ON IMPROVEMENT OF THE RUNGE—KUTTA—NYSTRÖM METHOD. I.

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1. §. Introduction

The so-called Runge—Kutta—Nyström method is very often applied to the numerical evaluation of the initial value problems of ordinary differential equations, particularly in the initial interval (s. [1], [2]).

The equation, or system of two or more equations, is usually reduced to the form

$$y^{(n)} = f(x\,;\ y\,;\ y';\ldots;\ y^{(n-1)}),$$

resp.

$$\frac{d}{dx} y_i = f(x\,;\ y_1\,;\ y_2\,;\ \ldots;y_n)\,;\ i = 1, 2, \ldots n$$

The evaluation of the function values f resp. f_i is a quite laborious and burdensome problem even when employing any of the most practical numerical method, by making use of modern electronic calculators. The question is important from that point of view whether the Runge—Kutta—Nyström method (further R. C. N.) can be improved by decreasing the number of the pivotal points at which the function values f resp. f_i are to be evaluated even at the price of increasing the number of elementary operations. This means that we intend to increase the degree of the R. C. N. method (over four), without increasing the number of the pivotal points at which the function values are to be evaluated.* But it can't be carried out without making modification in the procedure, because — as will be seen — increasing the degree of the unmodified R. C. N. method, immediately by leaps and bounds increases the number of the pivotal poinst at which the function values are to be evaluated. As an illustration: employing the unmodified R. C. N. method of fifth, sixth, seventh resp. eighth degree, we have to evaluate the function values at six, eight, eleven resp. fifteen points by each step,

* The degree of a procedure means the number of the terms equal in the Taylor expansion of the exact and of the approximative solution.

for first-order differential equation. Moreover, by increasing the degree, the parameter values of the procedure can only be computed with more and more difficulties.*

In the following, a modification of the original method is presented, partly to increase the degree almost as speedily as the number of the pivotal values increase, partly for the easier evaluation of the parameters. The fundamental idea for this modification is to make use of the partial derivatives of the function(s) $f(f_i)$ with respect to the numerical approach in the dependent variables.

In the first part of this work the modification will be presented — restricted to first-order equations — further it will be shown that the degree, both of the unmodified and of the modified R. C. N. method, at least theoretically, can be indefinitely increased. In the second part, the above-mentioned procedure will be generalized for the case of higher-order equations, finally the third part deals with the algoritmes, by means of which the parameters can, at least partly, be computed.

2. §. The Taylor series of the exact solution

21. Symbols ; reductions ; of the operator D. It may be assumed that the equation is given by the form

$$y' = f(x, y) \tag{21.1}$$

because the equation can surely be reduced to the above-mentioned form, in the surroundings of the points at which the approximate values of the solution are to be evaluated. Moreover, it will be assumed that $f(x, y)$ and its derivatives of a sufficient high order with respect to both variables are continuous in the surroundings of the point (x_0, y_0) for in the surroundings of singular points numerical methods are not used.

It is convenient to express the higher-order derivatives of the solution of the equation (21.1) by means of the operator

$$D_f = \frac{\partial}{\partial x} + f\frac{\partial}{\partial y} ; \; f = f(x, \; y) \tag{21.2}$$

as suggested by HEUN, further denoting it by D, if there is no misunderstanding to be feared. Let us suppose that the function

$$u = u\,(x, y) \tag{21.3}$$

* In the third part of this work, algoritmes will be shown as how to determine the parameters of the unmodified and modified R. C. N. method.

is differentiable in STOLZ's sense, in the domain T involving the point (x_0, y_0), if f is continuous in T. In this case u is differentiable along the solving-curve* of the equation (21.1) supposing that (x_0, y_0) is a point of the curve, as follows

$$\frac{d}{dx} u(x, y) \Big|_{y=y(x)} = \frac{\partial u}{\partial x} + y' \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} + f \frac{\partial u}{\partial y} = D_f u = Du \qquad (21.4)$$

It can easily be seen that

1° D is a linear operator, namely

$$D(u + v) = Du + Dv \qquad (21.5)$$

2° D is of differential-operator type, applied to products

$$D(uv) = vDu + uDv \qquad (21.6)$$

It is convenient to define the powers of D formally by

$$D^n = \sum_{k=0}^{n} \binom{n}{k} f^k \frac{\partial^n}{\partial x^{n-k} \partial y^k} \qquad (n = 1, 2, 3, \ldots) \qquad (21.7)$$

because by means of these the iterated values for D, and the higher-order derivatives of u with respect to y can be obtained in a very simple manner. It is easy to verify, that

$$D(D^n) = D^{n+1} + n\, Df\, D^{n-1} \frac{\partial}{\partial y} \qquad (21.8)$$

(see e. g. [3]).

22. *The Taylor series expansion.* The derivatives of the solution of the equation (21.1), and its Taylor expansion can also be written by means of the operator D (supposing that $f(x, y)$ and its derivatives of sufficient high-order are continuous at the specified point) is as follows

$$\frac{d}{dx} y = f \qquad (22.1)$$

$$\frac{d^2}{dx^2} y = Df \qquad (22.2)$$

* As f is continuous in T, there exists at least one solving curve through any point of T; if more exist through each point, the formula (21,4) is valid along each of them.

$$\frac{d^3}{dx^3} y = D^2 f + f_y Df \tag{22.3}$$

$$\frac{d^4}{dx^4} y = D^3 f + f_y D^2 f + f_y^2 Df + 3 Df_y Df \tag{22.4}$$

$$\frac{d^5}{dx^5} y = D^4 f + f_y D^3 f + f_y^2 D^2 f + 4 Df_y D^2 f + f_y^3 Df + + 7 f_y Df_y Df + 6 D^2 f_y Df + 3 f_{yy} (Df)^2 \tag{22.5}$$

$$\frac{d^6}{dx^6} y = D^5 f + f_y D^4 f + f_y^2 D^3 f + 5 Df_y D^3 f + f_y^3 D^2 f + 9 f_y Df_y D^2 f + + 10 D^3 f_y D^2 f + 10 f_{yy} D^2 f Df + f_y^4 Df + 12 f_y^2 Df_y Df + + 16 f_y D^2 f_y Df + 13 f_y f_{yy} (Df)^2 + 15 (Df_y)^2 Df + + 10 D^3 f_y Df + 15 Df_{yy} (Df)^2 \tag{22.6}$$

$$\frac{d^7}{dx^7} y = D^6 f + f_y D^5 f + f_y^2 D^4 f + 6 Df_y D^4 f + f_y^3 D^3 f + + 11 f_y Df_y D^3 f + 15 D^2 f_y D^3 f + 15 f_{yy} D^3 f Df + f_y^4 D^2 f + + 15 f_y^2 Df_y D^2 f + 25 f_y D_2 f_y D^2 f + 45 f_y f_{yy} D^2 f Df + + 24 (Df_y)^2 D^2 f + 20 D^3 f_y D^2 f + 60 Df_{yy} D^2 f Df + + 10 f_{yy} (D^2 f)^2 + f_y^5 Df + 18 f_y^3 Df_y Df + 31 f_y^2 D^2 f_y Df + + 38 f_y^2 f_{yy} (Df)^2 + 57 f_y (Df_y)^2 Df + 30 f_y D^3 f_y Df + + 75 f_y Df_{yy} (Df)^2 + 15 D^4 f_y Df + 81 D^2 f_y Df_y Df + + 63 f_{yy} Df_y (Df)^2 + 45 D^2 f_{yy} (Df)^2 + 15 f_{yyy} (Df)^3 ; \tag{22.7}$$

In the theorem which follows, the derivatives $\frac{d^n}{dx^n} y(x)$ are given.

22.1. Theorem. In the expansion of the higher-order derivatives of y
1° every term can only contain factors taking the following form

$$D^k \frac{\partial^l f}{\partial y^l} ; \quad k \geq 0 ; \; l \geq 0 \tag{22.8}$$

specially

$$D^\circ \frac{\partial^l f}{\partial y^l} \equiv \frac{\partial^l f}{\partial y^l} ; \; D^k \frac{\partial^\circ f}{\partial y^\circ} \equiv D^k f ; \; D^\circ \frac{\partial^\circ f}{\partial y^\circ} \equiv 1 \tag{22.9}$$

2° In the term taking the form of

$$\prod_{m=1}^{p} D^{l_m} \frac{\partial^{l_m} f}{\partial y^{l_m}} \cdot \prod_{M=1} D^{K_M} f \tag{22.10}$$

(there only exist terms of this type according to 1°, and $K_M \neq 0$; $M = 1, 2, \ldots, q$); (the arrengement is natural that is $\prod_{m=1}^{p} l_m \neq 0$) there is a relation between the "exponents" k_m, K_M, and the "ordial numbers" l_m

$$q = 1 + \prod_{m=1}^{p} l_m - p \tag{22.11}$$

from which immediately follows that

$$q \geqq 1$$

3° Whenever the term (22.10) occurs in the expansion of $y^{(n+1)}$, there is another relation between the exponents and the ordial numbers

$$\sum_{m=1}^{p} k_m + \sum_{m=1}^{p} l_m + \sum_{M=1}^{q} K_M = n \; (n = 1, 2, \ldots) \tag{22.13}$$

4° In the expansion of $y^{(n+1)}$ are those, and only those terms which suit the above-prescribed requirements of 1°—3°.

These propositions can be proved by means of the mathematical induction. Let us suppose that the propositions 1°—4° are valid until $n = N$, namely

$$y^{(N)} = \sum_i A_i \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{c_i} D^{K_M^{(i)}} f \tag{22.14}$$

and

$$1°.\; k_m^{(i)} \geq 0\,;\; l_m^{(i)} \geq 0\,;\; K_m^{(i)} \geq 1\,;$$

$$2°.\; q_i = 1 - p_i + \sum_{m=1}^{c_i} l_m^{(i)};\; (q_i \geq 1):$$

$$3°.\; \sum_{M=1}^{p_i} (k_m^{(i)} + l_m^{(i)}) + \sum_{M=1}^{i} K_M^{(i)} = N - 1\,;$$

4° The sum (22.14) contains all the terms which suit the above requirements of 1°—3° prescibed. $y^{(n+1)}$ can easily be expressed by means of D, using the relations (21.5)—(21.8). Hence it can be immediately seen that 1° is valid again. In order to prove 2°, let us consider all the terms derived from

$$\prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{K_M^{(i)}} f$$

Whenever D is applied to one of the factors of the first group — e. g. to the s-th one — it takes the following form

$$D^{k_s^{(i)}+1} \frac{\partial^{l_s^{(i)}} f}{\partial y^{l_s^{(i)}}} \cdot \left\{ \prod_{m=1}^{s-1} \cdot \prod_{m=\ +1}^{p_i} \right\} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{k_M^{(i)}} f +$$

$$(if\, k_s^{(i)} > 0) + k_s^{(i)} D^{k_s^{(i)}-1} \frac{\partial^{l_s^{(i)}+1} f}{\partial y^{l_s^{(i)}+1}} \left\{ \prod_{m=1}^{s-1} \cdot \prod_{m=s+1}^{pi} \right\} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} Df \cdot \prod_{M=1}^{q_i} D^{k_M^{(i)}} f \quad (22.15)$$

2° is evidently valid for the first term of the relation (22.15), because q_i, p_i and $\Sigma l_m^{(i)}$ are unchanged; similarly, valid also to the second term, because p_i is unchanged while both q_i and $\Sigma\, l_m^{(i)}$ increase — after regrouping — with the unit.

When D is applied to the S-th factor of the second group, it takes the following form

$$\prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot D^{K_S^{(i)}+1} f \cdot \left\{ \prod_{M=1}^{S-1} \cdot \prod_{M=S+1}^{q_i} \right\} D^{K_M^{(i)}} f +$$

$$+ K_S^{(i)} \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot D^{K_S^{(i)}-1} \frac{\partial f}{\partial y} \cdot Df \left\{ \prod_{M=1}^{S-1} \cdot \prod_{M=S+1}^{q_i} \right\} D^{K_M^{(i)}} f \quad (22.16)$$

Here 2° becomes valid again, because: in the first term p_i, q_i and $\Sigma\, l_m^{(i)}$ are unchanged; in the second term q_i is unchanged, while both p_i and $\Sigma\, l_m^{(i)}$ increase with the unit. By reason of the relations (22.15) and (22.16) it can be seen that q_i cannot be decreased, and so q_i remains ≥ 1. The sum $\Sigma\,(k_m^{(i)} + l_m^{(i)}) + \Sigma\, K_M^{(i)}$ increases in each of its terms with the unit, and so 3° remains valid in every term of y^{N+1}.

The validity of 4° can be proved as follows: every term of y^{N+1}, which has the form (22.10) and is in accordance to the properties 1°—3°, can be derived from one of the term of $y^{(N)}$ by means of the operation D.

Let the product then

$$\prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{K_M^{(i)}} f \quad (22.17)$$

one of the terms of the Taylor expansion for $y^{(N+1)}$, that is

$$1°.\ k_m^{(i)} \geq 0;\ l_m^{(i)} \geq 0;\ K_M^{(i)} \geq 1;$$

$$2°.\ q_i = 1 - p_i + \sum_{m=1}^{p_i} l_m^{(i)};\ (q_i \geq 1);$$

$$3°.\ \sum_{m=1}^{p_i} (k_m^{(i)} + l_m^{(i)}) + \sum_{M=1}^{q_i} K_M^{(i)} = N. \quad (22.18)$$

Let us moreover suppose that there is at least one factor — e. g. the S-th one — among the factors of the second group in (22.17), by which

$$K_S^{(i)} > 1 . \tag{22.19}$$

In this case, for instance, (22.17) is derived from the term

$$\prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot \left\{ \prod_{M=1}^{S-1} \cdot \prod_{M=S+1}^{r_i} \right\} D^{K_M^{(i)}} f \cdot D^{K_S^{(i)}-1} f \tag{22.20}$$

of the Taylor expansion for $y^{(N)}$ by means of the operation D. One must only realize that $y^{(N)}$ has a term such as (22.20). If this term satisfies the properties of 1°—3°, $y^{(N)}$ will surely have such a term, but, according to the conditions, even if the requirement 4° is statisfactory, supposing that $n = N$. It is easy to see that 1° is satisfactory ; 2° is realized for q_i, p_i and $\Sigma\, l_m^{(i)}$ are unchanged in comparison to (22.17) ; 3° is also valid, because $\Sigma\, l_m^{(i)}$ and $\Sigma k_m^{(i)}$ remain unchanged, $\Sigma\, K_M^{(i)}$ decreased with the unit, but the right-hand member of 3° must also be decreased with the unit, being (22.20) a term of $y^{(N)}$.

If it is impossible to find such an S, which satisfies (22.19), namely S has the form

$$\prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot (Df)^{q_i}, \tag{22.17*}$$

then — by reason of (22.18) — $p_i > 0$, supposing that $n \geq 3$. Now let us suppose that among the factors of the first group there exists one — e. g. s-th one — by which

$$k_s^{(i)} \geqq 1 \tag{22.21}$$

In this case, for instance, (22.17*) can be derived from the term

$$D^{k_s^{(i)}-1} \frac{\partial^{l_s^{(i)}} f}{\partial y^{l_s^{(i)}}} \cdot \left\{ \prod_{m=1}^{s-1} \cdot \prod_{m=s+1}^{p_i} \right\} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} (Df)^{q_i} \tag{22.22}$$

of $y^{(N)}$ by means of the operation D. This proves that $y^{(N)}$ has a term formed like (22.22) and is the same as above : 1° is evidently satisfied ; 2° is realized, because q_i, p_i and $\Sigma\, l_m^{(i)}$ are unchanged in (22.22) ; 3° is also valid, for $\Sigma\, K_M$ and $\Sigma\, l_m^{(i)}$ are unchanged and $\Sigma\, k_m^{(i)}$ is decreased with the unit, in comparison to (22.17*), for it is necessarily one of the terms talked about with the unit lower-order derivative.

If, finally, one can't find any s which satisfies (22.21), that is (22.17) takes the form

$$\prod_{m=1}^{p_i} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot (Df)^{q_i} \tag{22.17**}$$

then this latter product can, for instance, be derivated from the term

$$D\, \frac{\partial^{l_1^{(i)}-1} f}{\partial y^{l_1^{(i)}-1}} \prod_{m=2}^{p_i} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot (Df)^{q_i-1} \tag{22.23}$$

of $y^{(N)}$ by means of the operation D (as the second term) $y^{(N)}$ really contains the product (22.23), because it satisfies the properties 1°—3°, 1° is satisfied; the equality is again, valid, because both q_i and $\Sigma\, l_m^{(i)}$ are decreased with the unit in comparison to (22.17*); 3° is valid too, for $q_i = \Sigma\, k_m^{(i)}$ is decreased, $\Sigma\, k_m^{(i)} = 1$ is increased and $\Sigma\, l_m^{(i)}$ is decreased with the unit, so their sum is decreased with the unit.

The theorem is proved, because its propositions are evidently valid $N = 2, 3, \ldots, 7$, by reason of (22.1)—(22.7)

3. §. Numerical solution by Taylor series

31. Formal evaluation. Let us again, suppose that the function $f(x, y)$ and its derivatives of sufficient high order, with respect to both variables, are continuous in the domain T involving the point (x_0, y_0) and the point $[x_0 + \alpha;\ y_0 + \alpha f(x_0, y_0) + \beta]$ too.

31.1. Lemma. The above-mentioned condition requires that the Taylor expansion of the expression

$$\left.\frac{\partial^r f}{\partial y^r}\right|_{x_0 + \alpha;\ y_0 + \alpha f(x_0, y_0) + \beta} \tag{31.1}$$

can be given in the form

$$\frac{\partial^r f}{\partial y^r} + \alpha D \frac{\partial^r f}{\partial y^r} + \beta \frac{\partial^{r+1} f}{\partial y^{r+1}} + \frac{1}{2!}\left\{\alpha^2 D^2 \frac{\partial^r f}{\partial y^r} + 2\alpha\beta D \frac{\partial^{r+1} f}{\partial y^{r+1}} + \right.$$
$$\left. + \beta^2 \frac{\partial^{r+2} f}{\partial y^{r+2}}\right\} + \ldots + \frac{1}{n!}\sum_{s=0}^{n} \binom{n}{s} \alpha^{n-s} \beta^s D^{n-s} \frac{\partial^{r+s} f}{\partial y^{r+s}} + h_{n+1} \tag{31.2}$$

where the functions are to be taken at the point (x_0, y_0) and the remainder is h_{n+1}.

Proof: The proposition can be easily verified by rearrangement. The expansion only contains the operations D and $\frac{\partial}{\partial y}$ because the coefficient of f in the increment of y equals the increment of x.

The n-th term in the Taylor-series of the expression (31.1) has the form

$$G_n = \frac{1}{n!} \sum_{t=0}^{n} \binom{n}{t} \frac{\partial^{r+n} f}{\partial x^{n-t} \partial y^{r+t}} \alpha^{n-t} (\alpha f + \beta)^t =$$

$$= \frac{1}{n!} \sum_{t=0}^{n} \binom{n}{t} \frac{\partial^{r+n} f}{\partial x^{n-t} \partial y^{r+t}} \alpha^{n-t} \cdot \sum_{v=0}^{t} \binom{t}{v} \alpha^{v} f^{v} \beta^{t-v} =$$

$$= \frac{1}{n!} \sum_{t=0}^{n} \sum_{v=0}^{t} \binom{n}{t} \binom{t}{v} \alpha^{n-(t-v)} \beta^{t-v} f^{v} \frac{\partial^{r+n} f}{\partial x^{n-t} \partial y^{r+t}} \tag{31.3}$$

substituting z for $(t - v)$

$$t - v = z \qquad v = t - z; \tag{31.4}$$

So G_n takes the form

$$G_n = \frac{1}{n!} \sum_{t=0}^{n} \sum_{z=0}^{t} \binom{n}{t} \binom{t}{t-z} \alpha^{n-z} \beta^{z} f^{t-z} \frac{\partial^{r+n} f}{\partial x^{n-t} \partial y^{r+t}} . \tag{31.5}$$

In the following the double sequence of sums are inverted. Substituting s for t by the relations as follows

$$t - z = s; \; t = s + z \tag{31.6}$$

So

$$G_n = \frac{1}{n!} \sum_{z=0}^{n} \alpha^{n-z} \beta^{z} \sum_{t=z}^{n} \binom{n}{t} \binom{t}{t-z} f^{t-z} \frac{\partial^{r+n} f}{\partial x^{n-t} \partial y^{r+t}} =$$

$$= \frac{1}{n!} \sum_{z=0}^{n} \binom{n}{z} \alpha^{n-z} \beta^{z} \sum_{s=0}^{n-z} \frac{\binom{n}{s+z} \binom{s+z}{s}}{\binom{n}{z}} f^{s} \frac{\partial^{r+n} f}{\partial x^{n-(s+z)} \partial y^{r+(s+z)}} . \tag{31.7}$$

Now only that is to be considered that on one hand

$$\frac{\binom{n}{s+z} \binom{s+z}{s}}{\binom{n}{z}} = \frac{\frac{n!}{(s+z)!\,(n-s-z)!} \cdot \frac{(s+z)!}{s!\,z!}}{\frac{n!}{z!\,(n-z)!}} = \frac{n-z!}{s!\,(n-z-s)!} = \binom{n-z}{s} \tag{31.8}$$

and further, on the other hand

$$\sum_{s=0}^{n-z} \binom{n-z}{s} f^s \frac{\partial^{r+z+(n-z)}}{\partial x^{(n-z)-s} \cdot \partial y^{r+z+s}} = D^{n-z} \frac{\partial^{r+z} f}{\partial y^{r+z}}$$

by reason of the powers of D. Taking (31.8) and (31.9) into consideration, in (31.7) our proposition which had to be proved turns out to be right.

32. The terms of the expansion. In the following it will be shown that the power series of the numerical solution contains only terms, and in certain degrees all terms, that also occur in the power series of the exact solution — whether the unmodified or the modified R. C. N. method is used.

For characterising the steps, we intend to introduce the so-called step-distance, —indicating it by h — which serves the independent variable resp. its increment. So the series of the exact solution for the equation (21.1) takes the form

$$\Delta y = y(x_0 + h) - y(x_0) = hf + \frac{h^2}{2!} Df + \frac{h^3}{3!} (D^2 f + f_y Df) +$$

$$+ \ldots + \frac{h^n}{n!} \sum_i A_i^{(n)} \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot \prod_{M=1}^{q_i} D^{K_M^{(i)}} f + R_{n+1} \qquad (32.1)$$

where the general term is denoted in accordance with part 22. and $R_{n+1} = 0(h^{n+1})$.*

Let

$$k = hf(x_0 + \alpha;\ y_0 + \alpha f + \beta) \qquad (32.2)$$

and let us denote with β any linear combination of the terms at least of second order in h, which occurs in the expansion of Δy;

$$\beta = b_1^{(1)} h^2 Df + h^3 [b_1^{(2)} D^2 f + b_2^{(2)} f_y Df] + \ldots +$$

$$+ h^n \sum_i b_i^{(n)} \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{K_M^{(i)}} f + \ldots \qquad (32.3)$$

and with α_1 a similar linear combination, but only those coefficients of its terms can vanish, that contain the factor Df, finally let α the ratio of α_1 and hDf.

* Here and in the following let us suppose that $f(x, y)$ and its derivatives of sufficient high order, with respect to both variables, are continuous in the surrounding of the point (x_0, y_0).

$$\alpha = a_1^{(1)} h + a_2^{(2)} h^2 f_y + h^3 [a_3^{(3)} \cdot f_y^2 + a_4^{(3)} Df_y] + \ldots +$$

$$+ h^{n-1} \sum_i{}^* a_i^{(n)} \prod_{m=1}^{p_i} D^{K_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot \prod_{M=2}^{q_i} D^{K_M^{(i)}} f + \ldots \tag{32.4}$$

where Σ^* only extends to the terms of $y^{(n)}$ that have contained Df, and the second factor-group is indexed, so that $K_1^{(i)} = 1$.

32.1. Lemma. The Taylor expansion of the increment k (32.2) about h, only contains terms that occur in the expansion of the exact solution too.

Proof: We shall apply the lemma 31.1, substituting the expressions (32.4) resp. (32.3) for α resp. β, if specially $r = 0$. Let us consider e. g. the terms formed

$$h \alpha^{n-s} \beta^s D^{n-s} \frac{\partial^s f}{\partial y^s} \tag{32.5}$$

coming from the reduction of the expression

$$h [a_1^{(1)} h + a_2^{(2)} h^2 f_y + \ldots]^{n-s} \cdot [b_1^{(1)} h^2 Df + h^3 (b_1^{(2)} D^2 f +$$

$$+ f_y Df) + \ldots]^s \cdot D^{n-s} \frac{\partial^s f}{\partial y^s} . \tag{32.6}$$

We have to show that the propositions 1°—3° of the theorem 22.1 only hold for the terms above mentioned. It is trivial that these terms may contain only factors formed $D^k \frac{\partial^l f}{\partial y^l}$; similarly the condition $k \geq 0$, $l \geq 0$ is valid, too. The condition $k + l \geq 1$ is also trivial, because it holds for all factor of α and β, and if $n \geq 1$ for $D^{n-s} \frac{\partial^s f}{\partial y^s}$ too. 2° is also valid. According to (32.3) the equality

$$q_\beta = 1 + \sum_m{}'_\beta l_m - p_\beta \tag{32.7}$$

holds for any term of β. According to (32.4) the equality

$$q_\alpha = \sum_m{}'_\alpha l_m - p_\alpha \tag{32.8}$$

holds for any term of α. (Divided by Df).

So the equalities

$$q_{II} = s\, q_\beta = s + \sum_m{}'_{II} l_m - p_{II} \quad (s \geqq 1) \tag{32.9}$$

hold for any term of the second factor (coming from the raising tof s-th to power).

On the other hand the equation

$$q_I = \sum_m{}' r\, l_m - p_I \tag{32.10}$$

holds for any expanded term of the first factor. For any term of their product holds

$$q_r = \begin{cases} \sum_m{}'_r\, l_m - p_r, & \text{if } s = 0 \\ s + \sum_m{}_r\, l_m - p_r, & \text{if } s > 0 \end{cases} \tag{32.11}$$

Finally, the factor $D^{n-s} \dfrac{\partial^s f}{\partial y^s}$ increases q with the unit compared with q_r if $s = 0$, $\Sigma\, l_m$ and p remain unchanged, thus really

$$q = 1 + \sum_m{}' l_m - p.$$

If $s > 0$, q remains unchanged, that is equal to q_r but $\Sigma\, l_m = \Sigma_r\, l_m + s$ and $p = p_r + 1$, and so again

$$q = s + \sum_m{}' l_m - s - (p - 1) = 1 + \sum_m{}' l_m - p\,.$$

The validity of 3° can be shown in the same way; for any term of β containing h^b

$$\sum_m{}'_\beta\, (l_m + k_m) + \sum_M{}'_\beta\, K_M = b - 1 \tag{32.12}$$

thus for any term of β^s containing h^{sb}

$$\sum_m{}'_{II}\, (l_m + k_m) + \sum_M{}'_{II}\, K_M = sb - s\,. \tag{32.13}$$

Likewise for any term of α containing h^a

$$\sum_m{}'_a\, (k_m + l_m) + \sum_M{}_a\, K_M = a - 1\,, \tag{32.14}$$

thus for any term of α^{n-1} containing $h^{(n-s)a}$

$$\sum_m{}'_I\, (k_m + l_m) + \sum_M{}'_I\, K_M = (n - s)\, a - (n - s) \tag{32.15}$$

and so, for any term of $\alpha^{n-s}\beta^s$ containing $h^{sb+(n-s)a}$

$$\sum_m{}'_r (k_m + l_m) + \sum_M{}'_r K_M = sb - s + (n-s)\,a - (n-s) \qquad (32.16)$$

The result of the multiplication by $D^{n-s}\dfrac{\partial^s f}{\partial y^s}$ is

$$\sum_m{}'_r (k_m + l_m) = \sum_m{}' (k_m + l_m)\,;\ \sum_M{}'_r K_M + n = \sum_M{}' K_M \qquad (32.17)$$

if $s = 0$

$$\sum_m{}'_r k_m + n - s = \sum_m{}' k_m\,;\ \sum_m{}'_r l_m + s = \sum_m{}' l_m\,;$$

$$\sum_m{}'_r K_M = \sum_m{}' K_M\,;\ \sum_m{}'_r (k_m + l_m) + n = \sum_m{}' (k_m + l_m)\,;$$

if $s > 0$ (32.18)

and so for any term of (32.6) containing the factor $h^{sb+(n-s)a+1}$

$$\sum_m{}' (k_m + l_m) + \sum_M{}' K_M = n + sb - s + (n-s)\,a - (n-s) = sb + (n-s)\,a\,. \qquad (32.19)$$

This completes the proof for the theorem.

32.2. Lemma is more difficult to verify: The n-th partial sum of the Taylor expansion of (32.2) contains every term which occurs in its exact increment, if $a_1^{(1)} \neq 0$ in the expansion of α (32.4) and $b_i^{(k)} \neq 0$ in the expansion of β (32.3), supposing that $k \leq n-1$.*

Proof: The proposition will be verified based upon the lemma 31.1 using the formulas (32.4) and (32.3) and the conditions of the coefficients related to these formulas. Let us now consider an optional term in the expansion of Δy, if $s \leq n$

$$h^s \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \cdot \prod_{M=1}^{i} D^{K_M^{(i)}} f = k_i^{(s)} \qquad (32.20)$$

where then

$$1^\circ.\ k_m^{(i)} \geq 0\,;\ l_m^{(i)} \geq 0\,;\ K_M^{(i)} \geq 1\,;$$

$$2^\circ.\ q_i = 1 + \sum_{m=1}^{p_i^{(s)}}{}' l_m^{(i)} - p_i\,;\ q_i \geq 1\,;$$

$$3^\circ.\ \sum_{m=1}^{p_i^{(s)}}{}' (k_m^{(i)} + l_m^{(i)}) + \sum_{M=1}^{i^{(s)}}{}' K_M^{(i)} = s - 1\,. \qquad (32.21)$$

* That is, if every term really occurs in the expansion of β till the $(n-1)$-th power of h inclusive.

In the following it will be verified that the Taylor-expansion of k has also terms formed like (32.20) in case of $s \leq n$; e. g. such a term is to be found regrouped in the term of the form

$$h \frac{1}{(k_1^{(i)}+l_1^{(i)})!} \binom{k_1^{(i)}+l_1^{(i)}}{l_1^{(i)}} \cdot \alpha^{k_1^{(i)}} \beta^{l_1^{(i)}} D^{k_1^{(i)}} \frac{\partial^{l_1^{(i)}} f}{\partial y^{l_1^{(i)}}} \tag{32.23}$$

by proceeding in ascending integral powers of h.

Namely let us consider the term $a_1^{(1)}h$ from the series (32.4) of α. We shall verify that the other factors of (32.20) are dividable into $l_1^{(i)}$ groups that occur in the expansion of β (32.3) (e. g. in the special case when $l_1^{(i)} = 0$ the first factor group of (32.20) becomes identically equal to the unit). First we have to see that in (32.20) there exists at the most $l_1^{(i)}$ factors, even apart from the factor $D^{k_1^{(i)}} \frac{\partial^{l_1^{(i)}} f}{\partial y^{l_1^{(i)}}}$, namely also

$$h^{s-1-l_1^{(i)}-(k_1^{(i)}+l_i^{(i)})} \cdot \prod_{m=2}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{K_M} f \tag{32.20*}$$

can be divided into $l_1^{(i)}$ factor groups. But it can be immediately seen that the second factor group alone has at the most $l_1^{(i)}$ factors as $q_i \geq l_1^{(i)}$ by reason of the condition 2° in (32.21). A possibility for the division will be shown as follows, depending on the cases

a) $p_i - 1 < l_1^{(i)}$ resp. b) $p_i - 1 \geq l_1^{(i)}$.

In case a) we connect to each factor as many factors of the second factor group (and from h) as would comply with the requirements 1°—3° of the theorem 21,1 — so the expansion of Δy and the expansion of β will contain a so formed term, if the exponent h is not higher than $(n-1)$. Then we complete each of these from one among the factors of the second group to $l_1^{(i)}$ factors — so that these factors will also comply with the requirements 1°—3° of 21.1 and will occur in the expansion of β. We have only to show that all the factors of (32.20*) are only once used, moreover the sum of the exponents of h is equal to the exponent in (32.20*). It is easy to show, as by reason of the requirement 2° we have to exactly connect

$$1 + l_m^{(i)} - 1 = l_m^{(i)} \tag{32.24}$$

factors from the second factor group to $D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}}$ thus

$$\sum_{m=2}^{p_i} l_m^{(i)} \tag{32.25}$$

factors to the $(p_i - 1)$ groups in all, and so we still have

$$q_i - \sum_{m=2}^{p_i} l_m^{(i)} = 1 + l_1^{(i)} - p_i = l_1^{(i)} - [p_i - 1] \tag{32.26}$$

factors from the second factor group, just as many groups as we have to construct.

The exponent in the group

$$D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}}}{\partial y^{l_m^{(i)}}} \cdot \prod_{M=\gamma+1}^{\gamma+l_m^{(i)}} D^{K_M^{(i)}} f$$

(the required exponent of h by reason of 3°)

$$e_m = k_m^{(i)} + l_m^{(i)} + \sum_{M=\gamma+1}^{\gamma+l_m^{(i)}} K_M^{(i)} - 1 \tag{32.27}$$

and so the sum of the exponents

$$\sum_{m=2}^{p_i} (k_m^{(i)} + l_m^{(i)}) + \sum_{M=1}^{q_i} K_M - l_1^{(i)} = \sum_{m=1}^{p_i} (k_m^{(i)} + l_m^{(i)}) + \sum_{M=1}^{r_i} K_M - 2\, l_1^{(i)} - k_1^{(i)} =$$
$$= s - 1 - l_1^{(i)} - (k_1^{(i)} + l_1^{(i)}) \tag{32.28}$$

is equal to the exponent in (32.20*).

In case b) the procedure is similar, but here exactly one factor will be attached to every group but $p_i - l_1^{(i)}$ to the last group of the first factor group and so many factors from the second that every group would comply with the requirements 1°—2°. Similarly as in case a) one can see that (32.20*) can be divided exactly into $l_1^{(i)}$ groups, which comply with the requirements 1°—3° of the theorem 21.1 and which occur in Δy; one can immediately see that every exponent of h in every group in (32.20*) can not be higher than $(n - 1)$, if $s \leq n$, thus these terms occur in β and their product (32.20*) occurs in $\beta^{l_1^{(i)}}$. This completes the proof of our theorem — also considering (32.23).

The lemma which follows can be similarly verified, where α and β have the same significance as in (32.4) and (32.3).

32.3. Lemma. Let us consider the power series of

$$k^{(r)} = h \frac{\partial^r f}{\partial y^r} [x_0 + \alpha\,; y_0 + \alpha f + \beta]\,; \ (r = 1, 2, \ldots) \tag{32.29}$$

about h. In this case the expressions formed

$$[k^{(r)}]^s \cdot \beta^r \qquad (s = 1, 2, \ldots ; \; r = 1, 2, \ldots) \tag{32.30}$$

have an expansion β type, and the expression formed

$$[k^{(r)}]^s \cdot \beta^{r-1} \qquad (s = 1, 2, \ldots ; \; r = 1, 2, \ldots) \tag{32.31}$$

have an expansion α type.

4. §. On the improvement of the R. C. N. method

In the following it will be verified that both the unmodified and the modified R. C. N. method can be improved by optionally increasing the degree. The scheme of the unmodified Runge—Kutta—Nyström method is — using the greatest parameter number — as follows

$$\begin{aligned}
k_1 &= hf(x_0 ;\, y_0) \\
k_2 &= hf[x_0 + c_1^{(1)} h ;\; y_0 + c_1^{(1)} k_1] \\
k_3 &= hf[x_0 + c_1^{(2)} h ;\; y_0 + c_1^{(2)} k_1 + c_2^{(2)} (k_2 - k_1)] \\
&\vdots \\
k_n &= hf[x_0 + c_1^{(n)} h ;\; y_0 + c_1^{(n)} k_1 + c_2^{(n)} (k_2 - k_1) + \ldots + \\
&\quad + c_n^{(n)} (k_{n-1} - k_1)] ,
\end{aligned} \tag{4.1}$$

$$k = c_1 k_1 + c_2 k_2 + \ldots + c_n k_n \tag{4.2}$$

Where we evaluate the constants $c_i^{(k)}$; c_i : in all $n + \dfrac{n(n+1)}{2} = \dfrac{n(n+3)}{2}$ so, that the expansion of the k approximate "steps" correspond with the expansion of Δy till the highest degree inclusive as far as possible. According to the lemmas 32.1—32.2, it is to be noted that k contains all terms which occur in the n-th partial sum of the expansion of Δy, nevertheless in case $n > 4$ the maximum parameter $\dfrac{n(n+3)}{2}$ is not sufficient to make the equality of the coefficients sure in all such terms of the approximate and exact solution.

The maximum parameter number can be speedily increased by reason of the lemma 32.3, so that we include the evaluation of the $k_i^{(r)}$ values in the unmodified R. C. N. scheme, as follows e. g.:

$$
\begin{aligned}
k_1 \quad &= hf(x_0, y_0) \\
k_1^{(1)} &= hf_y[x_0 + c_1^{(1)} h; \; y_0 + c_1^{(1)} k_1] \\
k_2 \quad &= hf[x_0 + c_1^{(2)} h + d_1^{(2)} k_1^{(1)} h + d_1^{(3)} k_1^{(1)2} h + \ldots ; \; y_0 + c_1^{(2)} k_1 + \\
&\quad + d_1^{(2)} k_1 k_1^{(1)} + d_1^{(3)} k_1 k_1^{(1)2} + \ldots] \\
k_3 \quad &= hf[x_0 + c_1^{(3)} h + d_1^{(3)} k_1^{(1)} h + d_2^{(3)} k_1^{(1)2} h + \ldots ; \; y_0 + c_1^{(3)} k_1 + \\
&\quad + c_2^{(3)} (k_2 - k_1) + d_1^{(3)} k_1^{(1)} h + d_2^{(3)} k_1^{(1)2} h + \ldots + D_1^{(3)} k_1^{(1)} (k_2 - k_1) + \\
&\quad + D_2^{(3)} k_1^{(1)2} (k_2 - k_1) + \ldots] \\
&\vdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.3) \\
k_1^{(2)} &= hf_{yy}[x_0 + c_1^{(s)} h + d_1^{(s)} k_1^{(1)} h + \ldots ; \; y_0 + c_1^{(s)} k_1 + c_2^{(s)} (k_2 - k_1) + \ldots \\
&\quad + d_1^{(s)} k_1^{(1)} k_1 + d_2^{(s)} k_1^{(1)2} k_1 + \ldots + D_1^{(s)} k_1^{(1)} (k_2 - k_1) + \ldots] \\
k_{s+1} &= hf[x_0 + c_1^{(s+1)} h + d_1^{(s+1)} k_1^{(1)} h + \ldots + e^{(s+1)} k_1^{(2)} (k_2 - k_1) + \\
&\quad + \ldots ; \; y_0 + \ldots + e_1^{(s+1)} k_1^{(2)} \cdot k_1 (k_2 - k_1) + \ldots + \\
&\quad + E_1^{(s+1)} k_1^{(2)} (k_2 - k_1)^2 + \ldots] ;
\end{aligned}
$$

etc.; finally

$$
\begin{aligned}
k = {} & C_1 k_1 + \ldots + C_n k_n + C_{121}^{(1)} k_1^{(1)} (k_2 - k_1) + C_{131}^{(1)} k_1^{(1)} (k_3 - k_1) + \\
& + \ldots + C_{122}^{(1)} k_1^{(1)2} (k_2 - k_1) + \ldots + C_{221}^{(1)} k_2^{(1)} (k_2 - k_1) + \ldots + \\
& + C_{121}^{(2)} k_1^{(2)} (k_2 - k_1)^2 + C_{132}^{(2)} k_1^{(2)} (k_3 - k_1)(k_2 - k_1) + \ldots \qquad (4.4)
\end{aligned}
$$

It should be noted that the appearance that in the given way an infinity of parameters can be included even by finite degree, is only illusory, because the individual parameter will effect only after a partial sum of a certain degree. Even though, including the magnitudes of $k_i^{(r)}$ type in the scheme — one can read it in formulas /4,3/ — /4,4/ — on one hand the number of parameters increases very rapidly, but on the other hand — we shall see — that the system of equations — which requires the same coefficients of the individual members of K_t and of Δy — for the evaluation of parameters falls into several systems, they can be solved independently, but only after including the magnitudes $k_i^{(r)}$ — because the parameters in relating of the new type members appeares only on a certain degree in the Taylor-expansion of K_t.

We will show, as follows, that both the unmodified and the modified R. C. N. method can be improved to a wished degree. We should mention in advance that the theorem is not pronounced in the most exact form, that is : we don't say that one needs exactly so many steps of $k_i^{(r)}$ type for a given degree, which must be calculated in such an order of succession. The reason for this is : one can easily evaluate that making a certain number of auxiliary steps of $k_i^{(r)}$ type in a given order up to which exponent of h in the approximate increment of k_t will be found all terms occurring in the exact increment ;

how many independent parameters will be included in the method till the given degree, and whether their number is not smaller than the number of all terms of several type which occur in the exact solution till this degree — that is whether, supposing independent equations it is possible that the approximate and the exact increment will be equated by choosing suitable parameters. But in general it is nearly impossible to prove that the equations are really independent — and it may happen that making the same number of steps in several orders of succession the system of equations will sometimes be independent and sometimes not. As it is convenient to include in practice more than the required parameters in case of high orders of degree to make the system of equations manageable, and in eventual dependency cases with further steps to increase the number of occurring parameters, therefore it is sufficient to give the

4.5. theorem of fundamental importance by the value of which the R. C. N. method of optional degree can be given by means of sufficient $k_i^{(r)}$ numerical steps.

Before demonstrating of a special sharpening of the theorem 4.3, we will verify two important theorems and two lemmas.

4.1. theorem. The power series of the approximate increment of K_n, even of the k_n auxiliary step, in the unmodified R. C. N. method given by the formulas (4.1)—(4.2), contains all those, and only all those terms, inclusive, the n-th degree which occur in the power series of Δy (the exact increment) Proof: The power series of the increment k_2 has the form

$$k_2 = hf + h^2 C_1^{(1)} Df + \frac{h^3}{2!} C_1^{(1)^2} D^2 f + \ldots + \frac{h^{l+1}}{l!} C_1^{(1)^l} D^l f + \ldots \qquad (4.5)$$

by reason of the formula (31.2). Hence it can immediately be seen that the power series of $(k_2 - k_1)$ is of β type, in which all the terms, inclusive the quadratic ones, are contained which occur in the expansion of $(\Delta y - hf)$, too. By reason of the lemmas 32.1 and 32.2, k_3 contains only such, and inclusive the third partial sum, all such terms which occur in the expansion of Δy. The theorem will be verified by means of general induction. We suppose that the theorem is already verified in case of $n = 1; 2; \ldots s$:. Consequently, the sum

$$C_2^{(s)}(k_2 - k_1) + C_3^{(s)}(k_3 - k_1) + \ldots + C_s^{(s)}(k_s - k_1)$$

has an expansion of β type, that contains, inclusive, the terms of s-th degree all terms which occur in the power series of $(\Delta y - hf)$. So by reason of the lemmas 31.1—31.2 the propositions hold for k_{s+1} too. This completes the proof of the theorem.

4.2. Lemma. The power series of the difference between the approximate increment K_t in the R. C. N. method modified with the formulas (4.3)—(4.4) and between hf, is of β type.

This lemma follows immediately by means of similar mathematical induction with full knowledge of the lemmas 31.1 and 32.3.

4.3. Lemma. The coefficients of the $h^l D^{l-1} f$ formed terms in the approximate increment of K_t in the modified R. C. N. method given by the formulas (4.3)—(4.4) contains only the parameters $c_1^{(l)} (l = 1, 2, \dots n)$ and $C_l (l = 1, 2, \dots n)$, these parameters are absolutely independent of one another.

Proof: The first part of the proposition follows from the formula (31.2). It can be immediately seen by means of mathematical induction that the coefficient of the $h^l D^{l-1} f$ formed term in the power series of Δy is exactly $\frac{1}{l!}$, and so by reason of (31.2), the system of equations

$$C_1 + C_2 + \dots + C_n = 1$$

$$C_2 \cdot c_1^{(2)} + C_3 c_1^{(3)} + \dots = \frac{1}{2}$$

$$C_2 \cdot c_1^{(2)2} + C_3 c_1^{(3)2} + \dots = \frac{1}{3}$$

$$\vdots$$

$$C_2 [c_1^{(2)}]^r + \dots + C_n [c_1^{(n)}]^r = \frac{1}{r+1} \tag{4.6}$$

By reason of the well-known properties of the Van-der-Monde matrix also follows the complete independency. The direct consequence of this lemma is the

4.1. Corollary. In the modified R. C. N. method of n-th degree there occur at least $\left[\frac{n+1}{2}\right]$ auxiliary steps of k_s type (used in the unmodified method).

The k_1 auxiliary step gives one, all the others give two independent parameters by which the equations (4.6) can be satisfied. If the method is of n-th degree the number of the equations is exactly n.

4.4. Theorem. Let

$$k = hf[x_0 + \alpha_0 ; y_0 + \alpha_0 f + \beta_0] \tag{4.7}$$

where α_0 is an expression of optional α type, and β_0 is an expression of such β type the expansion of which contains all terms inclusive, the $n \geq 2$-th degree occurring in $(\Delta y - hf)$. Let moreover $k_i^{(1)}$ an optional auxiliary step of the modified R. C. N. method. The expansion of $k_i^{(1)}$ starts as follows

$$k_i^{(1)} = h f_y + \ldots \tag{4.8}$$

In this case

$$K = D(k - hf) + E k_i^{(1)}(k - hf) + F k_i^{(1)^2}(k - hf) + \ldots + G k_i^{(1)^s}(k - hf) + \ldots \tag{4.9}$$

is an expression of β type, the series of which contains all terms inclusive, the $(n+2)$-th degree occurring in $(\Delta y - hf)$.

Proof: By reason of the lemmas 32.1 and 32.3 it follows directly that K is really an expression of β type. On the other hand, by reason of the lemma 32.2, it follows that every term of $(\Delta y - hf)$ inclusive the $(n+1)$-th degree occurs. Let us now consider an optional term of $(n+2)$-th degree of Δy e. g. the term

$$h^{n-2} \cdot \prod_{m=1}^{p_i} D^{k_m^{(i)}} \frac{\partial^{l_m^{(i)}} f}{\partial y^{l_m^{(i)}}} \prod_{M=1}^{q_i} D^{K_M^{(i)}} \tag{4.10}$$

If $l_1^{(i)} \geq 2$, than we can verify, in the same way as in the proof of the lemma 32.2, that in K occur a term like (4.10). Then the factor group reduced according to (32.20*) is to be divided at least into two factors, so that no factor can be of higher degree than the n-th. These factors necessarily occur in the expansion of $\beta^{l_1^{(i)}}$ [See formula (3.23)]. If $l_1^{(i)} = 1$, but $k_1^{(i)} \geq 1$, then (32.20*) is a single factor in the expansion of β, but it occurs surely, because (32.20*) is at most of n-th degree. As the indexing was arbitrary, we verified the occurrence of (4.10) in K in all those cases if one could find such $1 \leq m \leq p_i$, to which

$$k_m^{(i)} + l_m^{(i)} \geq 2 \tag{4.11}$$

holds. After this we have only to verify the occurrence of

$$h^{n-2} (f_y)^{p_i} D^{n-1-p_i} f, \tag{4.12}$$

resp.,

$$h^{n-2} D^{n-1} f \tag{4.13}$$

in K, because the condition (4.11) being out of question, Δy may contain the terms formed like (4.12), resp. (4.13) according to the theorem 2.1. But the terms formed like (4.12) will surely occur in the term

$$[k_i^{(1)}]^{p_i} (k - hf) \tag{4.14}$$

of K, because $(k - hf)$ contains every term of $(\Delta y - hf)$ at the most of $(n+1)$-th degree, so specially $h^{n+2-p_i}D^{n+1-p_i}f$ $(p_i = 1, 2, \ldots)$ too. One can immediately read from the formula (31.1) that in case of $\alpha_0=0$ K contains a term like (4.13). This completes the proof of our theorem.

4.2. Corollary. The power series of the approximate increment of K_t, given by the formulas (4.3)—(4.4), contains every term of Δy, inclusive the n-th degree, if K_t contains at least $\left[\frac{n+1}{2}\right]$ auxiliary steps of k_i, and at least one of $k_i^{(1)}$ type. The demonstration can be made by mathematical induction — like that of the theorem 4.1, but using the theorem 4.4. Now the theorem 4.5 will be proved as follows.

The demonstration is based on the fact that the unmodified R. C. N. method given by the formulas (4.1)—(4.2) is the special case of the modified R. C. N. method given by the formulas (4.3)—(4.4), in that meaning that if certain parameters vanish in the latter one, we get the former. Consequently, if the theorem 4.1 holds for the unmodified R. C. N. method, it rather holds for the modified one, for the system of equations for parameters has surely that special solution-system, which we found presumed — in the special case. Now the so-called Eulerian broken-line method is a special case of the unmodified R. C. N. method — that is, if the sequence $C_1^{(1)}, C_1^{(2)}, \ldots; C_1^{(n)}$ is chosen to be the multiple of $\frac{h}{n}$ successively, $C_1^{(k)}$ $(i = 2, 3, \ldots k; \; k = 2, 3, \ldots n)$ and C_i are chosen to be equal to $\frac{h}{n}$. From this it follows that the theorem 4.1 holds for the unmodified R. C. N. method: if this statement were not true, that is, there would exist such a degree denoted by ν_0, for which the system of equations (considering it only until the ν_0-th power) would possess a contradict for any unmodified R. C. N. method with the auxiliary steps of a number n, no matter how great, then consequently we could find always, choosing the step-number n and the related parameter-system no matter how a set of the equations (21.1) which satisfies the strict conditions related to the function $f(x, y)$, and making a step by means of the chosen R. C. N. method, the error would have a least limit $H(\nu_0; K; h)$: H is independent of n. (That is, choosing any parameter group, there is a difference between the derivatives not of higher order than ν_0: e. g. the ν_1-th, in the expansion of the approximate and of the exact solution, in consequence of the assumed contradiction. Choosing a function the derivatives of which are small in absolute value, except the ν_1-th, and constructing the differential equation of (21.1) type, according to the assumptions, the error of this equation will be over the limit H.) But this leads to contradiction, because the Eulerian broken-line method, as a special R. C. N. method assures, in case of sufficient high step number, the existence of such a parameter group, by which the error in the set of equations (21.1) satisfying the above-mentioned conditions to $f(x, y)$, assures uniformly an arbitrarily small e.g. $\frac{H}{2}$ limit.

This completes the proof of our theorem; but we wish to mention that in the third publication of this theme the theorem 4.5 will be proved in a more strict form.

5. §. Examples

In the followings, we want to point out the practical "planning" of certain formulas of the modified R. C. N. method, and the practical method for the evaluation of the required parameter system. Moreover, we make a comparison between the obtained formula and the usual R. C. N. method, by means of a numerical example. It is to be emphasized that the formula (31.2) of the lemma 31.1 and the formula (32.6) were not given only for proving the mentioned theorems, but by using this formulas, the expansion serving for evaluation of the parameters can be made in an almost automatic way in practice too.

51. *Formulas of fifth degree in five points.* We give some simpler methods of fifth, resp., sixth degree, by means of parameters as follows. We will not enter into lengthy, tedious numerical evaluation details but we only notice, that it is advisable to outline the equations. This goes to show how the system of equations can be divided into groups.

The formulas of fifth degree in five points:

a)

$$k_1 = hf(x_0, y_0) \tag{51.1}$$

$$k_2 = hf\left(x_0 + \frac{4}{5}h; y_0 + \frac{4}{5}k_1\right) \tag{51.2}$$

$$k_1' = hf_y[x_0; y_0 + 3(k_2 - k_1)] \tag{51.3}$$

$$k_3 = hf\left[x_0 + \frac{1}{3}h; y_0 + \frac{1}{24}k_1 + \frac{7}{24}k_2\right] \tag{51.4}$$

$$k_4 = hf\left[x_0 + h; y_0 - \frac{37}{8}k_1 + \frac{27}{56}k_2 + \frac{36}{7}k_3 - \frac{1}{2}k_1'(k_2 - k_1)\right] \tag{51.5}$$

$$\begin{aligned} k = {} & \frac{5}{48}k_1 + \frac{125}{336}k_2 + \frac{27}{56}k_3 + \frac{1}{24}k_4 - \frac{29}{336}k_1'(k_2 - k_1) + \\ & + \frac{3}{28}k_1'(k_3 - k_1) - \frac{25}{336}k_1'^2(k_2 - k_1) + \frac{3}{28}k_1'^2(k_3 - k_1) - \\ & - \frac{1}{48}k_1'^3(k_2 - k_1) \end{aligned} \tag{51.6}$$

b)

$$k_1 = hf(x_0, y_0) \tag{51.7}$$

$$k_2 = hf[x_0 + a_2 h; y_0 + a_2 k_1] \tag{51.8}$$

$$k_3' = hf_y [x_0 + a_3 h; y_0 + a_3 k_1] \tag{51.9}$$

$$k_4' = hf_y [x_0 + a_4 h; y_0 + a_{41} k_1 + a_{42} k_2] \tag{51.10}$$

$$k_5 = hf[x_0 + a_5 h + \beta_{51} h k_3'; y_0 + a_{51} k_1 + a_{52} k_2 + \beta_{51} k_1 k_3'] \tag{51.11}$$

$$\begin{aligned} k = {} & R_1 k_1 + R_2 k_2 + R_5 k_5 + R_3 (k_2 - k_1) k_3' + R_4 (k_2 - k_1) k_1' + \\ & + R_6 (k_5 - k_1) k_3' + R_7 (k_5 - k_1) k_4' + R_8 (k_2 - k_1) k_3'^2 + \\ & + R_{10} (k_2 - k_1) k_4' + R_{11} (k_5 - k_1) k_3'^2 + R_{13} (k_5 - k_1) k_4'^2 + \\ & + R_{14} (k_2 - k_1) k_3'^3 ; \end{aligned} \tag{51.12}$$

Where

$a_2 = 0.35505103$	$R_1 = 0,11111111$
$a_3 = \frac{1}{3} = 0,33333333$	$R_2 = 0,51248580$
$a_4 = 0.6666667$	$R_3 = 0.46701860$
$a_5 = 0.84494897$	$R_4 = 0,33113104$
$a_{41} = 0,91990313$	$R_5 = 0,37640309$
$a_{42} = -0,25323646$	$R_6 = 0,21161946$
$a_{51} = 0.82731251$	$R_7 = -0,46810571$
$a_{52} = 0,01763646$	$R_8 = R_{10} = -0,19873047$
$\beta_{51} = 0,25938934$	$R_{11} = 0,04907363$
	$R_{13} = 0,16725358$
	$R_{14} = -0,13838645$

In both case, we planned the formula as follows. By reason of the formulas (22.1—7) we evaluated the minimum parameter-number in case of formula of fifth order (as many as the number of terms in the first five derivatives, that is : $1 + 1 + 2 + 4 + 8 = 16$). Moreover, by reason of formulas (32.20)—(32.31) we evaluated, that at least, how many steps of $k_i = k_i^{(0)}$ type are required to have all terms occurring in Δy till the fifth degree. As there exist at least 3, the practicable solution is 3 of $k_i^{(0)}$ and so many $k_i^{(1)}$ steps which can assure the occurring of 16 parameters. So we got the formula b) (because we had to include at least two $k_i^{(1)}$). First we included all possible parameters — which can theoretically occur considering the first five powers of the sum —, and when the equation was written down, we chose arbitrarily so many and such parameters as to solve the equation in the simplest way. The procedure is the same in case of a), resp., generally.

52. *A numerical example.* We show, for comparison, in case of the equation

$$y' = y - \frac{2x}{y}; \quad y(0) = 1$$

a) by $h = 1$ step

the several solutions, evaluated by means of

1° R. C. N. method of second degree.

$$y(1) \cong 1{,}833$$

2° the well-known R. C. N. method of fourth degree

$$y(1) \cong 1{,}772$$

3° the method of fifth degree shown in a)

$$y(1) \cong 1{,}744$$

4° the method of fifth degree shown in b)

$$y(1) \cong 1{,}735$$

and 5° integration. The exact solution is $y = \sqrt{1+2x}$

$$y(1) \cong 1{,}732$$

References

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Summary

We prove that approaching the solution of the equation

$$y' = f(x, y)$$

by means of the Runge—Kutta method, this method can be of optional order of degree, supposing that we have evaluated as many auxiliary increments, as is sufficient. We show that using the partial derivatives of function f, in respect to the dependent variable at the evaluation of the auxiliary increments, the method can again be of optional order of degree, but the number of the points rapidly decreases at which the function-value, resp., its derivatives are to be evaluated. We illustrate the above-mentioned by examples.

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