ON LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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In the recent years a book by ZYPKIN [2] was published about the difference equations of impulse and regulation technique. In this book, by using the so-called discrete Laplace transformation, an operational calculus for solving linear difference equations (and systems of difference equations) with constant coefficients was elaborated.

In this article we show a method for this difference equations which can be more easily treated and more generally applied than that of discrete Laplace transformation method. We are using MIKUSINSKI's method [4]. But in our treatment on operational calculus in connection with difference equations, the need of introducing abstract elements does not occur.

§ 1. Step functions and number sequences. Let n be a positive integer and $a(n)$ the value of $a(t)$ in $t = n$. ($a(n)$ may be also a complex number). $a(t)$ is called a *step function*, if

$$a(t) = a(n), \text{ if } n \leq t < n + 1 \tag{1}$$

$a(t)$ is called an *entrance function* if

$$a(t) \equiv 0, \text{ if } t < 0 \tag{2}$$

If $a(t)$ is a step function and if it is also an entrance function, it may be characterized by a number sequence $\{ a_0\, a_1, \ldots, a_n, \ldots \}$, where the relation

$$a_n = a(t) \big|_{t=n}. \tag{2'}$$

holds.

In the future we shall not make any difference between the step function $a(t)$ and the number sequence $\{ a(n) \}$ characterizing it. We are able to do this, because the one to one correspondence also holds on operations (sum, product) and limes, which will be introduced in § 3.

We define the *translation* of the function $a(n)$ by k by the equation

$$a(n-k) = \begin{cases} 0, & \text{if } n < k \\ a_{n-k}, & \text{if } n \geq k. \end{cases} \tag{3}$$

On the other hand the function $a(n+k)$ is defined by the equation

$$a(n+k) = \begin{cases} 0, & \text{if } n < 0 \\ a_{n+k}, & \text{if } n \geq 0. \end{cases} \tag{4}$$

Thus, the translation of $a(n+k)$ in positive direction by k does not lead to function $a(n)$.

§ 2. The difference equation and its solution. A *k-th* order linear difference equation with constant coefficients for the function $y(n)$ is

$$a_0 y(n) + a_1 y(n+1) + \ldots + a_k y(n+k) = f(n) \tag{5}$$

if $f(n)$ is a given function and $a_0, \ldots, a_k$ are given constants. The equation (5) has an unique solution, if the values of $y(n)$ are given at different k points, if — for example — the values

$$y(0) = y_0, \ldots, y(k-1) = y_{k-1} \tag{6}$$

are given. A very simple method for obtaining the solution of equation (5) which satisfies the conditions (6) is the following:

Taking $n = 0$, we put the values from conditions (6) in the equation (5). In the second step, taking $n = 1$ and making use of the conditions (6) and the value y_k (already determined), we determine y_{k+1}, etc. However, in that way we do not get a formula for the function $y(n)$, but only values of $y(n)$ in k, $k+1, \ldots$

In the following we modify the idea of the foregoing method, the substance of which was a successive translation in negative direction by $n, n+1, \ldots$ In our method by one translation we immediately obtain a formula for $y(n)$.

§ 3. The structure k. In the class of step functions [characterized by equations (1) and (2)], we define the addition, the subtraction of functions, the product of a function and a complex number in the usual sense.

The *product of two functions* is determined by the equations

$$a(n) * b(n) = c(n)_n$$

where

$$c\,(n) = \sum_{k=0}^{n} a\,(n-k)\,b\,(k) \tag{7}$$

It can be shown, that

$$[a\,(n) * b\,(n)] * c\,(n) = a\,(n) * [b\,(n) * c\,(n)]$$

$$a\,(n) * b\,(n) = b\,(n) * a\,(n)$$

$$[a\,(n) + b\,(n)] * c\,(n) = a\,(n) * c\,(n) + b\,(n) * c\,(n)$$

Still, we define *the limes:*

$$\lim_{k\to\infty} \{ a_k\,(n) \} = a\,(n)$$

if, and only if, the convergence holds for each n.

Examples about the limes.

I. If $a\,(n)$ is an arbitrary function and

$$a_k = t^k\,a\,(n)$$

where $t^k\,a\,(n)$ is determined by equation (9), then

$$\lim_{k\to\infty} a_k = 0$$

Since for arbitrary fixed n

$$t^k\,a(n) = 0 \text{ if } k > n$$

holds.

II. If

$$a_k\,(n) = \begin{cases} 0 & \text{if } n \neq k \\ c^k & \text{if } n = k \end{cases}$$

we obtain, that

$$\lim_{k\to\infty} a_k = 0$$

for, if n is arbitrary fixed $a_k\,(n) = 0$, if $k > n$ (Though $c^k \to \infty$!).

The class of step functions characterized by equations (1) and (2) in which the operation and limes are defined in the foregoing manner is called structure k.

§ 4. **Unit function.** If we define the function e by equation

$$e = e\,(n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

then

$$e * a(n) = a(n) \qquad\qquad c_1\, e \pm c_2\, e = (c_1 \pm c_2)\, e$$

$$\lim_{k \to \infty} (c_k\, e) = (\lim_{k \to \infty} c_k)\, e$$

hold for arbitrary function $a(n)$ and complex numbers c_1, c_2. Therefore we may identify the function e with the number 1 and we write for a complex number c the identity

$$c \equiv c\, e(n) \tag{8}$$

§ 5. **Translation function.** If we define t by equation

$$t = t(n) = \begin{cases} 0 \text{ if } n \neq 1 \\ 1 \text{ if } n = 1 \end{cases}$$

the equations

$$t^2 = t(n) * t(n) = \begin{cases} 0 \text{ if } n = 2 \\ 1 \text{ if } n = 2 \end{cases}$$

$$t^k = t(n) * t^{k-1} = \begin{cases} 0 \text{ if } n = k \\ 1 \text{ if } n = k \end{cases}$$

and

$$t^k(n) * a(n) = a(n-k) \tag{9}$$

hold, where $a(n-k)$ is defined by equation (3). From equations (9) and (4) we obtain

$$a(n) = t(n) * a(n+1) + a_0 \tag{10}$$

and

$$a(n) = t^k\, a(n+k) + t^{k-1}\, a_{k-1} + \ldots + t\, a_1 + a_0 \tag{11}$$

where $a_{k-1}, \ldots, a_1, a_0$ are defined by equations (8) and (2').

From § 3. we derive that for every function $a(n)$ the following series development hold in terms of $t(n)$:

$$a(n) = a_0 + a_1\, t(n) + \ldots + a_k\, t^k(n) + \ldots \tag{12}$$

where a_k is the value of $a(n)$ if $n = k$.

Hence

$$a(n) * b(n) = a_0 b_0 + (a_1 b_0 + a_0 b_1)\, t(n) + \ldots + (a_k b_0 + \ldots + a_0 b_k)\, t^k(n) + \ldots \tag{13}$$

§ 6. Inverse function. If $a(n)$ is given, and we can find $b(n) \in K$ that

$$a(n) * b(n) = e(n) \tag{14}$$

then the sequence $b(n)$ is called the inverse function of $a(n)$.

Since from equation (14) we obtain for the first k elements of $b(n)$, that

$$\begin{aligned} &a_0 b_0 && = 1 \\ &a_1 b_0 + a_0 b_1 && = 0 \\ &\vdots \\ &a_k b_0 + a_{k-1} b_1 + \ldots + a_0 b_k && = 0 \end{aligned}$$

the necessary and sufficient condition of the existence in K of the inverse function of $a(n)$ is

$$a_0 \neq 0.$$

We denote the inverse function of $a(n)$ by $a^{-1}(n)$ or $\dfrac{1}{a(n)}$.

§ 7. Rational functions of t. It is known that every rational function can be written in the form of a sum of a polynom and partial fractions δ. Therefore we can define every rational function of t by the following formulas.

$$a_0 + a_1 t + \ldots + a_n t^n = \{ a_0, a_1, \ldots, a_n, 0, 0, \ldots \} \tag{16}$$

$$\frac{1}{(1 - ct)} = 1 + ct + \ldots + c^k t^k + \ldots = \{ c^k \}^* \tag{17}$$

$$\frac{1}{(1-ct)^n} = \frac{1}{1-ct}\, \frac{1}{(1-ct)^{n-1}} = \left\{ \binom{n-1+k}{k} c^k \right\} \tag{18}$$

§ 8.. The solution of a difference equation. From equations (10) and (11) follows that equation (5) is equivalent with the linear algebraic equation

$$(a_0 t^k + a_1 t^{k-1} + \ldots + a_{k-1} t + a_k) * y(n+k) = f(n) + P_{k-1}(t)$$

* Since from equations (12) and (13) follows that $(1 - ct)(1 + ct + \ldots + c^k t^k + \ldots) = 1$.

where $a_k \neq 0$ and $P_{k-1}(t)$ is a polynom of degree $k-1$ the coefficients of which can be computed from (6). Hence

$$y(n+k) = \frac{f(n) + P(t)}{a_0 t^k + \ldots + a_k}, \text{ if } n \geq 0 \tag{19}$$

On the basis of formulas (16), (17), and (18) it can be shown that (19) is a function which belongs to K.

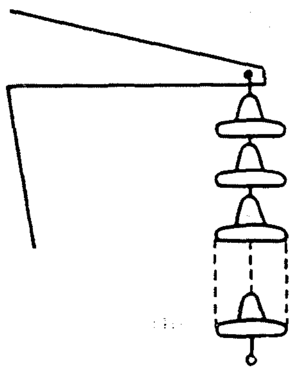

Fig. 1. The picture and the switch-on-picture of the high tension insulator

Observe that formula (19) can be applied for values $y(n)$, in the case $n \geq k$. Values of $y(n)$ in case of $n < k$ are given by the initial values (6).

§ 9. Examples.

I. Is given the difference equation

$$u(n+2) - u(n) = n!$$

with boundary conditions

$$u(0) = 0, \qquad u(5) = 1$$

Solution: The corresponding algebraic equation is

$$(1-t^2)\, u(n+2) - t\, u(1) - u(0) = n!$$

hence

$$u(n+2) = \frac{n! + tu(1) + u(0)}{1 - t^2}$$

Considering the identity

$$\frac{1}{1-t^2} = \frac{1}{2}\left[\frac{1}{1-t} + \frac{1}{1+t}\right] = \frac{1}{2}\{1+(-1)^n\}$$

and

$$t\,u(1) + u(0) = \{-u(0), u(1), 0, 0, \dots\}$$

and the first boundary condition, we obtain

$$u(n+2) = \sum_{k=0}^{n} (n-k)!\,[1+(-1)^k] + [1-(-1)^n]\,u(1)$$

From the second boundary condition follows that

$$u(5) = \sum_{k=0}^{3} (3-k)!\,[1+(-1)^k] + \frac{1}{2}[1-(-1)^3]\,u_1$$

hence

$$u_1 = u_5 - 7 = -6$$

Thus we obtain the *result*

$$u(n+2) = \sum_{k=0}^{n} (n-k)!\,[1+(-1)^k] - 6\,[1+(-1)^n], \text{ if } n \geq 0$$

$$u(1) = -6, \; u(0) = 0$$

This example cannot be solved by ZYPKIN's method, since the function $n!$ has no Laplace transform.

II. A high-tension insulator consists of a sequence of unit insulators which are connected by conductors. The first of these unit insulators are connected to a grounded console. The last unit insulator is connected to a high tension conductor in which alternating current of frequency ω flows. (See [2] p. 40. and the figure.)

The problem is to give the potential drop between the n-th and $n+1$-st unit insulator.

The potential in one unit insulator is constant. Thus — if we denote the capacity between two neighbouring members with C_1 and between the ground and the first unit insulator with C_2 — then the problem leads to the following difference equation :

$$u(n+2) - 2\left(1 + \frac{C_2}{2\,C_1}\right) u(n+1) + u(n) = 0$$

$$u(0) = 0 \qquad u(N) = u_L$$

where u_L is the potential drop between the last unit insulator and the ground.

Solution: Making use of the substitution

$$1 + \frac{C_2}{2C_1} = ch\,\tau \tag{20}$$

we obtain the linear algebraic equation [see (11)]

$$(1-2\,ch\,\tau\,t + t^2)\,u\,(n+2) - 2\,ch\,\tau\,u\,(1) + t\,u\,(1) = 0$$

From the identity

$$\frac{1}{1-2\,ch\,\tau \cdot t + t^2} = \frac{1}{e^{\tau} - e^{-\tau}}\left[\frac{e^{\tau}}{1-e^{\tau}t} - \frac{e^{-\tau}}{1-e^{-\tau}t}\right]$$

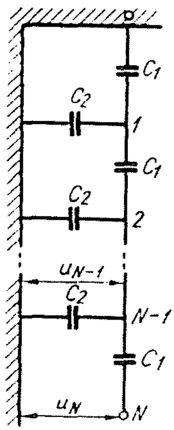

Fig. 2

and the equation (17), we get the formula

$$\frac{1}{1-2\,ch\,\tau \cdot t + t^2} = \frac{e^{\tau(n+1)} - e^{-\tau(n+1)}}{e^{\tau} - e^{-\tau}}$$

Hence follows that

$$u\,(n+2) = \frac{t-2\,ch\,\tau}{t^2 - 2\,ch\,\tau\,t + 1}\,u\,(1) = \frac{e^{\tau n} - e^{-\tau n}}{e^{\tau} - e^{-\tau}}\,u\,(1) -$$

$$-\frac{(e^{\tau} + e^{-\tau})\,(e^{\tau\cdot(n+1)} - e^{-\tau(n+1)})}{e^{\tau} - e^{-\tau}}\,u\,(1) = \frac{e^{\tau(n+2)} - e^{-\tau(n+2)}}{e^{\tau} - e^{-\tau}}\,u\,(1),$$

$$u\,(N) = \frac{e^{\tau N} - e^{-\tau N}}{e^{\tau} - e^{-\tau}}\,u\,(1) \quad \text{if} \quad N \geq 2$$

Thus the solution is (after replacing $n+2$ by n)

$$u(n) = \frac{e^{\tau n} - e^{-\tau n}}{e^{\tau N} - e^{-\tau N}} .$$

Because

$$\lim_{\tau \to 0} \frac{e^{\tau n} - e^{-\tau n}}{e^{\tau N} - e^{-\tau N}} = \frac{n}{N}$$

thus from (20)

$$u(n) = \frac{n}{N}$$

on small values of $C_2/2\,C_1$.

III. The grade of amplification in an amplifier with N members (see [2] p. 42). The problem leads to the following systems of difference equations:

$$z_1\, i(n) + u(n+1) - u(n) = 0$$

$$z_2\, i(n+1) - z_2\, i(n) + u(n+1) + S z_2\, u(n) = 0$$

S, z_1, z_2 constants and

$$u(0) = u_G \qquad\qquad i(N) = 0 \tag{22}$$

We show the method of solving only, without physical interpretation. The corresponding algebraic equation is (from (10))

$$z_1\, t\, i(n+1) + (1-t)\, u(n+1) = u_G - z_1\, i(0)$$

$$z_2(1-t)\, i(n+1) + (1 + S z_2 t)\, u(n+1) = z_2\, i(0) - S z_2\, u_G \tag{23}$$

hence

$$i(n+1) = \frac{z_2[(1 - S z_1)\, i(0)\, t + S u_G - i(0)] + u_G}{-z_2(1 - S z_1)\, t^2 + (z_1 + 2 z_2)\, t - z_2}$$

Making use of the substitutions

$$\sqrt{1 - S z_1} = A \tag{24}$$

and

$$\frac{1}{A}\left(1 + \frac{z_1}{2 z_2}\right) = ch\, \tau \tag{25}$$

we get the formula

$$i\,(n+1) = \frac{z_2\,[A^2\,i\,(0)\,t + S\,u_G - i\,(0)] + u_G}{-\,z_2\,(A^2\,t^2 - 2\,A\,ch\,\tau\,t + 1)}$$

where

$$A^2\,t^2 - 2\,A\,ch\,\tau\,t + 1 = (1 - Ae^{-\tau}\,t)\,(1 - Ae^{+\tau}\,t)$$

Thus with the method of partial fractions and using (17) we obtain

$$i\,(n+1) = A^n \cdot \frac{z_2\,[A\,i\,(0)\,e^{-\tau} + S u_G - i\,(0)] + u_G}{e^{-2\tau} - 1}\,e^{+\tau n} + \\ + A^n\,\frac{z_2\,[A\,i\,(0)\,e^{\tau} + S\,u_G - i\,(0)] + u_G}{e^{-2\tau} - 1}\,e^{-\tau n}$$

Considering the identities

$$\frac{1}{e^{2\tau} - 1} = \frac{e^{-\tau}}{e^{\tau} - e^{-\tau}}$$

$$\frac{1}{e^{-2\tau} - 1} = -\,\frac{e^{\tau}}{e^{\tau} - e^{-\tau}}$$

we get the more simple form

$$i\,(n) = A^n \left[\frac{(1 + z_2\,S)\,u_G - i\,(0)}{A\,sh\,\tau}\,sh\,n\,\tau + i\,(0)\,ch\,n\,\tau\right] \quad \text{if } n \geq 1. \qquad (26)$$

Substituting $n = N$ in (26), from (22) we get $i\,(0)$.

Using (24) and (25), we express $i\,(n)$ by the z_1, z_2, S constants. We get $u\,(n)$ from the algebraic equation (23) also in the described manner.

§ 10. A restriction of the method. If $P_n\,(t)$ is the polynom of the translation function and

$$P_n\,(0) = 0$$

we know from § 6, that $P_n\,(t)$ has no inverse in K. Thus if in case of a difference equation (or for a system of difference equations) the equivalent algebraic equation leads to a polynom with the foregoing behaviour, the above method cannot be applied. However this restriction is not essential, because in practice such difference equation does not occur. On the other hand, precisely this condition makes possible the foundation of operational calculus without the introduction of abstract elements.

Summary

The paper contains a new operator calculus for solving linear difference equation (and systems of difference equations) with constant coefficients. It is more general and more simple, than those described in [2] or [3]. To illustrate the method, we give three examples. Two of them are difference equations from the impulse and regulation techniques, and the third can not be solved by the application of finite Laplace or Dirichlet transforms.

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