1) and (1.2) make this analogy evident. In spite of this, vectors  $\mathbf{E}$  and  $\mathbf{H}$  are regarded by many as analogous. The reason is the historical origin, which is also shown by the fact that both quantities are named field strength vector. In accordance with the preceding discussion, vector  $\mathbf{B}$  should be rightly named magnetic field strength. At the same time Maxwell's equations given under (2.1) undeniably hint at this analogy, even formally. *E.g.* in case of no charge or current

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \quad (\text{Ie}), & \operatorname{rot} \mathbf{E} &= - \frac{\partial \mathbf{B}}{\partial t}, \quad (\text{IIe}) \\ \operatorname{div} \mathbf{B} &= 0 \quad (\text{IIIe}), & \operatorname{div} \mathbf{D} &= 0 \quad (\text{IVe}). \end{aligned} \quad (4.1)$$

This grouping decidedly hints at an  $\mathbf{E}-\mathbf{H}$ ,  $\mathbf{D}-\mathbf{B}$  analogy. This formalism, however, is artificial. Let us, namely, collect the equations containing the charge or current exciting the field, and those not containing these quantities :

$$\begin{aligned} \operatorname{rot} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J}, \quad (\text{If}), & \operatorname{div} \mathbf{D} &= \rho; \quad (\text{IVf}) \\ \operatorname{rot} \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \quad (\text{IIIf}), & \operatorname{div} \mathbf{B} &= 0. \quad (\text{IIIIf}) \end{aligned} \quad (4.2)$$

It is evident of this grouping that vectors  $\mathbf{H}$  and  $\mathbf{D}$  are those in close relationship with the exciting effect, while equations that can be simply reduced to zero relate to  $\mathbf{E}$  and  $\mathbf{B}$ . It is strictly formal, whether the formation of rotation or of divergence is prescribed in the mathematical formulation, as the same vector operation cannot lead to the vectorial current density and to the scalar charge density, respectively.

The question of formalism is finally decided by the relativistic formulation of the matter. The decomposition of the electromagnetic field into an electric and a magnetic field, namely, is known to be no objective operation, as it depends on the relative motion of the observer and of the charges. *E.g.* an observer fixed to the charge notices only the electrostatic field, in the case, however, that the charge and the observer have a relative displacement, then the observer may also notice the magnetic field.



The relativistic wording, in turn, knows only the electromagnetic field: the electrostatic and magnetic field strength values can be concentrated in a single antisymmetric field strength four-tensor  $\tilde{\mathfrak{F}}$ .

$$\tilde{\mathfrak{F}} = \begin{pmatrix} 0 & B_z & B_y & -\frac{j}{c} E_x \\ -B_z & 0 & B_x & -\frac{j}{c} E_y \\ B_y & -B_x & 0 & -\frac{j}{c} E_z \\ \frac{j}{c} E_x & \frac{j}{c} E_y & \frac{j}{c} E_z & 0 \end{pmatrix} \quad (4.3)$$

The rotation four-tensor of this tensor is zero, which exactly corresponds to the contracted form of Maxwell's equations II—III given under (4.2). The tensor of excitation can similarly be written:

$$\mathfrak{G} = \begin{pmatrix} 0 & H_z & -H_y & -jc D_x \\ -H_z & 0 & H_x & -jc D_y \\ H_y & -H_x & 0 & -jc D_z \\ jc D_x & jc D_y & jc D_z & 0 \end{pmatrix}. \quad (4.4)$$

The divergence four-vector of this vector supplies the current density four-vector  $\tilde{\mathfrak{S}}$ :

$$\tilde{\mathfrak{S}} = (J_x, J_y, J_z, jc \varrho). \quad (4.5)$$

This in turn, is the contracted form of equations I and IV given under (4.2).

The relativistic wording clearly expresses the fact that  $\mathbf{E}$  and  $\mathbf{B}$  on the one hand, and  $\mathbf{D}$  and  $\mathbf{H}$  on the other are analogous, as during the motion these values are transformed one into the other. It is also evident that Maxwell's equations I and IV, and II and III, respectively, are connected equations. This is another side of the same question.

It follows from the precedings that the equation system given under (3.1) contains physically well-defined vectors. Let us now examine how Maxwell's equations given under (2.1) can be deduced from this build-up, as the introduction of vectors  $\mathbf{D}$  and  $\underline{\mathbf{H}}$ , generally, greatly facilitates the computations.

## 5. Introduction of auxiliary quantities

The system of equations (3.1) describes the regularity of the electromagnetic field. In a given case, our task is to solve these equations in conformity with given conditions. For the solution, the introduction of new auxiliary

quantities is advisable. The essence of these is that for the sake of unification of formulation, scalar quantities (charge density and energy density) should be deduced from vectorial quantities, and even the other quantity exciting the field (current density) should be deduced from a vector. These equations do not contain any new physical assertions, they only define new quantities. Assertions relating to the regularity of the electromagnetic field are contained in equations I—V as well as in formula (3.2) for vectors  $\mathbf{P}$  and  $\mathbf{M}$ .

At first charge density is deduced as the divergence of a vector potential  $\mathbf{D}$ :

$$\varrho = \operatorname{div} \mathbf{D}. \quad (5.1)$$

This equation exactly corresponds to equation IV of the usual wording (2.1) but the interpretation is just the inverse of that one. There, the divergence of  $\mathbf{D}$ , where  $\mathbf{D}$  is regarded as a basic quantity, is taken as a new quantity defined by this relation, which will later be proved to be the charge density. Here, quite contrarily, vector  $\mathbf{D}$  has been introduced to describe charge density which is taken as a basic quantity. Equation (5.1) is substituted into the equation of continuity (1.4). Changing the order of partial derivation with respect to time and place, after reduction, we obtain

$$\operatorname{div} \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = 0. \quad (5.2)$$

As the expression inside the brackets is source-free, it can be deduced as the rotation of a vector potential:

$$\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \operatorname{rot} \mathbf{H}. \quad (5.3)$$

Thereby equation I of the customary system of equations (2.1) is arrived at. The value of  $\mathbf{H}$ , however, appears as a deduced quantity, like  $\mathbf{D}$  in the previous case. In this wording it is no wonder, therefore, that  $\mathbf{D}$  and  $\mathbf{H}$ , as vector potentials, cannot be generally given a direct physical meaning. The derivation (divergence or rotation) of both quantities being coupled to physically well-defined quantities (charge density and current density), it is obvious that the integrals of vector potentials supply integral values characteristic as shown by equations (2.2).

As can be seen, the auxiliary quantities in this way defined are in relationship with quantities exciting the field. Consequently they supply, beside the vectors characterizing field strength, data as to the excitation of the field, wherefore they can be named vectors of excitation.

Our third auxiliary equation produces a relationship between the variation of energy density and power density on one hand, and a vector quantity on the other, in the following form :

$$\frac{\partial w}{\partial t} + \mathbf{E} \mathbf{J} = - \operatorname{div} \mathbf{S}. \tag{5.4}$$

This equation will be discussed later.

Vector potentials introduced in this way, however, are not univocally defined. We may yet freely dispose of the rotation of  $\mathbf{D}$  and of the divergence of  $\mathbf{H}$ . By placing equation (5.1) into equation IV of system (3.1) we obtain

$$\operatorname{div} (\varepsilon_0 \mathbf{E} + \mathbf{P} - \mathbf{D}) = 0.$$

The quantity in the brackets can again be written as the rotation of a vector :

$$\varepsilon_0 \mathbf{E} + \mathbf{P} - \mathbf{D} = \operatorname{rot} \mathbf{C} \tag{5.5}$$

where  $\mathbf{C}$  can be any vector. If  $\mathbf{C}$  is taken as zero, then the well-known equation

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \tag{5.6}$$

will be arrived at. The relationships between the magnetic characteristics should now be examined. Equation (5.3) and the newly found equation (5.6) is substituted into equation I under (3.1) :

$$\operatorname{rot} \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} - \mathbf{H} \right) = 0.$$

The quantity in the brackets is free of turbulence, consequently, it can be written as the gradient of any  $\psi$  scalar value :

$$\frac{\mathbf{B}}{\mu_0} - \mathbf{M} - \mathbf{H} = - \operatorname{grad} \psi. \tag{5.7}$$

Taking  $\operatorname{grad} \psi$  as zero the well-known equation

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \tag{5.8}$$

will be obtained. By this choice the following equations for the rotation of  $\mathbf{D}$  and the divergence of  $\mathbf{H}$  will be arrived at, by virtue of equations II and III

under (3.1) :

$$\begin{aligned}\operatorname{rot} \mathbf{D} &= -\varepsilon_0 \frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{P}, \\ \operatorname{div} \mathbf{H} &= -\operatorname{div} \mathbf{M}.\end{aligned}\quad (5.9)$$

In the case of vector potentials, however, it is habitual to have, generally, the most simple equation for them. *E.g.* in the case of a quasistationary field, equation  $\operatorname{div} \mathbf{A} = 0$  is chosen besides the defining equation  $\mathbf{B} = \operatorname{rot} \mathbf{A}$ . In our case this is impossible, as by choosing  $\operatorname{rot} \mathbf{C} = 0$  and  $\operatorname{grad} \psi = 0$  disposition has already been made over the rotation of  $\mathbf{D}$  and the divergence of  $\mathbf{H}$ . There is, however, no obstacle in introducing, by suitably choosing  $\mathbf{C}$ , a vector potential  $\mathbf{D}^*$  the rotation of which is equal to zero :

$$\operatorname{div} \mathbf{D}^* = \varrho, \quad \operatorname{rot} \mathbf{D}^* = 0. \quad (5.10)$$

It follows from the first condition that the integral of  $\mathbf{D}^*$  for a closed surface also supplies the charge enclosed by that surface.

Forming the rotation of equation (5.5) gives :

$$-\varepsilon_0 \frac{\partial \mathbf{B}}{\partial t} + \operatorname{rot} \mathbf{P} = \operatorname{rot} \operatorname{rot} \mathbf{C}.$$

As known from the vector analysis :

$$\operatorname{rot} \operatorname{rot} \mathbf{C} = \operatorname{grad} \operatorname{div} \mathbf{C} - \Delta \mathbf{C}.$$

By choosing the divergence of any vector  $\mathbf{C}$  as zero, the following vectorial Poisson's equation will be arrived at :

$$\Delta \mathbf{C} = -\operatorname{rot} \mathbf{P} + \varepsilon_0 \frac{\partial \mathbf{B}}{\partial t}. \quad (5.11)$$

The solution of this equation :

$$\mathbf{C} = \frac{1}{4\pi} \int_V \frac{\operatorname{rot} \mathbf{P}}{r} dV - \frac{\varepsilon_0}{4\pi} \frac{\partial}{\partial t} \int_V \frac{\mathbf{B}}{r} dV. \quad (5.12)$$

Consequently, the relationship between  $\mathbf{D}^*$  and  $\mathbf{E}$  is :

$$\mathbf{D}^* = \varepsilon_0 \mathbf{E} + \mathbf{P} - \frac{1}{4\pi} \operatorname{rot} \int_V \frac{\operatorname{rot} \mathbf{P}}{r} dV + \frac{\varepsilon_0}{4\pi} \frac{\partial}{\partial t} \operatorname{rot} \int_V \frac{\mathbf{B}}{r} dV. \quad (5.13)$$

By substituting  $\mathbf{D}^*$  into the equation of continuity we obtain :

$$\operatorname{div} \left( \mathbf{J} + \frac{\partial \mathbf{D}^*}{\partial t} \right) = 0.$$

The expression, in the brackets, can be deduced as the rotation of a vector potential. Similarly to the preceding computation, let us suppose the condition equations relating to the vector potential designated by  $\mathbf{H}^*$  to be the most simple ones :

$$\operatorname{rot} \mathbf{H}^* = \mathbf{J} + \frac{\partial \mathbf{D}^*}{\partial t}, \quad \operatorname{div} \mathbf{H}^* = 0. \tag{5.14}$$

It follows from the first condition that the integral of  $\mathbf{H}^*$  for a closed curve also determines the excitation enclosed by the curve.

The divergence of equation (5.7) is :

$$\operatorname{div} \mathbf{M} = \operatorname{div} \operatorname{grad} \psi = \Delta \psi. \tag{5.15}$$

The solution of this Poisson's equation is :

$$\psi = - \frac{1}{4\pi} \int \frac{\operatorname{div} \mathbf{M}}{r} dV \tag{5.16}$$

Consequently, by virtue of equation (5.7) :

$$\mathbf{H}^* = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} - \frac{1}{4\pi} \operatorname{grad} \int \frac{\operatorname{div} \mathbf{M}}{r} dV. \tag{5.17}$$

The equations for both  $\mathbf{D}^*$  and  $\mathbf{H}^*$  can be further transformed, as the differential operator may be carried below the sign of integration. It is, however, not worth while to carry out this transformation. It is of no use, namely, to introduce vectors  $\mathbf{D}^*$  and  $\mathbf{H}^*$  in place of vectors  $\mathbf{D}$  and  $\mathbf{H}$ , respectively, as this would hardly mean making the computations more easy. Therefore we do not examine whether there are any singularities in the solution of equations (5.11) and (5.16). It can nevertheless be seen that these two vectors differ only in one or two additive members and they are identical in case of the fulfilment of some conditions which are often realized. The only aim of our discussion was to show that the main equations I—V express an objective regularity and that they cannot be chosen otherwise. The auxiliary equations, however, can only be regarded as definitions which could also be written in other ways if the viewpoints of practicability and tradition were not taken into account.

## 6. The power density vector

Let us now examine the auxiliary quantity  $\mathbf{S}$  introduced into equation (5.4) which is in connection with the variation of energy :

$$\frac{\partial w}{\partial t} + \mathbf{E} \mathbf{J} = - \operatorname{div} \mathbf{S}. \quad (6.1)$$

The first thing to be examined is : With which physical quantity is vector  $\mathbf{S}$  in relationship? For the sake of simplicity let us suppose that there is no convective current and the substance is isotropic. In this case, by expressing  $\mathbf{E}$  from the 3rd member of Maxwell's V equation and placing this value into (6.1) :

$$- \frac{\partial w}{\partial t} = \frac{J^2}{\gamma} - \mathbf{E}_e \mathbf{J} + \operatorname{div} \mathbf{S}. \quad (6.2)$$

By integrating this equation for a given volume we obtain the equilibrium of power :

$$- \frac{dW_{e.m.}}{dt} = \int_V \frac{J^2}{\gamma} dV - \int_V \mathbf{E}_e \mathbf{J} dV + \oint_A \mathbf{S} d\mathbf{A}. \quad (6.3)$$

The left side of the equation gives the decrease of the electromagnetic energy with time within the given volume. At the right side the first member is the Joule heat, the second is the power produced (or consumed) by the inserted fields, the third is the power flowing through the surface surrounding the given volume.

"Vector potential"  $\mathbf{S}$ , therefore, is Poynting's power flow vector the integral of which taken for a closed surface supplies the power flowing (radiating) through the surface.

Thereafter a relationship is to be found between energy density and the field characteristics. With this end in mind let us multiply Maxwell's II equation scalarly by  $\mathbf{H}$ , whilst equation (5.3) scalarly by  $\mathbf{E}$ . Afterwards the second equation is deduced from the first one :

$$\mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} + \mathbf{E} \mathbf{J} = \mathbf{E} \operatorname{rot} \mathbf{H} - \mathbf{H} \operatorname{rot} \mathbf{E} \equiv \operatorname{div} (\mathbf{E} \times \mathbf{H}). \quad (6.4)$$

By comparing this with equation (6.1) we may write :

$$\frac{\partial w}{\partial t} = \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \frac{\partial \mathbf{B}}{\partial t}, \quad (6.5)$$

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}. \quad (6.6)$$

As can be seen, in the expressions both for the variation of energy density and for the vector of power flow the vectors of both field strength and induction are included, and what is surprising they are not symmetrically arranged. This fact, however, means no contradiction to our opinion presented up to now. Neither the vector of energy density, nor that of radiation power have namely a direct physical meaning. Only the volume integral of energy density and the surface integral of the radiation vector can be interpreted physically. And it is known that both vectors  $\mathbf{D}$  and  $\mathbf{H}$  after integration supply a quantity having a physical content. Consequently, also the following equations are evidently possible :

$$\frac{dW_{e.m.}}{dt} = \int_V \left( \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} \right) dV, \tag{6.7}$$

$$P_{rad} = \int_A (\mathbf{E} \times \mathbf{H}) d\mathbf{A}. \tag{6.8}$$

Equation (6.6) is based on the following deliberation: If  $\text{div} (\mathbf{E} \times \mathbf{H}) = \text{div} \mathbf{S}$ , then  $\mathbf{E} \times \mathbf{H} = \mathbf{S}$ . In the general case, however,  $\mathbf{E} \times \mathbf{H}$  and vector  $\mathbf{S}$  differ in a divergence-free vector, consequently also equation

$$\mathbf{S}^* = \mathbf{E} \times \mathbf{H} + \text{rot} \mathbf{K} \tag{6.9}$$

is a possible solution, as  $\text{div} \mathbf{S}^* = \text{div} \mathbf{S} = \text{div} (\mathbf{E} \times \mathbf{H})$ , whichever vector  $\mathbf{K}$  may be. In other words equation (6.1) only determines the divergence of  $\mathbf{S}$ , but we have free disposition over its rotation. In accordance with (6.6), however, we obtain univocally :

$$\text{rot} \mathbf{S} = \text{rot} (\mathbf{E} \times \mathbf{H}) \tag{6.10}$$

Similarly to the previous case, a rotation-free vector  $\mathbf{S}^*$  could be determined in this case too.

According to relativistic meditations, however, the rotation of vector  $\mathbf{S}$  is not simply a formal problem. A determined impulse moment pertains namely to the energy flow. If therefore we want vector  $\mathbf{S}$  actually to determine power flow in any point, then its rotation should correspond to the impulse moment of the electromagnetic field. On this basis the validity of equation  $\text{rot} \mathbf{S} = \text{rot} (\mathbf{E} \times \mathbf{H})$  can be proved. This also means that in equation (6.9)  $\text{rot} \mathbf{K} = 0$ , consequently equation (6.6) supplies the value  $\mathbf{S}$  at a given place.

It should be noted that from a relativistic viewpoint equation (6.1) is the fourth component of the divergence of the energy-impulse tensor.

## 7. Energy density

Finally it should be briefly discussed, why the derivation in respect to time of energy density and not energy density itself are included both in equation IV under (3.1) and in equation (6.1). Formally, this fact is sufficiently evident, as the great majority of physical laws can be written in the form of differential equations.

On examining energy, the variations should be determined as energy itself, that is a quantity that cannot be measured. We can only measure the labour necessary to arrive from a state into another given state, or inversely the labour that is freed when the system is transformed from one state into another. If the basic state can be univocally defined and the introduced labour can be divided into parts one of which is changing energy content and the other transformed into heat, then the corresponding work values (that may be negative values too) can be identified with the energy of the system.

Let us now examine the variation of electromagnetic energy density. Equation (6.5) is integrated in respect to time :

$$\int_{t_0}^t \frac{\partial w}{\partial t} dt = \int_{t_0}^t \left( \mathbf{E} \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \frac{\partial \mathbf{B}}{\partial t} \right) dt.$$

Denominations

$$w(0) = w_0, \quad \mathbf{D}(0) = \mathbf{D}_0, \quad \mathbf{B}(0) = \mathbf{B}_0$$

are introduced :

$$w - w_0 = \int_{\mathbf{D}_0}^{\mathbf{D}} \mathbf{E} d\mathbf{D} + \int_{\mathbf{B}_0}^{\mathbf{B}} \mathbf{H} d\mathbf{B}. \quad (7.1)$$

In the most simple case the polarizations are the homogeneous linear functions of field intensity according to equation (2.3), that is

$$\mathbf{D} = \varepsilon \mathbf{E}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}. \quad (7.2)$$

In this case we obtain :

$$\begin{aligned} w - w_0 &= \frac{1}{2} \varepsilon (E^2 - E_0^2) + \frac{1}{2} \mu (H^2 - H_0^2) = \\ &= \frac{1}{2} \varepsilon (E^2 - E_0^2) + \frac{1}{2\mu} (B^2 - B_0^2). \end{aligned} \quad (7.3)$$



Be  $E_0 = 0$ ,  $B_0 = 0$ ,  $H_0 = 0$  and to this state an energy density  $w_0 = 0$  is logically ordered. In this way the well-known formula for energy density is arrived at :

$$w = \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 = \frac{1}{2} \varepsilon E^2 + \frac{1}{2\mu} B^2 . \quad (7.4)$$

If there is a permanent polarization too, we may write on the basis of equation (3.2)

$$\mathbf{D} = \varepsilon \mathbf{E} + \mathbf{P}_0, \quad \mathbf{B} = \mu \mathbf{H} + \mu_0 \mathbf{M}_0 . \quad (7.5)$$

In this case the following formula is obtained for the change of energy density :

$$\begin{aligned} w - w_0 &= \frac{1}{2} \varepsilon (E^2 - E_0^2) + \frac{1}{2} \mu (H^2 - H_0^2) = \\ &= \left[ \frac{D^2 - D_0^2}{2\varepsilon} - \frac{\mathbf{D} - \mathbf{D}_0}{\varepsilon} \cdot \mathbf{P}_0 \right] + \left[ \frac{B^2 - B_0^2}{2\mu} - \frac{\mathbf{B} - \mathbf{B}_0}{\mu} \cdot \mu_0 \mathbf{M}_0 \right] . \end{aligned} \quad (7.6)$$

To which state should  $w_0 = 0$  now be ordered? The most simple form is obtained by choosing  $E_0 = 0$ ,  $H_0 = 0$ ,  $w_0 = 0$  and expressing energy density by  $E$  and  $H$ . In this case, namely

$$w = \frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 . \quad (7.7)$$

As  $\mathbf{D}_0 = \mathbf{P}_0$  and  $\mathbf{B}_0 = \mu_0 \mathbf{M}_0$ , the other two equations will become much more complicated. Though the above choice is the most simple, it is not the most logic one. It would be more obvious to denominate that state as free of energy in which there are no charges or currents, that is in case of  $D_0 = 0$  and  $H_0 = 0$  is  $w_0 = 0$ . Computing now with  $D$  and  $H$ , we obtain :

$$w = \left[ \frac{D^2}{2\varepsilon} - \frac{\mathbf{D} \cdot \mathbf{P}_0}{\varepsilon} \right] + \frac{1}{2} \mu H^2 . \quad (7.8)$$

It should be emphasized, however, that this choice is similarly arbitrary, as we have no physical ground to state that the energy content of a polarized substance is zero.

As far as polarization is a (inhomogeneous) linear function of field intensity, however, the processes are reversible and the zero level may be taken at will. In respect to computation technique this fact is of no importance just as the choice of the zero level of potential energy is also arbitrary.

In case of a homogenous non-linear relationship (*e.g.* soft magnetic materials) the processes are reversible, the zero level of energy density can univocally be fixed to the state  $E_0 = 0$ ,  $D_0 = 0$  and  $H_0 = 0$ ,  $B_0 = 0$ , respectively, while energy density can in principle be computed in the following form :

$$w = \int_0^D \mathbf{E} \, d\mathbf{D} + \int_0^B \mathbf{H} \, d\mathbf{B} . \quad (7.9)$$

It can also be easily plotted in a graph as the area between the characteristic curve and the axes  $D$  and  $B$ , respectively, this being well-known for the case of ferromagnetic materials.

The situation is the most complicated when processes are irreversible, that is, the polarization is a non-linear and non-homogeneous function of field intensity. This is the situation *e.g.* in case of hard magnetic materials (permanent magnetic materials). In such a case we should account for the thermodynamic effects due to variations in the electromagnetic field, as *e.g.* hysteresis loss. In this case no rational and usable supposition can be taken for the zero level of energy density and only the variation of energy density can be described. This latter can simply be graphically interpreted by making use of the well-known method of the energetic analysis of the hysteresis loop.

Energy is influenced by thermodynamic effects even in the most simple cases. The total energy of any system, namely, can be divided into thermal and electromagnetic energies. If polarization and consequently permittivity and permeability are functions of time, then the variations of electromagnetic and thermal energy are reciprocally influencing one another. Becker [3] has proved that in case of  $\mathbf{P} = \epsilon_0 \kappa_e \mathbf{E}$  and of an isothermal transformation, the electromagnetic energy exactly equals the free energy (as named in the thermodynamic sense). In case of an electric field the total energy density is :

$$w = \left( 1 + \frac{T}{\epsilon_r} \frac{d\kappa_e}{dT} \right) w' , \quad (7.10)$$

where  $w' = \frac{1}{2} \epsilon E^2$ , the density of the free energy. The expression  $\epsilon E^2/2$  can therefore only be regarded as electric energy density if susceptibility (and consequently permittivity) do not depend on temperature. For many materials  $\kappa_e = k/T$ , therefore the total energy density is

$$w = \frac{1}{2} \epsilon_0 E^2 < \frac{1}{2} \epsilon E^2 . \quad (7.11)$$

While the strength of the electric field increases from 0 to  $E$ , the value of heat produced or dissipated, in respect to the unit of volume is

$$q = \frac{1}{2} T \frac{dz}{dT} \varepsilon_0 E^2. \quad (7.12)$$

If susceptibility decreases with temperature the material sends out heat during the building up of the field. The corresponding magnetic effect can be expressed by the exchange of signs.

### Summary

Maxwell's equation can be built up in two ways (beyond the Lorentz' conception). The one is to introduce quantities not defined in detail beside the quantities describing the field, which are related in an empirical way to the quantities describing the field. This method is practical from the viewpoint of computation technique. Its disadvantage is that it somewhat confuses the fact of the field being produced by charges and currents actually present or fixed to the substance.

The other way is to express directly the relationship between the quantities describing the field, and actual and fixed charges or currents. This method better emphasizes the physical essence of the matter, nevertheless it is too general, especially for the calculation of simple problems. Therefore the introduction of some new quantities was practicable, these have, however, only the role of auxiliary quantities for computation purposes (**D**, **H** and **S**). Maxwell's equations written in this form therefore express an objective regularity, the equations relating to these quantities, however, are only definitions, determined from the point of view of practicability only.

In the course of the discussion of the energy equation we have stated that in a general case only the variation of energy density can be described by a rule, if the thermodynamical effects or those of the material structure are not taken into account. In practice, however, the possibility of the application of both methods has its limitations. Therefore on computing energy, the course of the variation of the electromagnetic field should be taken into account.

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