

ABOUT THE STAR-LIKE FUSION REACTOR

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1. Energy balance

As is known from literature [1, 2, 3, 4], one of the most difficult problems in realizing thermonuclear reactors is that of confining the gas plasma of a temperature of about 10^8 °K, the only solution seems to be the generating of a sufficiently strong magnetic field, most conveniently by means of its own magnetic field of high intensity current flowing through the plasma, representing an electromagnetic wall. It cannot be envisaged to directly confine it into a vessel having any actual material wall; the idea has arisen to surround the central core of a several million degrees temperature by a very wide gas layer — similarly to the stars — to ensure heat isolation. The impossibility of this arrangement, because of the thermal conductivity of the plasma exceeding that of any metal, is pointed out in literature.

In the following we treat at least in principle quantitatively the dimensions for realising, a steady energy production in DT gas at constant pressure, confined in a spherical container having a wall of a high, but technically not impossible temperature; in other words we consider the behaviour of the isobar DT star.

This star has particular properties. As it is to be expected and subsequently verified, the phenomena of gravitation is of no importance, because of the small mass of the star. The STEFAN-BOLTZMANN distribution law does not hold for the radiation, as it is a space radiation proportional to \sqrt{T} , as stated simultaneously in several places in literature [1, 3]. The radiation pressure term too, vanishes in the equilibrium equations. In contrast to the common stars, here we precisely know the composition of the star, and further the dependence of the fusion energy production upon density and temperature, is well known. Thus for the energy equilibrium an exact expression may be obtained.

We depart from the equation for this energy balance. The difference between the fusion power generated within a sphere of radius r and the radiated power is equal to the energy lost from the surface by conduction:

$$\int_0^r [p_f(T, \varrho) - p_r(T, \varrho)] 4 \pi r^2 dr = -4 \pi r^2 \kappa(T, r) \frac{dT}{dr} \quad (1)$$

where p_f and p_r are the specific fusion and radiation powers, ϱ is the plasma density; κ is the thermal conductivity.

For these quantities the functions are plotted as follows: The fusion energy production according to [1] is:

$$p_f = W_0 N_D N_T (\sigma v_{DT})_{av}$$

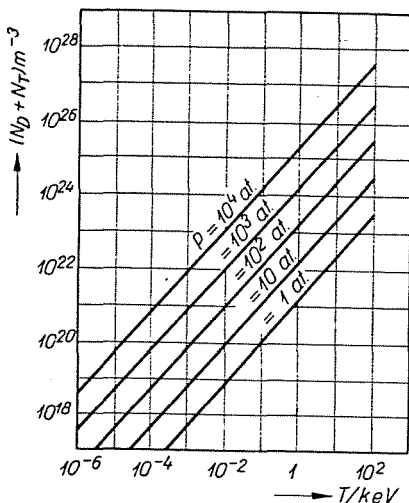


Fig. 1. Plasma-pressure dependence on temperature and density

where $W_0 = 0,2 \cdot 17$ MeV, is the fraction of energy released per reaction imparted to charged reaction products. Thus the energy of neutrons may be considered as lost in regards to heating effect. $(\sigma v_{DT})_{av}$ is the probability, that a particle will react and N_D and N_T are the densities of the deuterium and tritium nuclei, respectively. The value of $(\sigma v_{DT})_{av}$ can be obtained from the diagram.

According to [6] the thermonuclear power production may be expressed as:

$$p_f = \alpha N^2 T^{-2/3} e^{-\beta} T^{-1/3}$$

A quantitative formula of this form is given by [5]

$$p_f = 1,88 \cdot 10^{-24} (N_D + N_T)^2 T^{-2/3} e^{-3,9 T^{-1/3}} \frac{W}{\text{cm}^3}$$

where differing from [5], T has to be substituted in keV. The formulas of [1] and [5] do not give the same results, the difference, however, does not essentially affect the result.

The radiation is the *Bremstrahlung* of electrons. Its value obtained by [1] and [3] is identical, but in discrepancy with the numerical value given by [5] is

$$p_r = 0,54 \cdot 10^{-30} (N_D + N_T)^2 T^{1/2} \frac{W}{cm^3}$$

Here T has to be substituted in keV, and $N_D + N_T$ in cm^3 .

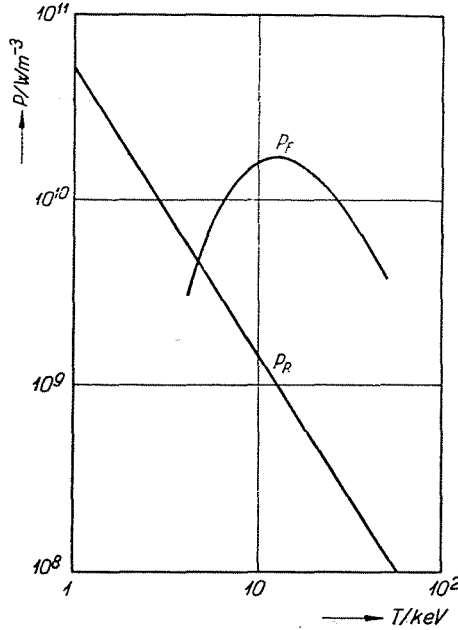


Fig. 2. Specific fusion and radiation energies at constant pressure $p = 1$ kat

Density and temperature are related through pressure. Considering that ions and electrons are involved equally in producing pressure, the equation for gas takes on the form :

$$p = (N_e + N_D + N_T) \frac{mc^2}{3} = (N_e + N_D + N_T) kT$$

If the pressure is measured in 1000 at (kat) and the temperature in keV, it becomes :

$$(N_D + N_T) = 3,06 \cdot 10^{23} p/T$$

Here the density is given in m^{-3} . This relationship is shown in Fig. 1.

The curve of the produced and that of the dissipated energy have both to be replotted against constant pressure instead of constant density. In the

graph in Fig. 2 the curve plotted in [1] has been used. The radiation power is expressed by :

$$p_r(T, p) = 5 \cdot 10^{10} p^2 T^{-3/2} W m^{-3}$$

This can also be seen in Fig. 2. At first the fact may seem surprising, that the radiated power decreases with increasing temperature ; this only means the influence of the decrease in density to be dominant at constant pressure.

The thermal conduction coefficient may be evaluated in first approximation on basis of the equation for the kinetic gas theory :

$$\kappa = \frac{k}{2} N_e \lambda v = \frac{k}{2} N_e \frac{v}{(N_D + N_T) \sigma}$$

where λ is the mean free path for electrons ; v is the electron velocity ; σ is the effective cross section for collision. The value of the latter is according to [1] :

$$\sigma = \frac{6 \cdot 10^{-19}}{W^2} cm^2$$

where W is the kinetic energy of electrons.

Thus for κ the following expression is obtained :

$$\kappa = 1,68 \cdot 10^{13} T^{3/2} \frac{W}{m \cdot keV}$$

This relationship is shown by curve *b*, in Fig. 3. Here we have plotted the thermal conduction curve for the case of the effective collision cross section assumed to be constant, independently of the energy and to be just $r_0^2 \pi$ where r_0 is the BOHR radius. It can be seen that at low temperatures the curve gives an extremely low conductivity.

For *a* as well as for *b* the conductivity is obtained as independent of density, and consequently of pressure : at low densities the small number of conducting electrons are compensated by a proportionally longer mean free path. According to the SPITZER—HÄRM theory [7] thermal conduction also depends even if in a smaller degree, on density.

We did not as yet consider the electric field due to the diffusion of electrons, which tends to decrease the diffusion and thus the thermal conductivity too. In general there is a flux of electric charges and also of current density. In the steady state the generated field just compensates for the excess diffusion of electrons, thus the current density becomes zero. For this case the thermal

conduction expressed by SPITZER—HÄRM and calculated in units used by us, becomes :

$$\kappa = \frac{6,43 \cdot 10^{13}}{\ln(qC^2)} T^{3/2} \frac{W}{m \cdot keV}$$

where

$$qC^2 = \frac{3}{\sqrt{\pi}} \frac{1}{e^3} \left(\frac{4\pi\epsilon_0}{2} \right)^3 \sqrt{\frac{1}{N_e}} T^{3/2} =$$

$$= 3,59 \cdot 10^{17} \frac{1}{\sqrt{N_e}} T^{3/2} = 6,51 \cdot 10^5 \sqrt{\frac{1}{P}} T^3$$

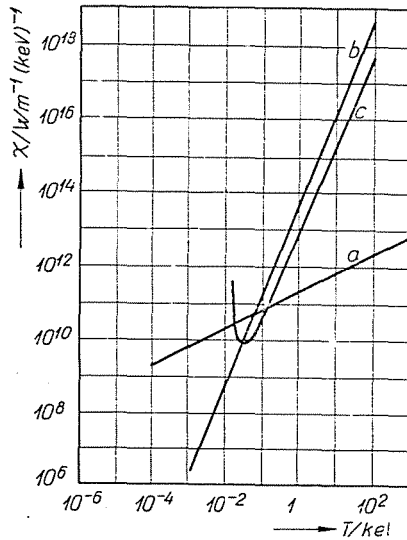


Fig. 3. The variation of thermal conductivity as a function of temperature

- a) calculated according to the kinetic gas theory with the Bohr-radius as a radius for effective cross-section
- b) calculated according to the kinetic gas theory, but with an effective cross-section, depending on the energy
- c) calculated according to the Spitzer—Härm formula

where e is the electron charge ; ϵ_0 is the dielectric constant in vacuum. For thermal conduction a relation almost similar to [7] is obtained, with the difference that by the expression $\ln(qC^2)$ density and pressure are also involved. The curve c , in Fig. 3 shows the value of κ evaluated for a pressure of $p = 1$ kat. As it can be seen the error is not too important, even if we calculate with the more simple curve b .

2. Transformation of the energy equation into an integral equation

Let us write our equation as follows :

$$\frac{1}{r^2} \int_0^r \{p_f [T(s), p] - p_r [T(s), p]\} s^2 ds = \kappa(T, p) \frac{dT}{dr}$$

By integrating both sides from 0 to r , that is from the corresponding temperature T_0 to temperature T .

$$\int_0^r \frac{1}{t^2} \int_0^t (p_f - p_r) s^2 ds dt = - \int_{T_0}^T \kappa \frac{dT}{dr} dr$$

The right hand side integrating by parts :

$$\left[-t^{-1} \int_0^t (p_f - p_r) s^2 ds \right]_0^r + \int_0^r s (p_f - p_r) ds = \int_T^{T_0} \kappa dT$$

Considering that

$$\lim_{t \rightarrow 0} \frac{1}{t} \int_0^t (p_f - p_r) s^2 ds = 0$$

following integral equation is obtained :

$$\int_0^r s \left(1 - \frac{s}{r}\right) (p_f - p_r) ds = \int_T^{T_0} \kappa dT$$

From this integral equation a simple law of similarity may be derived as a good approximation ; for approximation let us assume the value of κ to be independent of density.

Our equation may then be rewritten as follows :

$$\int_0^r sp \left(1 - \frac{sp}{rp}\right) \frac{(p_f - p_r)}{p^2} d(sp) = \int_T^{T_0} \kappa dT$$

on the other hand

$$p_f(T, p) \equiv p^2 p_f^*(T)$$

$$p_r(T, p) \equiv p^2 p_r^*(T)$$

thus introducing the new variable $z = pr$

$$\int_0^z s \left(1 - \frac{s}{z}\right) [p_f^*(T) - p_r^*(T)] ds = \int_T^{T_0} \kappa dT \quad (2)$$

Consequently the relations depend only on the product pr and not separately on p and r , respectively.

Let us substitute for the temperature the new variable

$$y = \int_0^T \kappa dT = \frac{2}{7} \cdot 1,68 \cdot 10^{13} T^{7/2} \tag{3}$$

by using above approximation. The right hand side of equation 2., then becomes:

$$\int_T^{T_0} \kappa dT = \int_0^{T_0} \kappa dT - \int_0^T \kappa dT = y(0) - y(z)$$

Further, let us introduce the symbol:

$$p_f^*(T) - p_r^*(T) = F(y)$$

our integral equation may be written as:

$$y(z) + \int_0^z s \left(1 - \frac{s}{z}\right) F(y) ds = y(0) \tag{4}$$

In the case of calculating the thermal conduction by means of the SPITZER—HÄRM formula, y depends not only on T , but also on p , and thus the law of similarity does not strictly apply. Considering the portion of the function κ above the curve a . (Fig. 3) — incidentally it is also the condition at which the formula may be applied — the approximate value of y is:

$$y = \int_0^T \kappa(T, p) dT \sim \frac{7,95 \cdot 10^{12}}{\ln(qC^2)} T^{7/2} \frac{W}{m}$$

3. Convergence of the integral equation iteration

The kernel of the equation being positive, for y following limit may be given:

$$y_m - y_0 - \frac{z^2}{6} \text{Max } F(y) \leq y \leq y_0$$

As point of departure for the iteration let us assume the temperature to be constant:

$$y^0(z) \equiv y(0)$$

$$y^{(n+1)}(z) = y(0) - \int_0^z s \left(1 - \frac{s}{z}\right) F[y^{(n)}(s)] ds \tag{5}$$

For the error remaining after each step following limit may be given :
As definition let it be that

$$M(z) = \text{Max}_{y_m < y < y_0} \left| \frac{dF(y)}{dy} \right|; \quad M(z) \text{ monotonous growing}$$

$$\begin{aligned} |y^{(n+1)}(z) - y^{(n)}(z)| &= \left| \int_0^z s \left(1 - \frac{s}{z}\right) \left\{ F[y^{(n)}] - F[y^{(n-1)}] \right\} ds \right| \\ &\leq \int_0^z s \left(1 - \frac{s}{z}\right) \left| F[y^{(n)}] - F[y^{(n-1)}] \right| ds \\ &\leq \int_0^z s \left(1 - \frac{s}{z}\right) M(s) |y^{(n)}(s) - y^{(n-1)}(s)| ds \\ &\leq M(z) \int_0^z s \left(1 - \frac{s}{z}\right) |y^{(n)}(s) - y^{(n-1)}(s)| ds \end{aligned}$$

By inserting the product $\delta^{(0)}(z, m)$

$$\delta^{(0)}(z, m) = |y^{(1)}(z) - y^{(0)}(z)| = \frac{z^2}{6} y_0 = \frac{z^2}{6} F_0$$

$$\delta^{(n+1)}(z, m) = M(m) \int_0^z s \left(1 - \frac{s}{z}\right) \delta^{(n)}(s, m) ds$$

it is evident that

$$|y^{(n+1)}(z) - y^{(n)}(z)| \leq \delta^{(n)}(z, z)$$

by Laplace transformation with respect to the variable z

$$\begin{aligned} z \overline{\delta^{(n+1)}(z, m)} &= \frac{M(m)}{p^2} z \overline{\delta^{(n)}(z, m)} \\ z \overline{\delta^{(n)}(z, m)} &= \left(\frac{M(m)}{p^2} \right)^2 z \overline{\delta^{(0)}(zm)} \\ &= \left(\frac{M(m)}{p^2} \right)^2 \frac{z^3}{6} F_0 \\ &= \left(\frac{M(m)}{p^2} \right)^2 \frac{F_0}{p^4} \end{aligned}$$

$$z \delta^{(n)}(z, m) = F_0 [M(m)]^n \frac{z^{2n+3}}{(2n+3)!}$$

$$\delta^{(n)}(z, z) = \frac{F_0}{z [M(z)]^{3/2}} \frac{[z \sqrt{M(z)}]^{2n+3}}{(2n+3)!}$$

$$|y - y^{(n)}| \leq \sum_{h=0}^{\infty} |y^{n+h} - y^{n+h-1}| \leq \sum_{h=0}^{\infty} \delta^{(n+h)}(z, z)$$

$$|y(z) - y^{(n)}(z)| \leq \frac{F_0}{z [M(z)]^{3/2}} \sum_{n=h}^{\infty} \frac{[z \sqrt{M(z)}]^{2h+3}}{(2h+3)!}$$

4. Quantitative discussion of the first order approximations

Let the temperature be constant in a zero order approximation

$$y^0(z) = y(0) = y_0$$

then according to equation (5) :

$$y(z) \equiv y^{(1)}(z) = y_0 - \frac{F_0 z^2}{6}$$

or in a somewhat rewritten form :

$$y(z) = y_0 \left[1 - \left(\frac{z}{z_{kr}} \right)^2 \right]$$

where as a definition :

$$z_{kr} = \sqrt{\frac{6 y_0}{F_0}} = \sqrt{\frac{6 y(T_0)}{p_f(T_0) - p_1(T_0)}}$$

and z_{kr} is called the characteristic radius.

Introducing the variable T from equation (3):

$$T(z) = T_0 \left[1 - \left(\frac{z}{z_{kr}} \right)^2 \right]^{3/2}$$

For $z_{kr} = (pr)_{kr}$ the temperature falls to zero. The true value of (pr) which — let us say — at a temperature of $T \approx 2000^\circ \text{K}$ is allowable, — is certainly smaller. The plot of $T(z)$ is shown in Fig. 4.

The value of the characteristic radius depends — though in a small degree — also on the density given by the exact value of α . Further, again slightly, it depends on the value of T_0 taken at point $r = 0$. The following Table shows this dependence for 1 kat.

p	kat	1	1	1
T_0	keV	5	10	20
$N_D + N_T$	m^{-3}	$6,12 \cdot 10^{22}$	$3,06 \cdot 10^{22}$	$1,53 \cdot 10^{22}$
p_f	Wm^{-3}	$5,1 \cdot 10^9$	$1,6 \cdot 10^{10}$	$1,26 \cdot 10^{10}$
p_r	Wm^{-3}	$4 \cdot 10^9$	$1,6 \cdot 10^9$	$5 \cdot 10^8$
$p_f - p_r$	Wm^{-3}	$1,1 \cdot 10^9$	$1,1 \cdot 10^{10}$	$1,2 \cdot 10^{10}$
y	Wm^{-1}	$2,8 \cdot 10^{14}$	$2,85 \cdot 10^{15}$	$2,9 \cdot 10^{16}$
r_{kr}	m	1250	1250	3850

It can be seen that even by choosing a pressure of 1 kat, the radius will still be of the order of magnitude of km. s. At lower pressures this value obtained would be proportionally greater.

The high power production too, is due to the great size. In zero order approximation will be :

$$P_f = \frac{4\pi r_a^3}{3} p_f(T_0)$$

for $T_0 = 10$ keV and $p = 1$ kat it is given as

$$P_f = 3,68 \cdot 10^{16} W$$

The radiation power in first approximation is :

$$\begin{aligned} P_r &= \int_0^{r_k} 4\pi r^2 p_r(r) dr \approx 4\pi r_k^3 p_r(T_0) \int_0^1 \frac{r}{r_k} \left(\frac{T_0}{T}\right)^{3/2} d\left(\frac{r}{r_k}\right) = \\ &= \frac{3}{2} B\left(\frac{3}{2}; \frac{4}{7}\right) \frac{4\pi}{3} r_k^3 p_r(T_0) = 2,01 \frac{4\pi}{3} r_k^3 p_r(T_0) \end{aligned}$$

where

$$B(x, y) = \frac{\Gamma(x) \cdot \Gamma(y)}{\Gamma(x+y)}$$

and it becomes by substituting the foregoing values :

$$P_r = 3,7 \cdot 10^{16} W$$

About 10% of the total power production leaves in the form of radiation, the rest by thermal conduction.

The energies here involved are rendered useless from a technical point of view by their order of magnitude. Taking into account, that a power plant of 200 MW is considered to be a large one, we can see what technical problems would have to be solved, in order to remove an output of 1000 MW from every m^2 of the surface of a sphere with 1 km radius. It has to be noted that also the neutron energy is to be considered.

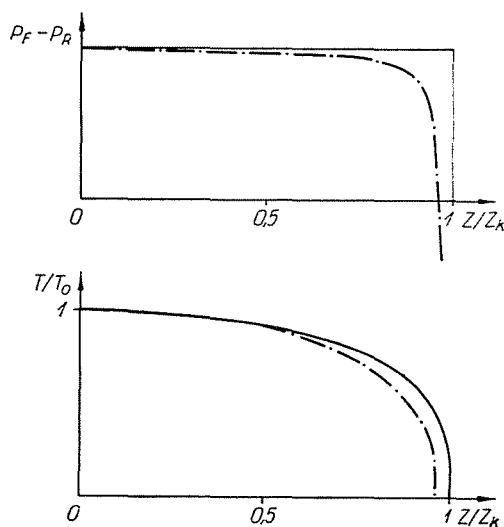


Fig. 4. Curves plotted for the first order approximation. The relative position of the exact curve is shown by the dashed line

Even if the conduction of 1000 MW could be realized — though with great difficulties — how and for what uses could the remaining energy be disposed of? This seems impossible on an earthly scale.

The conditions calculated at the first approximation do not essentially alter by further approximations. In Fig. 1 we have plotted the difference $p_f - p_r$, obtained by the first approximation. It can be seen that it is constant excepted in a thin shell; thus further iteration does not lead to a decisive alteration. For the exact solution the curve runs somewhat below the curve obtained for the first approximation (Fig. 4).

The value of the voltage taken at the center of the sphere and the point z_{kr} can be simply determined. According to SPITZER—HÄRM following relation may be written for current density j :

$$j = 0 = \sigma E + a \Delta T = -\sigma \frac{dU}{dr} + a \frac{dT}{dr}$$

From this

$$U = - \int_0^{T_0} \frac{a}{\sigma} dT = 0,72 \frac{k T_0}{e}$$

Let be $T_0 = 10$ keV then $V = 7,2$ kV.

This is a good value from a technical point of view : the tension of high power generators being of the same order of magnitude. In principle, if we could introduce an electrode in the center of the high temperature core, energy could be taken out of it as from a galvanic cell working according to diffusion law. A tension of the same character would occur at the steady state operation of a plasma surrounded by an electromagnetic wall, due to the small temperature-gradient.

5. Control of assumptions

Our speculations were based on the following assumptions :

- a) The composition of the gas does not change.
- b) The gas is in plasma state.
- c) There is a Maxwellian distribution of velocity.
- d) The radiation absorption being negligible, the plasma is transparent.
- e) The pressure is constant.
- f) The mean free path is so short that the thermal conduction may be calculated by applying the formula.
- g) The reaction products reach an equilibrium with the gas at the place of their origin.

ad a) Let us examine the consumption time of the total amount of gas in the case of constantly producing the initial energy.

$$\theta = \frac{\frac{N_D + N_T}{2}}{N_D N_T (\sigma v_{DT})_{av}} = \frac{2}{(N_D + N_T) (\sigma v_{DT})_{av}} = 0,67 \text{ sec}$$

From this it can be seen that our assumption is valid for only a few msecs at the most. At a lower energy production an equilibrium is reached, owing to the increasing radiation effects of the reaction products at a smaller temperature gradient, thus the critical radius will increase.

ad b) The gas is at such a high temperature in the major part of the volume, that the plasma state is maintained. The conditions are uncertain in the neighbourhood of the external wall, where at a few thousand degrees, also neutral particles may be found. Of course, this effects the thermal conductivity.

The events here occurring may somewhat enlarge the shell, at the same time, however, neither the dimensions, nor the produced energy will undergo any essential alteration.

ad c) The essential difference from the Maxwellian distribution is the following: many reactions occur per unit time resulting in a great number of high energy reaction products. By transferring their energies to the *D* and *T* nuclei, the distribution density will be shifted towards higher energies. As the temperature T_0 is chosen for the sake of stability near to the temperature for the maximum energy production, no appreciably higher production will be obtained, thus the decreasing in size cannot be expected.

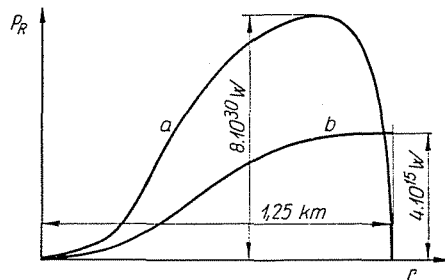


Fig. 5. The total radiated power a) according to the Stefan—Boltzmann law; b) according to the law of space radiation

ad d) In Fig. 5 we have plotted the total radiated power as evaluated according to the Stefan—Boltzmann law and on basis of the space radiation, respectively. It can be seen that our sphere is extremely transparent: the Stefan—Boltzmann formula giving a value that is higher by several orders of magnitude. Thus the absorption may again be neglected, except for a thin shell. The two radiation curves show an intersection at $T = 14 \text{ eV}$.

ad e) For some ten tons of material, obviously, the gravitation effects may be neglected. In order to estimate the radiation pressure let us suppose the total energy to be absorbed in the external wall. Thus

$$\Delta p < \frac{2}{c} \frac{P_r}{4 \pi r_k^2} \sim 0,093 \text{ at}$$

which may be completely neglected, as compared to 1000 at.

ad f) The thermal conduction law is only valid, as long as the relative change per unit mean free path is small, that is:

$$\left| \lambda \frac{d(\ln T)}{dr} \right| \ll 1$$

By determining the value of λ from [1] this condition may be written as :

$$\begin{aligned} \left| \lambda \frac{d(\ln T)}{dr} \right| &= \left| p \lambda \frac{d(\ln T)}{dz} \right| = \frac{T_0^3}{z_k} \frac{z}{z_k} \left[1 - \left(\frac{z}{z_k} \right)^2 \right]^{-1/2} = \\ &= 2,97 \cdot 10^{-12} \left[1 - \left(\frac{T}{T_0} \right)^2 \right]^{1/2} \left(\frac{T_0}{T} \right)^{1/2} \end{aligned}$$

for $T_0 = 20$ keV.

This value will be the unity for a temperature of 18 eV, thus immediately next to the wall, anywhere else it will be considerably smaller.

ad g) Similarly the reaction products, except for neutrons, will be stopped in the major part of the space on a path along which T does not appreciably change.

The mean free path of the reaction product ${}^4_2\text{He}$ is, in fact, of the order of several 10 meters. Approximation holds here only for a smaller fraction of space.

6. Conclusions

It can be seen that our assumptions hold, and that the results obtained for dimensions and power production may be considered true as regards to the order of magnitude. An improvement towards the approach of more real values cannot be expected from further refinement of calculations. By letting our imagination run loose, we may suggest the possibility of an artificial star, besides the artificial planet to be realised in the near future. In principle, thus a "black" wall temperature and radius might be found at which the total power output will be dissipated, and the problem of removing such an extremely large power does not arise. For the present, however, the resulting wall temperature makes the realisation quite illusory, even in the case of a sudden improvement in the heat resistivity of materials.

Summary

The possibility of the steady state of a deuterium tritium plasma-mixture, confined at constant pressure in a spherical container, is considered.

The temperature chosen for the plasma sphere center is in the vicinity of the temperature belonging to the maximum of thermonuclear energy production; the temperature of the external wall being fixed at about a few thousand degrees. Although the dimensions obtained are on an earthly scale, technically the realization of a fusion reactor of this type is found to be impossible.

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