STEINER TYPE INEQUALITIES IN PLANE GEOMETRY

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In the course of an investigation into the magnitude of the principal frequency Λ of a stretched membrane of area A and periphery L, it was found for a convex membrane that Λ can be appraised by the quantities A and L. To put it in more exact terms, if one seeks a solution of the partial differential equation

$$\Delta u + \Lambda^2 u = 0$$

u vanishing on the periphery of a convex plane domain D of area A, periphery L, the first principal frequency Λ satisfies the double inequality [8]

$$\frac{1}{2} \frac{L}{A} < \Lambda < \sqrt{3} \frac{L}{A}.$$

Apart from various applications of Courant's principle, not easily treatable in forms of concise inequalities, this double inequality is, as far as I know, the first simple two-sided — though admittedly not too sharp — existimation of the principal frequency of a membrane connecting it with geometrical data of the domain D. Moreover these data are the simplest geometrical quantities attached to a given plane domain.

If one wants to get rid of the restriction of the convexity of the domain D, one can see by means of examples that the left-hand side of the above double inequality fails to hold [9]. Yet one can show that

$$\Lambda < \sqrt{3} \frac{L}{A}.$$

for every membrane [9].

For showing this one has to prove an elementary geometrical property of plane figures, or more generally, plane point sets. Though this property

(Theorem I of this paper) can be expressed in quite simple terms and a partial analogy of it for convex domains was known to J. STEINER (and proved by quite elementary means) about a century ago, yet the demonstration of our theorem presented quite unexpected difficulties, mostly of topological nature.

These difficulties were later overcome, or rather, got round in an ingenious way by B. SZŐKEFALVI-NAGY, who gave an independent proof of Theorem I, to be published in the Acta Scientiarum Mathematicarum. His proof is self-contained, it does not rely on other results. Yet it seems worth while to present the following proof to show how closely the topics of this paper are interrelated with other investigations, notably with those of H. HADWIGER.

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The outer parallel point set S_ϱ of a closed plane point set S is defined as the union of all closed circular disks of radius ϱ whose centres are points of S. The inner parallel point set $S_{-\varrho}$ ($\varrho < r$ where r is the radius of the greatest circle which can be inscribed in S) is the closure of the set of the centres of all those closed circular disks of radius ϱ which lie entirely in the interior of S.

The point sets S we shall deal with are all closed. We suppose throughout the whole paper for any point set to be met with, that their area A and the length L of their boundaries B exist in Minkowski's sense [10]. More precisely: if B' is a part of B, then the limits[1]

$$\lim_{\varepsilon\to+0} \frac{\text{area of } B'_\varepsilon S}{\varepsilon} \quad \text{and} \quad \lim_{\varepsilon\to+0} \frac{\text{area of } (B'_\varepsilon - S)}{\varepsilon}$$

exist and have the same finite value L', the length of the part B' of B. In particular

$$L = \lim_{\varepsilon\to+0} \frac{A - A_{-\varepsilon}}{\varepsilon} = \lim_{\varepsilon\to+0} \frac{A_\varepsilon - A}{\varepsilon}.$$

We suppose further that

$$\lim_{\varepsilon\to+0} L_{-\varepsilon} = \lim_{\varepsilon\to+0} L_\varepsilon = L. \tag{1}$$

[1] The notations A, B, L will be used consistently in such a manner that e.g. A_ε means the area of the set S_ε, $L^1_{-\varrho}$ the length of the boundary of the set $S^1_{-\varrho}$, $(B^1_{-\varrho})_\sigma$ the boundary of $(S^1_{-\varrho})_\sigma$ etc. Further we will denote by $C_s(T)$ a closed circular disk of radius s, centre T.

Other notations are as follows. If S_1, S_2 are point sets and P a point, then

$S_1 + S_2$ is the set of all those points which belong either to S_1 or to S_2;

$S_1 - S_2$ is the set of all those points of S_1, which do not belong to S_2;

$S_1 S_2$ is the set of all those points which belong simultaneously to S_1 and S_2 (the common part of S_1 and S_2);

$S_1 \subset S_2$ or $S_2 \supset S_1$ means that each point of S_1 belongs to S_2, too;

$P \in S_1$, $P \notin S_2$ means that the point P is an element of S_1, but not of S_2.

If S is a convex domain, then Steiner's equalities hold :

$$A_\varrho = A + \varrho L + \pi \varrho^2, \; L_\varrho = L + 2 \varrho\pi. \tag{2}$$

In search of the extension of the validity of Steiner's equalities, H. HADWIGER [4, 5] defined the notions of under-convexity and over-convexity. A closed point set S is said to be under-convex of degree a if for any $0 < \varkappa < a$ and for every point T of the plane, $SC_\varkappa(T)$, the common part of S and $C_\varkappa(T)$ is void or simply connected, yet if $\varkappa > a$ one can find at least one point T for which this is not true. If S is convex, its degree of under-convexity is ∞. We may extend the definition of under-convexity to $a = 0$ and say that the domain S is under-convex of degree 0 if to every $a > 0$ one can find a $\varkappa < a$ and a point T, so that $SC_\varkappa(T)$ consists of disconnected parts. (According to this, the degree of under-convexity of a non-convex polygonal domain is 0.) On the other hand S is said to be over-convex of degree β if $\bar{S}^*$, the closure of the complementary set S^* of S, is under-convex of degree β.

HADWIGER proved that if the simply connected closed domain S is under-convex of degree a and over-convex of degree β then (2) holds for $-\beta < \varrho < a$.

Another result of HADWIGER [4, 5] is, that if S is a simply connected domain, not containing infinity, then

$$A_\varrho \leqq A + \varrho L + \pi \varrho^2 \tag{3a}$$

and he calls it an inequality of Steiner's type. A similar formula follows from investigations of B. SZ.-NAGY [11] and G. BOL. [1]. They found that if S is a convex domain, then

$$A_{-\varrho} \geqq A - \varrho L + \pi \varrho^2 \quad (0 < \varrho < r) \tag{3b}$$

moreover $L_{-\varrho} \leqq L - 2\pi\varrho$.

In the following we shall prove

Theorem I : If S is any simply connected domain, not containing infinity and subjected to the conditions given above, then

$$L_{-\varrho} \leqq L - 2\pi\varrho \quad (\varrho < r) \;\text{(4a)} \quad \text{and} \quad L_\varrho \leqq L + 2\pi\varrho \quad (\varrho > 0) \quad \text{(4b)}$$

from which (3a) *and* (3b) *follow for any simply connected domain* by an integration with respect to ϱ.[2]

[2] A relation substantially equivalent with formula (4b) — which has however no significance in the membrane problem mentioned in the introduction — was quite recently proved by G. Fast [12].

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Consider the open set $\tilde{S}_{-\varrho}$ of the centres of all closed circular disks of radius ϱ which lie in the interior of the simply connected plane domain S. The set $\tilde{S}_{-\varrho}$ may consist of several disconnected parts $\tilde{S}^1_{-\varrho}, \tilde{S}^2_{-\varrho}, \ldots, \tilde{S}^n_{-\varrho}$. The closure of $\tilde{S}^i_{-\varrho}$ will be denoted by $S^i_{-\varrho}$ and termed a *component of the inner parallel point set* $S_{-\varrho}$ of S.[3] It is easy to see that any $S^i_{-\varrho}$ is a simply connected domain. We prove the following

Lemma I : if S is under-convex of degree a, then any component $S^i_{-\varrho}$ of $S_{-\varrho}$ is under-convex of degree at least $\varrho + a$.

Suppose there exists a circular disk $C_{\varrho+a-\varepsilon}(O)$ such that $S^i_{-\varrho}\, C_{\varrho+a-\varepsilon}(O)$ consists of at least two disconnected parts. Then there are at least two arcs

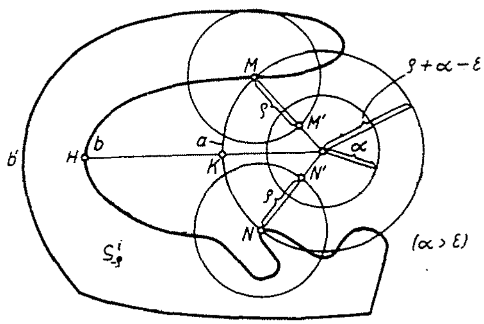

Fig. 1

on the periphery of $C_{\varrho+a-\varepsilon}(O)$ which do not belong to $S^i_{-\varrho}$ with the exception of their end points. It will be shown that if S is finite then each of these arcs can be connected with infinity by a path not going through $C_{\varrho+a-\varepsilon}(O)$ *i.e.* $S^i_{-\varrho}$ is disconnected.

For suppose the contrary. Let a be an arc of $C_{\varrho+a-\varepsilon}(O)$ not belonging to $S^i_{-\varrho}$ excepted its end points M and N, from where one cannot attain infinity in the manner described above. M and N divide the periphery $B^i_{-\varrho}$ of $S^i_{-\varrho}$ in two parts b and b' one of which, say b, has the property that the finite domain bounded by b, OM and ON does not contain interior points of $S^i_{-\varrho}$. The half-ray from O, going through the middle point K of a should meet b at H.

We define the domain δ as the domain bounded by b, and the straight line segments OM and ON, and state that *B cannot have points in the interior of $\delta - \delta\, C_a(O)$*. For if P_0 were such a point, it could be connected with infinity by a path p_0 having no common point with S, excepted P_0. We distinguish two cases :

[3] Of course $S^i_{-\varrho}$ and $S^j_{-\varrho}$ $(i \neq j)$ may have common boundary points. It can be shown, that the number N of those points, which belong to at least two $S^i_{-\varrho}$'s is limited and a common upper limit of N and n is $A/(\pi\varrho^2)$.

1. If $a \leqq \varepsilon$ (this includes the case $a = 0$) then OM and ON are contained in $C_\varrho(M)$ resp. $C_\varrho(N)$, hence no point of them is outside S and p_0 cannot reach infinity from P_0.

2. If $a > \varepsilon$, let M' and N' be the points of intersection of OM with $C_\varrho(M)$ resp. of ON with $C_\varrho(N)$. Then p_0 has to intersect either OM' or ON' for MM' and NN' are in S. So $p_0\, C_a(O)$, if it exists, is not void, its points do not belong to S, and it divides $C_a(O)$ in at least two disconnected parts. One of these parts contains M' and the other N', as p_0 enters $C_a(O)$ on a point of the arc whose central angle is MON and leaves it finally through a point not belonging to this arc. The two points M' and N', both belonging to S, cannot be connected in the inside of $C_a(O)$ by a path lying entirely in $C_a(O)$. Hence $C_a(O)$ is a disk of radius a for which $SC_a(O)$ is disconnected, contrary to our assumption.

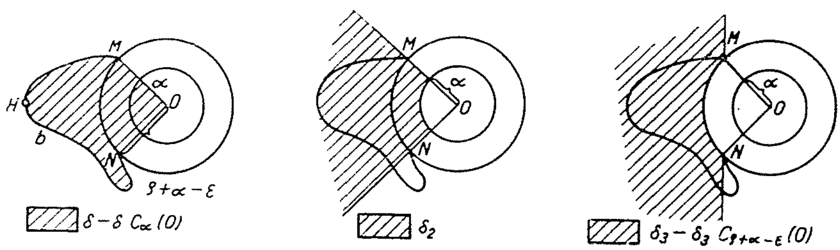

Fig. 2

Now we will seek a point P_H of B the distance of which from H is ϱ. We will see that such a point does not exist at all, *i.e.* H does not exist. We define the angular domain δ_1 as limited by the infinite rays from O, going through M, resp. N and containing K in its interior. P_H cannot be in $\delta_2 = \delta_1 - \delta_1\, C_a(O)$. For supposing the contrary, it cannot be in $\delta - \delta\, C_a(O)$ so at least one interior point of the straight line segment $H P_H$ contains a point H' of $S^i_{-\varrho}$ which is impossible since $H' P_H < \varrho$.

Neither can P_H be in the interior of $C_\varrho(M)$ or $C_\varrho(N)$. But $\delta_2 + C_\varrho(M) + C_\varrho(N)$ contains the interior of the parallel point set of radius ϱ of the arc a. Hence $P_H K \geqq \varrho$.

Finally let δ_3 be the half plane limited by the straight line going through M and N and containing K. $P_H \notin \delta_3 - \delta_3\, C_{\varrho+a-\varepsilon}(O)$ hence the angle HKP_H is greater than $\pi/2$ and as $P_H K \geqq \varrho$ it follows that $P_H H > \varrho$ in contradiction with the definition of P_H.

If the simply connected domain S contains infinity, a similar analysis shows that lemma I is true in this case too.

Recalling the definition of over-convexity we can now enounce

Lemma II. If S is over-convex of degree β, then any connected component S_ϱ^i of S is over-convex of degree at least $\varrho + \beta$.

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We proceed now to the proof of formula (4a) in the case when S is under-convex of degree $a > 0$. One can construct a component of its internal parallel domain $S_{-\varrho}$ by taking a circular disk of radius ϱ lying entirely in S and moving it continuously in every possible manner so that it remain always in the interior of S. (The moving disk must have no common points with B.) The part of the plane covered by the centre of the moving disk in its various positions s an open point set $\tilde{S}_{-\varrho}^1$ and its closure is the component $S_{-\varrho}^1$ of $S_{-\varrho}$. The closure of the area covered by the whole disk is $(S_{-\varrho}^i)_\varrho$. Clearly $(S_{-\varrho}^1)_\varrho \subset (S_{-\varrho})_\varrho \subset S$. (Cfr. Hadwiger [3] p. 17.)

The boundary $(B_\varrho^{-1})_\varrho$ of $(S_{-\varrho}^1)_\varrho$ consists at least partly of points of B. If $(B_{-\varrho}^1)_\varrho$ has other points too, it can be shown that these consist of circular arcs $a'_1, a'_2, \ldots, a'_h$ and the length of none of these arcs is greater than $\pi\,\varrho$.[4] The parts of B not common with $(B_{-\varrho}^1)_\varrho$ will be denoted by $b_1, b_2, \ldots, b_h$ so that b_g and a'_g have common end points. The length of b_g is greater than that of a_g. For b_g connecting two points of the periphery of $C_\varrho(O_g)$, is outside of $C_\varrho(O_g)$, so it is longer than the shorter arc of $C_\varrho(O_g)$ connecting these same points.

It may happen that S_ϱ does not have other points than those of $S^1{}_\varrho$. Then using our lemma and Hadwiger's theorem on under-convex point sets, we conclude that

$$L \geqq (L_{-\varrho}^1)_g = L_{-\varrho} + 2\,\varrho\,\pi$$

and have shown the inequality (4a) in the case when $S_{-\varrho}$ consists of one component only.

If, on the other hand, $S_{-\varrho}$ has other points too than those of $S^1_{-\varrho}$, let P_2 be one of them. Connect the point P_2 with $(B^1_{-\varrho})_\varrho$ by a curve lying in S and outside of $(S^1_{-\varrho})_\varrho$. The connecting curve ends in a point of the circular arc,

[4] For let A_g be a point of $(B^1_{-\varrho})_\varrho$ not belonging to B and let $O_g \in S^1_{-\varrho}$ be the centre of that circle of radius ϱ whose periphery contains A_g. $C_\varrho(O_g)$ has to contain at least two boundary points of B, otherwise one could find in any neighbourhood of O_g a point $O'_g \in S_{-\varrho}$ such that $C_\varrho(O'_g)$ would contain A_g in its interior. Let $P_g \in B$, $Q_g \in B$ be two points of $C_\varrho(O_g)$, such that the open arcs $\widehat{P_gA_g}$, resp. $\widehat{Q_gA_g}$ do not contain points of B. Then these arcs certainly belong to $(S^1_{-\varrho})_\varrho$ and the combined length of them cannot be greater than $\pi\,\varrho$. For, supposing the contrary, either the arc $\widehat{P_gA_g}$ or the arc $\widehat{Q_gA_g}$ is less than $\pi\,\varrho$ and one could find in any neighbourhood of O_g a point O''_g lying on the periphery of $C_\varrho(P_g)$ or of $C_\varrho(Q_g)$, such that $A_g \in C\,(O''_g) \subset (\tilde{S}^1_{-\varrho})_\varrho$. On the other hand O_g is perfectly determined as the nearest point of $S^1_{-\varrho}$ to the chord P_gQ_g and it follows hence that no point of the arc $a'_g = \widehat{P_gA_gQ_g}$ can be an interior point of $S^1_{-\varrho}$.

say a'_1 whose end points are P_1 and Q_1. Then place a disk of radius ϱ on P_2 and move it again in every possible manner, so that it should never have common points with B. We again get an open point set, say $\tilde{S}^2_{-\varrho}$ which has no common point with $S^1_{-\varrho}$. Its closure is another component $S^2_{-\varrho}$ of $S_{-\varrho}$. Repeat this procedure until all components $S^2_{-\varrho}, S^3_{-\varrho}, \ldots, S^n_{-\varrho}$ of $S_{-\varrho}$ and their outer parallel domains $(S^2_{-\varrho})_\varrho, \ldots, (S^n_{-\varrho})_\varrho$ are found.

We will use an induction for the proof of (4a) which was shown to be valid for $n=1$. We draw the chord P_1Q_1 connecting the end points of the arc a'_1. As $P_1 Q_1 \leqq 2\varrho$ no point of this chord lies in the interior of $S_{-\varrho}$. $P_1 Q_1$ divides the interior of S in two parts Σ' and Σ'' both of which contain at least one, and so at most $n-1$ components of $S_{-\varrho}$. The parts of the boundary B belonging to Σ' resp. Σ'' will be denoted by B' resp. B'', their lengths by L' resp. L''.

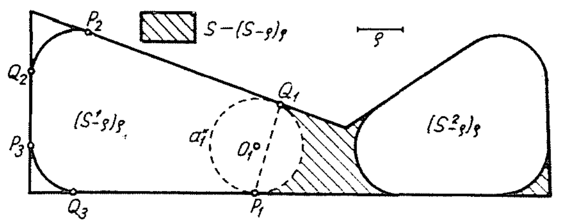

Fig. 3

Note that by virtue of our construction that part of $S_{-\varrho}$ which lies in Σ' is entirely determined by B' and the remaining part of $S_{-\varrho}$ depends solely on the shape of B''. Hence we have a considerable liberty in deforming the curve B'' (B') into a new position B''_0 (B'_0) so that the domain Σ'_0 (Σ''_0) bordered by B' and B''_0 (by B'' and B'_0) is such that $(\Sigma'_0)_{-\varrho}\, \Sigma' = S_{-\varrho}\, \Sigma'$ $[(\Sigma''_0)_{-\varrho}\, \Sigma'' = S_{-\varrho}\, \Sigma'']$.

All that is wanted of B''_1 is that it should not have common points with the interior of the disk $C_\varrho(O_1)$. If this condition is fulfilled then there does not exist such a point $P' \in \Sigma'$ for which $C_\varrho(P')\, B' = 0$ and $C_\varrho(P')\, B''_1 \neq 0$.

If we denote by a''_1 that arc of the periphery of $C_\varrho(O_1)$ which completes a'_1 to the whole circumference, then we may choose B''_0 to be the arc a''_1. Again there is no point P'' of Σ'' for which $C_\varrho(P'')$ has no common point with B'', but has a common point with a'_1.

So we conclude

$$\Sigma' S_{-\varrho} = [\Sigma' + C_\varrho(O_1)]_{-\varrho}$$

$$\Sigma'' S_{-\varrho} = [\Sigma'' + C_\varrho(O_1)]_{-\varrho}$$

Let us now denote the total length of the boundaries of $\Sigma' S_{-\varrho}$ and $\Sigma'' S_{-\varrho}$ by $L'_{-\varrho}$ and $L''_{-\varrho}$. As $\Sigma' S_{-\varrho}$ and $\Sigma'' S_{-\varrho}$ contain less than n components, we may use our inductive assumption and write

$$L' + L'' \leqq B' - 2\varrho\pi + B'' - 2\varrho\pi + 2\varrho\pi.$$

This is formula (4a) with the proviso that S is under-convex of degree $\alpha > 0$.

If the degree of under-convexity of S is 0, then according to our lemma S_ε is under-convex of degree at least S_ε and as $S_{-\varrho} = (S_{-\varepsilon})_{-(\varrho-\varepsilon)}$ $(0 < \varepsilon < \varrho)$ (cfr. Hadwiger [3] p. 17) we can state that $L_{-\varrho} \leqq L_{-\varepsilon} - 2(\varrho - \varepsilon)\pi$. Using the formula (1) we have again (4a).

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For proving the inequality (4b) we suppose at first that the simply connected closed domain S is over-convex of degree β and we construct its outer parallel domain S_ϱ by taking all closed circular disks of radius ϱ which

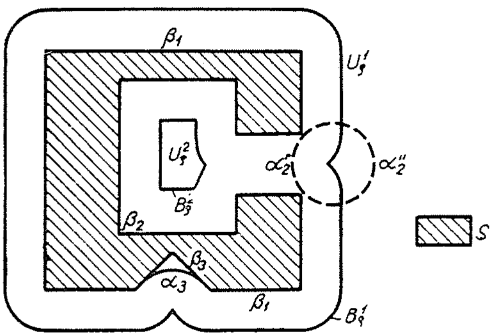

Fig. 4

do not contain points of S. The complementary set of the union U_ϱ of the centres of all these disks is S_ϱ.

First take a disk sufficiently far from S, and move it continuously in every possible manner so that it should have no common points with S. Then a simple connected part U_ϱ^1 of U_ϱ is constructed which extends to infinity. The boundary $B_{\varrho\varrho}$ of the closure of $(U_\varrho^1)_\varrho$ partly consists of points of B and it may happen that $B_{\varrho\varrho}$ contains other points too ; it can be shown that these lie on circular arcs $\alpha_2', \alpha_3', \ldots, \alpha_\lambda'$ of radius ϱ the central angle of which is at most π. Let $\beta_\varkappa$ be that connected part of B, which does not belong to $B_{\varrho,\varrho}$ and has common end points with $\alpha_\varkappa'$. The remaining part of B will be denoted by β_1. Obviously, β_λ is not shorter than α_λ hence the length $L_{\varrho\varrho}$ of $B_{\varrho,\varrho}$ is not greater than L.

If $U_\varrho^1 = U_\varrho$ we argue that the boundary B_ϱ of S_ϱ is the same as the boundary of U_ϱ and $L_{\varrho\varrho}$ is the length of the boundary of the closure of $(U^*)_{-\varrho}$. Hence

$$L \geqq L_{\varrho\varrho} \geqq L_\varrho - 2\varrho\pi$$

and formula (4b) is proved if S_ϱ is a simply connected domain.

If B_ϱ, the boundary of S_ϱ consists of several disconnected parts, then one of these, say B_ϱ^1 is the same as the boundary of U_ϱ^1 and the others lie in the domains bounded by $\beta_\varkappa$ and $\alpha'_\varkappa$.

Deforming B in such a way that each $\beta_\varkappa$ is replaced by $\alpha_\varkappa$, and terming $S^{(1)}$ that point set, the boundary of which is the deformed boundary B, one sees that $S^{(1)}$ is the complementary domain of U_ϱ^1 and so $S^{(1)}$ is the inner parallel point set of radius ϱ of the closure of $(U^1)^*$. Let now $\alpha''_\varkappa$ be the circular arc which completes α'_k to an entire circumference. Denoting by $S^{(\varkappa)}$ the domains bounded by $\beta_\varkappa$ and $\alpha''_\varkappa$ it can be shown that the* boundary of the inner parallel domain of radius ϱ of $S^{(\varkappa)}$ coincides with that part of B_ϱ, which is surrounded by $\beta_\varkappa$ and $\alpha'_\varkappa$. Using the letters $\beta_\varkappa$, $\alpha'_\varkappa$, $\alpha''_\varkappa$ for the notation of the length of these curves we have

$$L^{(1)} \leqq \beta_1 + \sum_\varkappa \alpha'_\varkappa + 2\varrho\pi$$

and

$$L^{(\varkappa)} \leqq \beta_\varkappa + \alpha''_\varkappa - 2\varrho\pi \qquad (\varkappa = 2, 3, \ldots \lambda)$$

By addition it follows

$$\sum_\varkappa L^{(\varkappa)} = L_\varrho \leqq L + (\lambda - 1)\cdot 2\varrho\pi + 2\varrho\pi - 2(\lambda-1)\varrho\pi = L + 2\varrho\pi.$$

If the degree of over-convexity of S is 0, then we prove (4b) first for S_ε which is over-convex of degree at least ε. As $(S_\varepsilon)_{\varrho-\varepsilon} = S_\varrho$ $(0 < \varepsilon < \varrho)$ we have

$$L_\varrho < L_\varepsilon + 2(\varrho - \varepsilon)\pi .$$

Hence using (1), the inequality (4b) follows again.

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These results can be generalized to domains of k-tuple connectivity. We have in this case

Theorem II. If S is a k-tuply connected domain, then

$$L_{-\varrho} \leqq L + 2(k-2)\pi\varrho \qquad (\varrho \leqq r) \tag{5a}$$

$$L_\varrho \leqq L + 2\pi\varrho \tag{5b}$$

Hence it follows that e.g. for a ring-shaped domain $L_{-\varrho} \leqq L$.

For a k-tuply connected domain can be completed by the addition of $k-1$ simply connected domains, say $S_2, S_3, \ldots, S_k$ to a simply connected domain S_1. The length of the boundaries B_i of S_i will be denoted by L_i.

Now $S_{-\varrho}$ consists of all those points which are simultaneously part of $(S_1)_{-\varrho}$, $[(S_2)_\varrho]^*, \ldots, [(S_k)_\varrho]^*$ so

$$B_{-\varrho} \subset (B_1)_{-\varrho} + (B_2)_\varrho + \ldots + (B_k)_\varrho.$$

From this

$$L_{-\varrho} \leqq (L_1)_{-\varrho} + (L_2)_\varrho + \ldots + (L_k)_\varrho \leqq (L_1 - 2\pi\varrho) + (L_2 + 2\pi\varrho) + \ldots + (L_k + 2\pi\varrho) = L + 2(k-2)\pi\varrho.$$

The less informative inequality (5b) is derived essentially in the same way.

Both (5a) and (5b) are in a sense best possible inequalities. Let namely S be a circular disk, out of which $k-1$ circular disks of radius ε are cut out. If $\varrho < \varepsilon$ then $L_{-\varrho} = L + 2(k-2)\pi\varrho$ and if $\varrho > \varepsilon$ then $L_\varrho = L + 2\pi\varrho$.

7

A corollary of the inequality (4a) is the extension of the *isoperimetric inequality of Bonnesen* to non-convex domains, namely that if A is the area of a simply connected domain S, L the length of its periphery, r the radius of the greatest inscribed circle, then

$$L^2 - 4\pi A \geqq (L - 2\pi r)^2$$

or

$$A \leqq L r - \pi r^2. \tag{6}$$

This is special case of an inequality found by L. FEJES TÓTH [2]. We use an argument due to HADWIGER [7] according to which $L_{-\varrho} \leqq L - 2\pi\varrho$ integrated in the interval $0 \leqq \varrho \leqq r$ yields

$$\int_0^r L_{-\varrho}\, d_\varrho \leqq \int_0^r (L - 2\pi\varrho)\, d\varrho$$

or as the left hand side of this inequality is A, we have (6).

Summary

The following theorem and its generalizations are proved under conditions specified in the foregoing paper.

Let S be a simply connected plane domain, L the length of its boundary, A its area. C_ϱ and $C_{-\varrho}$ are plane curves, not necessarily connected, lying outside and inside respectively of S, consisting of the set of points the nearest distance of which from the boundary points of S is ϱ. If L_ϱ and $L_{-\varrho}$ are the lengths of the curves C_ϱ and $C_{-\varrho}$ respectively, further A_ϱ an. $A_{-\varrho}$ are the areas included by C_ϱ and $C_{-\varrho}$, then the inequalities (3a), (3b), (4a), (4b) hold.

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