THE INFLUENCE OF THE LAYOUT AND DYNAMIC CHARACTERISTICS OF SERVOMECHANISMS ON THE TEMPERATURE CONDITIONS OF SEPARATELY EXCITED D.C. SERVOMOTORS USED IN THE SERVOMECHANISMS

By

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Nomenclature

In general the time functions are denoted with small letters, the Fourier or Laplace-transforms with capitals.

u terminal voltage of the d. c. motor
U transform of u
u_0 steady-state value of the terminal voltage of the d. c. motor
W heat generated in the armature of the motor during the transient phenomenon
Θ inertia of the motor
v speed of the motor
$\mathfrak{v}$ transform of v
v_∞ steady-state value of v
m load torque of the d. c. motor
M transform of m
i armature current of the d. c. motor
I transform of i
i_∞ steady-state value of i
T_m electromechanical time constant of the motor (See Eq. 42c)
T_v time constant of the armature of the d. c. motor (See Eq. 42b)
R_m resistance of the armature of the d. c. motor
R_a additional resistance in the armature circuit
R total resistance of the armature circuit
Φ main flux of the d. c. motor
K_m gain of the d. c. motor (See Eq. 42a)
Y_m transfer function of the d. c. motor relating to the terminal voltage (See Eq. 44)
Y_t transfer function of the d. c. motor relating to the load torque (See Eq. 45)
Y_v transfer function of the internal feedback used in the servomechanism (See Fig. 3c)
Y_r transfer function of the gear train (See Eq. 6)
Y resultant transfer function of the open-loop of the servomechanism
Y^* resultant transfer function of the closed-loop of the servomechanism for which holds

$$Y^* = \frac{Y}{1 + Y}$$

I. Introduction

In high quality servomechanisms of larger output a frequently used final control element is the separately excited direct current servomotor. The operating conditions of a servomotor used as a final control element greatly differs from the normal operating conditions of electric motors to which the characteristic data of the motors (rated voltage, current, speed, output, losses, etc.) are related. In cases where motors rated for normal operating conditions are to be used in servomechanisms, it must be carefully examined, under what conditions the motors can be used as servomotors. Furthermore it has to be determined how a motor suitable for a certain servomechanism can be selected on the basis of its rated data.

If a motor is to be used in a servomechanism it is necessary to known in the first place the values of the time constants. Taking these into consideration the general layout of the control system should be selected and the servomechanism be designed.

In addition to this there is need of a method for giving information on the size (normal rated output) of the motor to be selected. It must be remembered that these motors are constantly in transient operation, and thus the load torque is not characteristic for the temperature conditions of the motor.

Among the losses arising in the transient operation of the servomotor only the copper losses differ considerably from the losses in steady-state conditions. The iron losses depend exclusively on the speed and the motor must not even temporarily be run at a speed considerably higher than its rated speed. Therefore the difference between the temperature rise of servomotors and that of motors in normal operation must be determined on the basis of copper losses.

It is a well-known fact that if a voltage u_0 is suddenly applied to a single running separately excited d. c. motor from a voltage source having negligible internal resistance, the amount of heat generated in the rotor is equal to the kinetic energy accumulated in the rotating parts.

$$W = \frac{1}{2}\,\theta\,\nu_{\infty}^{2}\,. \tag{1}$$

In expression (1) it is assumed that the inductivity L of the armature circuit is zero and the motor is not loaded with any external torque.

It can be proved that taking the inductivity L of the armature into consideration and in case of a constant load torque m, the heat arising during the starting period is

$$W = \frac{1}{2}\,\theta\nu_{\infty}^{2} + \frac{1}{2}\,i_{\infty}^{2}\,L + 2\,m\,\nu_{\infty}\,T_{m} \tag{2}$$

in addition to the heating power assumed constant, which causes in the armature a resistance R_m by the steady-state current i_∞ corresponding to the external load torque m in steady-state. (See App. II.)

If thus a unit-step voltage is switched on a separately excited d. c. motor, the heat generated during the starting period consists of three parts in addition to the steady-state copper loss $i_\infty^2 R_m$. One part of this heat is the kinetic energy accumulated in the rotating parts, a second part is the magnetic energy of the rotor, a third part is equal to the mechanical energy consumed by the driven object during the time $2T_m$.

To estimate the order of magnitude of the individual terms of the equation, let us introduce the relative voltage drop γ connecting voltage u_0 and current i_∞.

$$\gamma = \frac{i_\infty R_m}{u_0} = \frac{i_\infty}{i_{rz}} = \frac{m}{m_{rz}} \tag{3}$$

where i_{rz} and m_{rz} are the short circuit (starting) current and torque respectively corresponding to voltage u_0.

So after some simple conversions not detailed herewith, in place of Eq. (2), the following can be obtained for the heat generated during the starting period :

$$W = \frac{1}{8}\frac{u_0^2}{R_m} T_m \left[(1-\gamma)^2 + \frac{T_v}{T_m}\gamma^2 + (4\gamma - 4\gamma^2)\right]. \tag{4}$$

The sequence of the three terms between square-brackets in the last expression corresponds to that of the terms in Eq. (2).

If the relative ohmic voltage-drop is sufficiently small, the second and third term of the expression between the square brackets are small, as compared to the first term. Therefore the heat generated is with good approximation equal to the kinetic energy accumulated in the rotor

$$\frac{1}{2}\theta \nu_\infty^2 = \frac{1}{2}\frac{u_0^2}{R_m} T_m (1-\gamma)^2 \tag{5}$$

II. The amount of heat generated under transient conditions in the armature circuit of motors used in servomechanisms

If the d. c. motor constitutes an element of a servomechanism, the copper losses under transient conditions depend on the characteristics of the reference input. Without giving up the claim for generality, the block diagram of the control system may be assumed to be in accordance to Fig. 1. A possible feedback varying with frequency and spanning the motor (transfer element Y_m) can be incorporated in Y_v, those feedbacks, however, not spanning the motor can be converted into series elements according to the known rules of the block algebra, and can thus be incorporated in Y_e.

Y_r is the transfer function of the gear train coupled to the motor. The integrating effect of the motor can be transferred to this element by regarding the angular velocity ν as the output quantity of the motor. Thus the transfer function is

$$Y_r = \frac{X_k}{\nu} = \frac{K_r}{p}. \tag{6}$$

The d. c. motor with constant excitation is characterized as a linear element by two transfer functions. One of them describes the variation of the angular velocity with the terminal voltage, the other that with the torque.

As is known the two functions are

$$Y_m = \frac{K_m}{p^2 T_m T_v + pT_m + 1} \tag{7}$$

and

$$Y_t = \frac{-R_m K_m^2 (1 + pT_v)}{p^2 T_m T_v + pT_m + 1}\,. \tag{8}$$

(See App. II.)

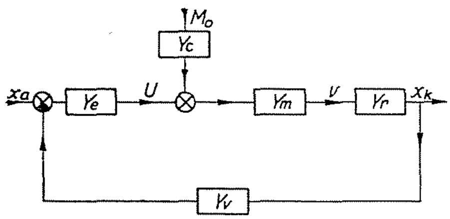

Fig. 1

Consequently for the Laplace-transforms of the speed the following can be written:

$$v = U \frac{K_m}{p^2 T_m T_v + pT_m + 1} - M \frac{R_m K_m^2 (1 + pT_v)}{p^2 T_m T_v + pT_m + 1}\,. \tag{9}$$

From the fundamental equations of the d. c. motor, however, the variation of the transform I of the armature current with the transform U of the terminal voltage and with the transform M of the load torque can also be determined in the following way (See App. II.):

$$I = \frac{U}{R_m} \frac{pT_m}{p^2 T_m T_v + pT_m + 1} - M \left(\frac{K_m pT_m (1 + pT_v)}{p^2 T_m T_v + pT_m + 1} - K_m \right) =$$

$$= U \frac{pT_m}{R_m K_m} Y_m + M Y_m\,. \tag{10}$$

If the servomotor is the element of a servomechanism, the independent variable is not the terminal voltage u but the reference input x_a. In this case the variation of the transform I of the armature current i with the transforms X_a and M of the reference input and of the torque, respectively, may be determined. According to the block diagram given in Fig. 1

$$U = X_a \frac{Y_e}{1 + Y_e Y_m Y_r Y_v} - M \frac{Y_t Y_m Y_r Y_v Y_e}{1 + Y_e Y_m Y_r Y_v} =$$
$$= X_a \frac{Y_e}{1+Y} - MY_t \frac{Y}{1+Y}, \qquad (11)$$

if $Y = Y_e Y_m Y_r Y_v$.

By substituting the above expression of U in the expression of I as given in Eq. (10) we have got

$$I = X_a \frac{pT_m}{R_m K_m} \cdot \frac{Y_e Y_m}{1+Y} - M \frac{pT_m}{R_m K_m} \frac{Y_t Y_m Y}{1+Y} + M Y_m . \qquad (12)$$

In a given case knowing the time variations of the reference input and of the torques, furthermore the transfer functions, a rational fractional function is obtained for I. By inversely transforming this we obtain the time fuction of the current:

$$\mathcal{L}^{-1}[I(p)] = i(t) .$$

This may serve as a basis for determining the heat generated in the armature of the motor.

It is known, however, that the inverse transformation cannot be carried out by purely algebraic means on account of the high order of the denominator of the reference function. There is, however, no need for it as Raleigh's theorem for Fourier transform gives a possibility to determine without inverse transformation directly from the transform the inproper integral of the square of time function.

It is also known, however, that the Fourier transformation can be defined only for absolute integrable functions $f(t)$, i. e. for those, which

$$\int_{-\infty}^{+\infty} f(t)\, dt$$

is finite. Therefore our examination must be carried out by adapting such a reference function at which the current $i(t)$ of the motor approaches a steady-state value i_∞. For in this case the absolute integrable function

$$f(t) = i(t) - i_\infty \qquad (13)$$

of Raleigh's theorem may be applied. The current $i(t)$ approaches a finite value in case of a step or constant-velocity reference input.

In this case the heating power $P(t) = i(t)^2 R_m$ generated in the rotor is as indicated in Fig. 2 when plotted against time.

The steady-state value of $P(t)$ is

$$P_\infty = i_\infty^2 R_m .$$

As the total heat generated during the starting period the infinite range integral of the power

$$P(t) - P_\infty$$

will be defined. (See Fig. 2)

$$W = \int_0^\infty [P(t) - P_\infty] \, dt = R_m \int_0^\infty [i(t)^2 - i_\infty^2] \, dt . \tag{14a}$$

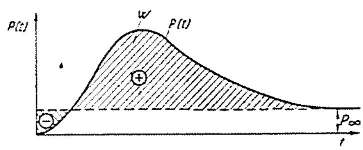

Fig. 2

Thus this is the heat generated on account of the transient phenomenon in addition to the heating power caused by the current i_∞ corresponding to the given load.

On the basis of Eq. (14*a*) we shall express W in the following form:

$$W = R_m \int_0^\infty (i^2 - i_\infty^2) \, dt = R_m \int_0^\infty [(i - i_\infty)^2 + 2 i_\infty (i - i_\infty)] \, dt . \tag{14b}$$

As W is expressed now as the integral of the function $f(t) = i - i_\infty$, the first term of Eq. (14*b*) can be determined by means of Raleigh's theorem while the second term, by means of the known final value and initial value theorem, holding for Fourier transformations.

Our further investigations will be continued in that case, where the external load of the motor is zero. For in this case $i_\infty = 0$, hence the second term between the square brackets is also equal to zero. This neglection gives a good approximation of the actual conditions, provided the size of the servomotor is correctly selected, in view of the required accuracy of the control.

If for some reason the term $2 i_\infty (i - i_\infty)$ in Eq. (14*b*) is required, it must be taken into account that integral of infinite range can be calculated by the method given in App. II. (See the derivation of Eq. (55).)

Let us therefore examine the servomechanism having a layout as shown in Fig. 1 to which a constant-velocity input of a slope v is applied. The speed of the reference input has to be of the value to which the rated speed of the motor corresponds in steady state.

As the transform of the reference input is

$$X_a = \frac{v}{p^2},$$

furthermore

$$M = 0,$$

therefore according to Eq. (12)

$$I = \frac{v}{p^2} \frac{p T_m}{R_m K_m} \frac{Y_e Y_m}{1 + Y}. \tag{15}$$

After some alterations, taking into consideration that $Y = Y_e Y_m Y_r Y_v$ and $Y_r = \frac{K_r}{p}$ we have

$$I = \frac{v T_m}{R_m K_m K_r} \cdot \frac{1}{Y_v} \cdot \frac{Y}{1 + Y}. \tag{16}$$

But $\frac{v}{K_r} = \nu_\infty$ (being at least a control system of type 1 under discussion, at which the speed of the controlled variable is also v in steady state), furthermore

$$\frac{1}{Y_v} \frac{Y}{1 + Y} = Y^* (p) \tag{17}$$

is the resultant reference function of the closed-loop. Therefore

$$I(p) = \frac{\nu_\infty T_m}{R_m K_m} Y^* (p). \tag{18}$$

Because a load torque $m = 0$ has been assumed, the steady-state value of the armature current is equal to zero as can be seen from Eq. (18) by means of the final value and initial value theorem. Consequently, the second term between the square-brackets in Eq. (14b) giving the heat during the transient phenomenon is zero.

As $I(p)$ expressed by Eq. (18) is the transform of an $i(t)$ which is absolute integrable, $j\omega$ can simply be substituted for p in the expression of $I(p)$, thus the Fourier-transform of the current being obtained.

$$I(j\omega) = \frac{\nu_\infty T_m}{R_m K_m} Y^* (j\omega). \tag{18a}$$

According to Raleigh's theorem holding for Fourier transforms, if $F(j\omega)$ is the transform of some absolute integrable function $f(t)$, then

$$\int_{-\infty}^{+\infty} [f(t)]^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| F(j\omega) \right|^2 d\omega . \tag{19}$$

Accordingly, the total heat generated in the armature is

$$W = R_m \int_0^{\infty} [i(t)]^2 \, dt = \frac{R_m}{2\pi} \int_{-\infty}^{+\infty} |I(j\omega)|^2 \, d\omega =$$

$$= R_m \frac{\nu_\infty^2 T_\infty^2}{R_m^2 K_m^2} \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y^*(j\omega)|^2 \, d\omega . \tag{20}$$

If it is considered that the electromechanical time constant in the last expression is

$$T_m = \theta R_m K_m^2 ,$$

then after some simple conversion we obtain

$$W = \nu_\infty^2 \theta T_m \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y^*(j\omega)|^2 \, d\omega . \tag{21}$$

For computing improper integrals of such a form known formulae can be found in the literature provided, $Y^*(j\omega)$ is a rational fractional function the numerator of which is of lower order than the denominator.

In App. I/a the value of the integrals

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y^*(j\omega)|^2 \, d\omega$$

is indicated. In our case for $Y^*(p)$ which is nothing else but the transfer function of the closed-loop, the following can be stated:

1. $$Y(0) = 1 .$$

2. The order of the numerator is lower by at least two, than that of the denominator.

The statement 1 concludes from the type number of the control system. As for statement 2 derivates from the fact that the servomotor and the succeeding gear train constitute a series element of the system, in the resultant transfer function of which the numerator is of an order lower of at least two, than the denominator. As for the transfer function of the remaining elements of the system, the order of its numerator is not higher than that of its denominator.

Consequently, in our case in the expressions given in App. I/*a* $B_{n-1} = 0$ and $A_0 = B_0$. For numerical calculations these expressions can be converted into simpler forms. To this end the resultant gain K of the closed-loop, the value of which can be proved to be 1 in case of a system of type 1 (See App. III), will be lifted out of the expressions.

$$K = \frac{A_0}{(A_1 - B_1)} \tag{22}$$

Thus the mentioned expressions can be converted into the following form :

$$J_k = \frac{K}{2} F_k = \frac{A_0}{2(A_1 - B_1)} F_k . \tag{23}$$

Hence the total heat generated in the armature circuit

$$W = \frac{\theta \, v_\infty^2}{2} T_m K F_k . \tag{24}$$

The values of F_k are given in App. I/*b*.

As an example, the heat generated in the armature circuit has been determined for the case of servomotors, as according to Figs. 3*a*, 3*b*, 3*c*, provided a constant-velocity reference input is applied. The numerical values chosen are indicated in the figures. The values of $T_m K F_k$ are given in the following table.

$T_m = 0{,}25$ sec

	K	F_k	$T_m K F_k$	W
2a	10/sec	2,02	3,03	$3{,}03 \frac{\theta v_\infty^2}{2}$
2b	30/sec	1,61	7,26	$7{,}26 \frac{\theta v_\infty^2}{2}$
2c	30/sec	1,88	8,46	$8{,}46 \frac{\theta v_\infty^2}{2}$

Hence it can be seen that the heat generated during the transient phenomenon is in case of *a)* appr. 3 times, in case of *b)* appr. 7,3 times, in case of *c)* appr. 8,5 times, as much as, the kinetic energy accumulated in the rotating parts.

The heat calculated above is generated in the entire armature circuit, thus in case of a motor being supplied from an amplidyne, this is the total heat generated in the armature of the motor, plus in that of the amplidyne. This heat is evidently divided between the motor and the amplidyne in the ratio of their resistances.

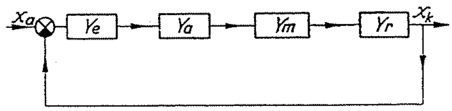

Fig. 3/a

$$Y_e = K_e;\; Y_m = \frac{K_m}{1 + p\,T_m};\; Y_a = \frac{K_a}{(1 + p\,T_{a1})\,(1 + p\,T_{a2})}\; Y_r = \frac{K_r}{p}$$

$$K_e = 0{,}1;\; K_a = 10;\; K_m = 2\,\frac{1}{\mathrm{Vsec}};\; K_r = 5\ \mathrm{V}$$

$$T_{a_1} = 0{,}03\ \mathrm{sec};\; T_m = 0{,}15\ \mathrm{sec};\; T_{a2} = 0{,}03\ \mathrm{sec};$$

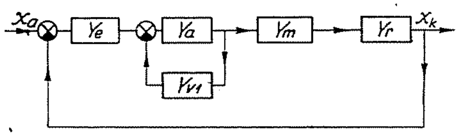

Fig. 3/b

$$Y_{v1} = K_{v1}$$

$$K_e = 0{,}6;\; K_a = 10;\; K_m = 2\,\frac{1}{\mathrm{Vsec}};\; K_r = 15\ \mathrm{V};\; K_{v_1} = 0{,}5$$

$$T_{a_1} = 0{,}03\ \mathrm{sec};\; T_m = 0{,}15\ \mathrm{sec};\; T_{a2} = 0{,}03\ \mathrm{sec};$$

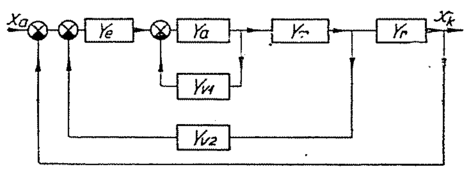

Fig. 3/c

$$Y_{v_2} = K_{v2}$$

$$K_e\,1{,}8;\; K_a = 10;\; K_m = 2\,\frac{1}{\mathrm{Vsec}};\; K_r = 20\ \mathrm{V};\; K_{v_1} = 0{,}5;\; K_{v2} = 0{,}5$$

$$T_{a_1} = 0{,}03\ \mathrm{sec};\; T_{a2} = 0{,}03\ \mathrm{sec};\; T_m = 0{,}15\ \mathrm{sec}$$

III. The determination of the optimum transfer function in view of the temperature conditions of the motor

By means of the expression derived above, the copper loss in the servomotor of a servomechanism designed according to specified control qualities (i. e. response time, overshoot, etc.) can be determined when following-up a unit constant-velocity reference input. It can be of interest to determine the layout at which the amount of heat will be at minimum.

Accordingly, we could proceed by determining the minimum value of F_k while varying some parameters according to the conventional method for computing extreme values. Unfortunately, by this procedure such a value of the parameter concerned is obtained, that cannot be realized either technically, or at which the control quality is unacceptable.

Therefore another approach should be made for the solution of the problem. The values characteristic for the quality of the control (response time, overshoots, etc.) should be specified, and an output signal should be sought for, by which these requirements are just satisfactory, the copper loss in the rotor of the servomotor being at the same time a minimum. Such transfer elements at which this ideal output signal would arise cannot be realized in practice. If, however, the amount of heat is determined that could be generated in this

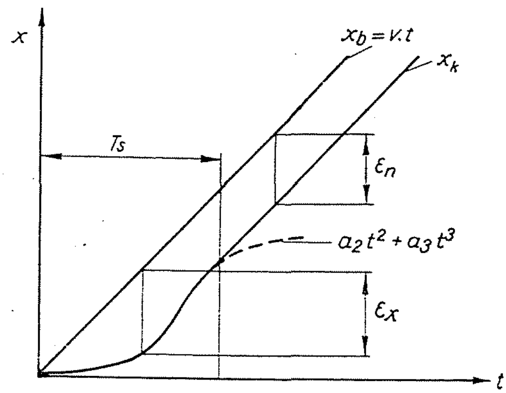

Fig. 4

ideal case, this gives a good basis of comparison to decide, whether for the sake of improving the temperature conditions of the motor is it advisable to make alterations in the layout or parameters of the control system selected in view of the quality of the control, and to see at all whether considerably better results could be obtained. If not, the parameters selected on the basis of the quality of the control should be retained.

A) *The analytical expression of the optimum output signal*

The current of the idly running motor is proportional to the acceleration of the output signal. Because

$$i = K_m \theta \frac{d v}{dt} = \frac{K_m \theta}{K_r} \frac{d^2 x_k}{dt^2},$$

where K_r is the transfer coefficient of the gear train succeeding the motor. Our task is to determine such a

$$x_k = x_k(t)$$

function for which the integral

$$W = \int_0^\infty i^2(t)\, R_m\, dt = \frac{R_m\, \Theta^2\, K_m^2}{K_r^2} \int_0^\infty \left(\frac{d^2\, x_k}{dt^2}\right)^2 dt \tag{25}$$

is a minimum.

It can be assumed that the follow-up of such an input is dealt with for which the steady-state value of the current is zero (e. g. in case of a unit-step or a unit constant-velocity input), hence from the output signal it can be assumed to attain its steady-state value or its derivative in time T_s unknown for the present, while from there it can be substituted by a straight line. (See Fig. 4) As for $t > T_s$ the acceleration is zero, therefore in the above expression the limit of integration is t_s instead of ∞. Under these circumstances our task is actually the solution of a variation calculus problem.

We shall apply the *Euler-Poisson* equation according to which the function

$$y = y\,(x)$$

for which the integral

$$H = \int_{x_2}^{x_1} F\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}\right) dx \tag{26}$$

is a minimum, can be expressed by the following differential equation :

$$\frac{dF}{dy} - \frac{d}{dx}\frac{dF}{dy'} + \frac{d^2}{dx^2}\frac{dF}{dy''} = 0\,. \tag{27}$$

Here $\qquad y' = \dfrac{dy}{dx} \qquad$ and $\qquad y'' = \dfrac{d^2 y}{dx^2}\,.$

In our case

$$x = t$$

$$y = x_k\,, \quad y' = \frac{dx_k}{dt}\,, \quad y'' = \frac{d^2x_k}{d.^2}$$

$$H = W$$

and

$$F = \frac{R_m\, \Theta^2\, K_m^2}{K_r^2}\left(\frac{d^2\, x_k}{dt^2}\right)^2 .$$

Because in our case F depends only on its second order derivative, therefore

$$\frac{dF}{dy} = 0,$$

$$\frac{dF}{dy'} = 0,$$

$$\frac{dF}{dy''} = \frac{R_m \theta^2 K_m^2}{K_r^2} \cdot 2 \frac{d^2 x_k}{dt^2}. \tag{27}$$

By substituting these values into the above differential equations we get

$$\frac{d^2}{dt^2} \frac{R_m \theta^2 K_m^2}{K_r^2} \cdot 2 \frac{d^2 x_k}{dt^2} = 0 \tag{28}$$

that is

$$\frac{d^4 x_k}{dt^4} = 0 . \tag{29}$$

The general solution of this differential equation is as follows :

$$x_k = a_3 t^3 + a_2 t^2 + a_1 t + a_0 . \tag{30}$$

The constants in this differential equation can be determined by taking the following boundary conditions as a basis (See Fig. 4) :

1 if $t = 0$ $x_k = 0$

2 if $t = 0$ $\frac{dx_k}{dt} = 0$

3 if $t = T_s$ $x_k = vT_s - \varepsilon_n$

4 if $t = T_s$ $\frac{dx_k}{dt} = v .$

The first two conditions express that at the first instant the output signal as well as its derivative are equal to zero, the latter being so, on account of the inertia of the motor. The conditions 3 and 4 express that the output signal continuously and with a continuous tangent passes through to the steady state substituted with the straight line.

Based on these conditions the following expressions are obtained for the values of the constants :

$$a_0 = 0 ,$$

$$a_1 = 0 ,$$

$$a_2 = \frac{2 v T_s - 3 \varepsilon_n}{T_s^2} = \frac{v}{KT_s^2} (2 KT_s - 3) \tag{31}$$

$$a_3 = \frac{2 \varepsilon_n - vT_s}{T_s^3} = \frac{v}{KT_s^3} (2 - KT_s) . \tag{32}$$

In the expressions of a_2 and a_3 it is implied that a control system of the type 1 is under discussion, therefore the following relation holds true between the velocity of the reference input and the steady-state error:

$$\frac{v}{\varepsilon_n} = K$$

where K is the gain of the open-loop.

The ideal output signal chosen, according to the mentioned above is shown in Fig. 4. It can be well seen that the error has a maximum value. Let us denote the quotient of this maximum and the steady-state error with γ.

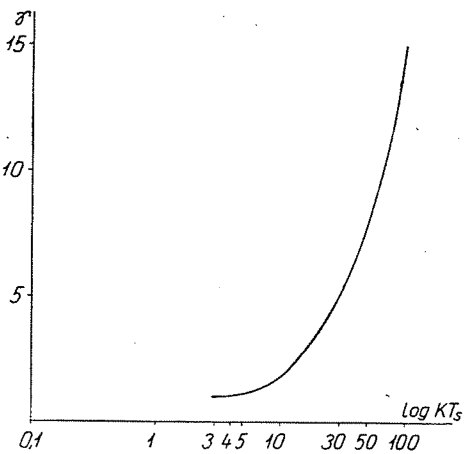

Fig. 5

It can be verified in a simple way (See App. IV) that the value of the quotient is

$$\frac{\varepsilon_x}{\varepsilon_s} = \gamma = \frac{(KT_s)^2\,(4\,KT_s - 9)}{27\,(KT_s - 2)^2} \tag{33}$$

The relation between γ and KT_s as given by Eq. (33) is shown in Fig. 5.

B) *The heat generated in case of the optimum output signal*

In order to determine the ideal output signal suitable for a given control task, beside the gain K either the response time T_s or the value of γ characteristic for the maximum dynamic error must be accessed beforehand.

As according to the above discussion, the current of the servomotor is proportional to the second order derivate of the output signal, hence by differenciating Eq. (30) twice, and by substituting it into Eq. (25), for the total heat generated during the starting period, in this ideal case

$$W = \frac{R_m \Theta^2 K_m^2}{K_r^2} \int_0^{T_s} (2a_2 + 6a_3 t)^2 \, dt$$

is obtained. By carrying out the integration and substituting the values of the constants from Eqs. (31) and (32), furthermore taking into consideration that

$$\frac{v}{K_r} = \nu_\infty$$

and

$$\theta R_m K_m^2 = T_m$$

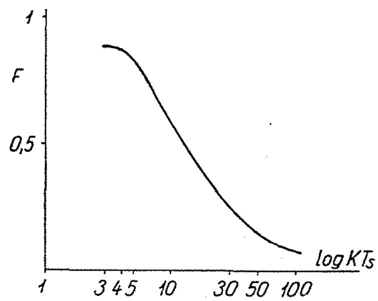

Fig. 6

we have

$$W = \frac{\theta \nu_\infty^2}{2} \frac{T_m}{T_s} \cdot \frac{8\left[(KT_s)^2 - 3\,KT_s + 3\right]}{(KT_s)^2} . \tag{34}$$

In order to make a comparison possible of the heat generated in case of the practicable control system with that calculated above, it is worth while to carry out the following simple alteration :

$$W = \frac{\theta \nu_\infty^2}{2} T_m K \cdot \frac{8\left[(KT_s)^2 - 3\,KT_s + 3\right]}{(KT_s)^3} =$$

$$= \frac{\theta \nu_\infty^2}{2} T_m \, K \cdot F_i \tag{35}$$

where

$$F_i = 8 \; \frac{(KT_s)^2 - 3\,KT_s + 3}{(KT_s)^3} . \tag{36}$$

In Fig. 6 the variation of F_i with KT_3 as expressed by the above equation is shown. By comparing this with Fig. 5, F_i can be plotted against γ. (See

Fig. 7) From this it can be seen that the maximum value of F_i is 8/9 for $\gamma = 1$. For $\gamma = 1{,}5$, $F_i = 0{,}67$.

Taken into account the afore-mentioned it can be stated that in case of an output signal that is optimum concerning the temperature rise of the motor, and in case a control system advantageous as for overshoots, the value of F_i may be chosen to be between 0,7~0,9. Therefore the minimum amount of heat generated in the armature of the servomotor is

$$W = 0{,}7 \sim 0{,}9 \cdot T_m K \frac{\theta \nu_\infty^2}{2} .$$

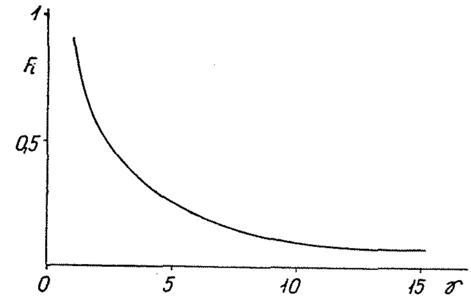

Fig. 7

It is evident that this cannot be achieved in practice, because such transfer elements are not available, by means of which the ideal output signal could be produced. Calculations carried out in several cases show that in practice the value of F_i can be reduced to appr. 1,4.

The calculation procedure shown above can be used not only to determine the temperature conditions of servomotors, being an element in a closed-loop, but it can also be used to examine the temperature conditions of motors in intermittent operation, the armatures of which are supplied through elements varying with frequency. This is true e. g. for Ward-Leonard drives of intermittent operation and of those controlled by amplidynes.

Appendix I/a

$$Y^*(p) = \frac{B_{n-1} p^{n-1} + \ldots + B_0}{A_n p^n + \ldots + A_0} \; *$$

$$J = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |Y^*(j\omega)|^2 \, d\omega$$

*James—Nichols—Phillips: Theory of Servomechanisms. McGraw-Hill Book Company, Inc. 1947.

$$J_1 = \frac{B_0^2}{2 A_0 A_1}$$

$$J_2 = \frac{B_1^2 + \dfrac{A_2 B_0^2}{A_0}}{2 A_1 A_2}$$

$$J_3 = \frac{A_1 B_2^2 + A_3 B_1^2 - 2 A_3 B_0 B_2 + \dfrac{A_2 A_3 B_0^2}{A_0}}{2 A_3 (A_1 A_2 - A_0 A_3)}$$

$$J_4 = \frac{B_3^2 (A_1 A_2 - A_0 A_3) + B_2^2 A_1 A_4 - 2 B_1 B_3 A_1 A_4 - B_1^2 A_3 A_4 - 2 B_0 B_2 A_3 A_4 + \dfrac{A_4 B_0^2}{A_0} (A_2 A_3 - A_1 A_4)}{2 A_4 (A_1 A_2 A_3 - A_1^2 A_4 - A_0 A_3^2}$$

Appendix I/b

$$F_1 = 1 ,$$

$$F_2 = 1 ,$$

$$F_3 = \frac{\dfrac{B_1^2}{A_0} + A_2}{\dfrac{A_1 A_2 - A_0 A_3}{A_1 - B_1}}$$

$$F_4 = \frac{B_2^2 \dfrac{A_1}{A_0} - B_1^2 \dfrac{A_3}{A_0} - 2 B_2 A_3 + (A_2 A_3 - A_1 A_4)}{\dfrac{A_1}{A_1 - B_1} (A_2 A_3 - A_1 A_4) - \dfrac{A_0 A_3^2}{A_1 - B_1}}$$

$$F_5 = \frac{B_3^2 \dfrac{A_{12}}{A_0} + (B_2^2 - 2 B_1 B_3) \dfrac{A_{14}}{A_0} + (B_1^2 - 2 B_0 B_2) \dfrac{A_{34}}{A_0} + A_2 A_{34} - A_4 A_{14})}{\dfrac{A_1}{A_1 - B_1} A_2 A_{34} - A_4 A_{14} + \dfrac{A_3}{A_1 - B_1} A_{34} - \dfrac{A_5}{A_1 - B_1} A_{14}}$$

where the following substitutions were carried out in F_5:

$$A_{12} = A_1 A_2 - A_0 A_3 ,$$

$$A_{14} = A_1 A_4 - A_0 A_5 ,$$

$$A_{34} = A_3 A_4 - A_2 A_5 .$$

Appendix II

The voltage equation of the d. c. motor is

$$u = R_m i + L_m \frac{di}{dt} + c \Phi \nu \tag{37}$$

and its torque equation is

$$c \Phi i = m + \Theta \frac{d \nu}{dt} . \tag{38}$$

By rewriting Eqs. (37) and (38) into Laplace-transforms, with initial value equal to zero we have

$$U = (R_m + pL_m)\, I + c\,\Phi\,\nu \tag{39}$$

$$c\,\Phi\, I = M + p\,\Theta\,\nu \tag{40}$$

By eliminating I from the last two equations and solving them for ν, the transfer function of the motor is obtained as follows:

$$\nu = U \frac{c\,\Phi}{p\,\Theta\,(R_m + pL_m) + c^2\,\Phi^2} - M \frac{R_m + pL_m}{p\,\Theta\,(R_m + pL_m) + c^2\,\Phi^2} \tag{41}$$

If in the above equation the following substitutions are carried out

$$\frac{1}{c\,\Phi} = K_m \tag{42a}$$

$$\frac{L_m}{R_m} = T_v\,, \tag{42b}$$

$$\frac{\Theta\,R_m}{c^2\,\Phi^2} = \Theta\,R_m\,K_m^2 = T_m\,, \tag{42c}$$

the equation assumes the following form:

$$\nu = U \frac{K_m}{p^2\,T_m\,T_v + p\,T_m + 1} - M \frac{R_m\,K_m^2\,(1 + pT_v)}{p^2\,T_m\,T_v + pT_m + 1} = U\,Y_m + M\,Y_t\,Y_m \tag{43}$$

where

$$Y_m = \frac{K_m}{p^2\,T_m\,T_v + pT_m + 1} \tag{44}$$

$$Y_t = -\,RK_m\,(1 + pT_v)\,. \tag{45}$$

On the basis of the above written the block diagram given in Fig. 8 can be substituted for the motor as a transfer element.

Now the transfer functions for current I will also be determined. To this end let us express I from Eq. (40):

$$I = \frac{M}{c\,\Phi} + \frac{p\Theta}{c\Phi}\,\nu = K_m\,M + p\,\frac{\Theta\,R_m}{c^2\,\Phi^2}\,\frac{c\,\Phi}{R_m}\,\nu = K_m\,M + \frac{pT_m}{K_m\,R_m}\,\nu \tag{46}$$

Then substituting ν from Eq. (41) we have

$$I = \frac{U}{R_m}\,\frac{pT_m}{p^2\,T_m\,T_v + pT_m + 1} - M\left(\frac{K_m\,pT_m\,(1 + pT_v)}{p^2\,T_m\,T_v + pT_m + 1} - K_m\right) =$$

$$= U\,\frac{pT_m}{R_m\,K_m}\,Y_m + M\;Y_m. \tag{47}$$

Based on these, the time variation of the velocity, and that of the current of the motor can be determined. Both have two components: one of them varies with the voltage, the other with the current. While deriving the expressions we have had to assume that the linear relations expressed by the initial equations (37) and (38) actually exist between the individual variables, furthermore the constants therein are truly constants (this refers in the first place to Φ and L_m), and the voltage u and load torque m are known variables, varying exclusively with time. This latter conditions must particularly be emphasized in case of the torque m as e. g. the bearing friction or the windage loss depending on the speed, and the torque caused by the eddy-currents cannot evidently be taken into consideration in this expression.

Note : If a damping torque $m_d = k\,\nu$ varies linearly with the speed exists, for the functions relating to the angular velocity and to the current, it can be proved similarly to the above stated that instead of Eqs. (44), (43), (47) the following expressions hold good :

$$Y'_m = \frac{K_m}{p^2\,T_m\,T_v + p\,(a\,T_v + T_m) + 1 + a}$$

$$\nu = U\,Y'_m + M\,Y_t\,Y'_m \tag{48}$$

$$I = U\,\frac{p\,T_m + a}{R_m\,K_m}\,Y'_m + M\,Y'_m$$

where

$$a = \frac{k\,R_m}{c^2\,\Phi^2}\,. \tag{49}$$

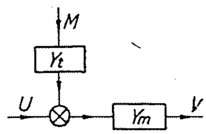

Fig. 8

In our further discussion those resistances, that are proportional to the speed will not be taken into consideration. In place of the reactive frictional torques, however, we shall assume for the time being, an active external torque.

As the constant torque requires a steady-state current $i_\infty = I_0$ to be maintained, therefore, the heat generated during the switching-on period will be defined as the following integral : (See Fig. 2)

$$W = \int_0^\infty (i^2\,R_m - i_\infty^2\,R_m)\,dt\,. \tag{50}$$

Let us express W as follows :

$$W = R_m \int_0^\infty (i^2 - i_\infty^2)\,dt = R_m \int_0^\infty \left[(i - i_\infty)^2 + 2i_\infty\,(i - i_\infty)\right]\,dt\,. \tag{51}$$

To determine this Raleigh's theorem will be used according to which if $F(j\omega)$ is the Fourier-transform of some absolute integrable function $f(t)$, then

$$\int_{-\infty}^{+\infty} f(t)^2\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |\,F\,(j\,\omega)\,|^2\,d\,\omega\,. \tag{52}$$

According to the final value and initial value theorem it follows from Eq. (47) that as a result of a unit-step change of U and M

$$i_\infty = m_0\,K_m$$

(where m_0 is the value of the applied torque).

Consequently, by substituting $U = \frac{u_0}{p}$ and $M = \frac{m_0}{p}$ we have

$$L[i - i_\infty] = u_0 \frac{T_m}{R_m K_m} Y_m + \frac{m_0}{p} Y_m - \frac{m_0 K_m}{p} =$$

$$= \frac{u_0}{R_m} \frac{T_m}{p^2 T_m T_v + pT_m + 1} - m_0 K_m \frac{T_m (1 + pT_v)}{p^2 T_m T_v + pT_m + 1} =$$

$$= \frac{T_m \left[\frac{u_0}{R_m} - m_0 K_m (1 + pT_v)\right]}{p^2 T_m T_v + pT_m + 1} . \tag{53}$$

By substituting $j\omega$ for p in Eq. (53), we obtain the Fourier-transform of $i - i_\infty$. And to this the integral (52) is to be applied. That is

$$\int_0^\infty (i - i_\infty)^2 \, dt = \frac{T_m^2}{2\pi R_m^2} \int_{-\infty}^{+\infty} \left| \frac{- m_0 K_m R_m T_v j\omega + (u_0 - m_0 K_m R_m)}{(j\omega)^2 T_m T_v + j\omega T_m + 1} \right|^2 d\omega .$$

For the determination of improper integrals of such a form closed expressions are available. (See App. I/a.)

In our case using the symbole given in App. I/a

$$\begin{aligned}
B_1 &= - m_0 K_m R_m T_v = - i_\infty R_m T_v , \\
B_0 &= u_0 - m_0 K_m R_m = u_0 - i_\infty R_m , \\
A_2 &= T_m T_v , \\
A_1 &= T_m , \\
A_0 &= 1 .
\end{aligned}$$

Hence

$$\int_0^\infty (i - i_\infty)^2 \, dt = \frac{T_m^2}{R_m^2} \frac{i_\infty^2 R_m^2 T_v^2 + T_m T_v (u_0^2 - 2u_0 i_\infty R_m + i_\infty^2 R_m^2)}{2 T_m^2 T_v} .$$

By multiplying the last equation by R_m, after some simple conversions we obtain

$$R_m \int_0^\infty (i - i_\infty)^2 \, dt = \frac{1}{2} \frac{u_0^2}{R_m} T_m + \frac{1}{2} i_\infty^2 R_m T_v - u_0 i_\infty T_m + \frac{1}{2} i_\infty^2 R_m T_m . \tag{54}$$

By means of the final value and initial value theorem, the second term between the square-brackets in Eq. (51) can very easily be integrated.

$$\int_0^\infty (i - i_\infty) \, dt = \left[\int (i - i_\infty) \, dt\right]_\infty - \left[\int (i - i_\infty) \, dt\right]_0 .$$

But according to the final value and initial value theorem,

$$\left[\int (i - i_\infty)\, dt\right]_\infty = \left[\frac{1}{p}\mathscr{L}(i - i_\infty)\cdot p\right]_0 = [\mathscr{L}(i - i_\infty)]_0$$

and similarly

$$\left[\int (i - i_\infty)\, dt\right]_0 = [\mathscr{L}(i - i_\infty)]_\infty .$$

Making use of the Laplace-transform of $(i - i_\infty)$ as written in Eq. (53) we have

$$\int_0^\infty (i - i_\infty)\, dt = T_m \frac{u_0}{R_m} - m_0 K_m T_m = T_m \frac{u_0}{R_m} = i_\infty T_m .$$

Hence after simple alteration we obtain

$$2 i_\infty R_m \int_0^\infty (i - i_\infty)\, dt = 2 u_0 i_\infty T_m - 2 i_\infty^2 R_m T_m . \tag{55}$$

Let us substitute the expressions given in Eqs. (54) and (55) into Eq. (51):

$$W = \frac{1}{2}\frac{u_0^2}{R_m} T_m + \frac{1}{2} i_\infty^2 R_m T_v + u_0 i_\infty T_m - \frac{3}{2} i_\infty^2 R_m T_m . \tag{56}$$

Let us convert the above expression in the following way:
As according to Eq. (41) the steady-state speed is

$$v_\infty = u_0 K_m - m_0 R_m K_m^2 ,$$

or

$$u_0 = \frac{v_\infty}{K_m} + m_0 K_m R_m = \frac{v_\infty}{K_m} + i_\infty R_m$$

therefore the first term of Eq. (56) is

$$\frac{1}{2}\frac{u_0^2}{R_m} T_m = \frac{1}{2} u_0^2 \Theta K_m^2 = \frac{1}{2}\Theta v_\infty^2 + m_0 v_\infty T_m + \frac{1}{2} i_\infty^2 R_m T_m . \tag{56a}$$

For the second term

$$\frac{1}{2} i_\infty^2 R_m T_v = \frac{1}{2} i_\infty^2 L_m . \tag{56b}$$

In the third term the power $u_0 i_\infty$ consumed by the supply network in steady state can be written as the sum of the mechanical output $m_0 v_\infty$ and of copper loss $i_\infty^2 R_m$:

$$u_0 i_\infty T_m = m_0 v_\infty T_m + i_\infty^2 R_m T_m . \tag{56c}$$

By substituting Eqs. (56*a*), (56*b*), (56*c*) into Eq. (56) the total heat generated during the starting period is obtained:

$$W = \frac{1}{2}\Theta v_\infty^2 + \frac{1}{2} i_\infty^2 L + 2 m_0 v_\infty T_m . \tag{57}$$

Appendix III

The verity of Eq. (22) can be seen in the following way:

If the transform of the error of the servomechanism is denoted E, the error transfer function of the system is

$$\frac{E}{X_a} = Y_\varepsilon^*(p) = 1 - Y^*(p) = 1 - \frac{B_{n-1}p^{n-1} + \ldots + B_1 p + B_0}{A_n p^n + \ldots + A_1 p + A_0} =$$

$$= \frac{A_n p^n + (A_{n-1} - B_{n-1})p^{n-0} + \ldots + (A_1 - B_1)p + B_0 - A_0}{A_n p^n + \ldots + A_1 p + A_0}.$$

Considering that on account of $Y_\varepsilon^*(0) = 0$ holds $B_0 - A_0 = 0$, the transform of the error caused by unit constant-velocity input is

$$E(p) = \frac{1}{p^2} \frac{p\left[A_n p^{n-1} + (A_{n-1} - B_{n-1})p^{n-2} + \ldots + (A_1 - B_1)\right]}{A_n p^n + \ldots + A_1 p + A_0}.$$

The steady-state value ε_∞ of the error can on one hand be computed from the above expression by means of the final value and initial value theorem, on the other hand, however, it is known that its reciprocal is equal to the reciprocal of the resultant gain in case of a system type 1. Therefore

$$\varepsilon_\infty = \left[p\,E(p)\right]_{p=0} = \frac{A_1 - B_1}{A_0} = \frac{1}{K}.$$

Appendix IV

Let us introduce the expression $a = \frac{1}{K T_s}$. With this Eqs. (31) and (32) assume the following forms:

$$a_2 = \frac{v}{T_s}(2 - 3a) \tag{58a}$$

$$a_3 = \frac{v}{T_s^2}(2a - 1). \tag{58b}$$

According to Fig. 4 and Eq. (30) the error of the control is

$$\varepsilon = vt - a_3 t^3 - a_2 t^2. \tag{59}$$

The extreme value of the error can be calculated from the following expression:

$$\frac{d\varepsilon}{dt} = v - 3a_3 t^2 - 2a_2 t = 0$$

$$t_{12} = \frac{a_2 \pm \sqrt{a_2^2 + 3a_3 v}}{-3a_3} =$$

$$= \frac{\frac{v}{T_s}(2 - 3a) \pm \sqrt{\frac{v^2}{T_s^2}(4 - 12a + 9a^2) + \frac{v^2}{T_s^2}(6a - 3)}}{3\frac{v}{T_s^2}(1 - 2)} =$$

$$= T_s \frac{2 - 3a \pm (3a - 1)}{3(1 - 2a)}.$$

Hence

$$t_1 = T_s \frac{1}{3\,(1-2\,a)} \tag{60a}$$

$$t_2 = T_s \tag{60b}$$

Let us build the second-order derivative of Eq. (59)

$$\frac{d^2\,\varepsilon}{dt^2} = -\,2\,a_2 - 6\,a_3\,t$$

the value of which for $t = T_s$ is

$$\left[\frac{d^2\,\varepsilon}{dt^2}\right]_{t=T_s} = \frac{v}{T_s}\,(2-6\,a)\,,$$

This is positive, in other words the error has a minimum value for T_s if

$$a < \frac{1}{3}\,.$$

In this case according to Eq. (60a) the error has its maximum value at

$$t_1 < T_s\,.$$

Consequently, only those curves can be of interest to us for which a is smaller than $1/3$. Let us comput the quotient γ of the maximum and minimum value of the error.

The maximum error ε_x according to Eq. (59) is

$$\varepsilon_x = v t_1 - a_3\,t_1^3 - a_2\,t_1^2 = v\,T_s \frac{1}{3\,(1-2\,a)} - \frac{v}{T_s^2}\,(2\,a-1)\,\frac{T_s^3}{27\,(1-2\,a)^3} -$$

$$-\,\frac{v}{T_s}\,(2-3\,a)\,\frac{T_s^2}{9\,(1-2\,a)^2} = v\,T_s\,\frac{4-9\,a}{27\,(1-2\,a)^2}\,.$$

As, however, $v = \varepsilon_n K$, (ε_n is the minimum value of the error, and in this case the steady-state value of it, too) is therefore

$$\varepsilon_x = \varepsilon_n\,KT_s\,\frac{4-9\,KT_s}{27\left(1-\dfrac{2}{KT_s}\right)^2} = \varepsilon_n\,(KT_s)^2\,\frac{4\,KT_s-9}{27\,(KT_s-2)^2}$$

furthermore

$$\gamma = \frac{\varepsilon_x}{\varepsilon_n} = (KT_s)^2\,\frac{4\,KT_s-9}{27\,(KT_s-2)^2}\,.$$

Summary

The article shows how to give an explicite form to the improper integral of the square current of a d. c. servomotor used in automatic feedback control systems. Thus the heat generated in the armature circuit of the servomotor can be determined.

This method of variation calculus is suitable for determining the least quantity of heat arising in the armature circuit in case of given parameters describing the behaviour of the control system. This optimal value existing among the ideal circumstances gives a good basis for considering, whether it is worth to change the layout of the realized control for reducing the quantity of heat arising in the armature circuit.

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