EXPLANATION OF THE SO-CALLED "SOUND DETONATION"

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A few months ago a great detonation was heard in the neighbourhood of the Polytechnic University in Budapest. It caused no damage. The explication of this phenomenon given by the newspapers was by no means clear. The explosion was probably effected by a jet engine. The real explanation is as follows.

The question is: What should be the path and speed of a jet plane if there is a point in space to which the sound waves emitted by the jet engine along its course arrive simultaneously (from every point of its path).

First the movement will be assumed to be a two-dimensional one. The following denotations will be used (t. Fig. 1):

equation of the orbit in polar coordinates: $r = r(\varphi)$,

initial position of the airplane (at time $t = 0$) : P_0,

position at time $t : P$,

center of the sound detonation occurring[1] at time $t' > t : A$,

speed of the sound: c,

speed of the airplane (assumed to be constant): v,

length of the arc $\widehat{P_0P}$: s,

length of the radius vector of P_0 resp. P: r_0 resp. r.

Thus we have

$$r_0 = c\,t', \quad r = c\,(t' - t), \quad t = \frac{s}{v}.$$

Hence

$$r = c\,t' - c\frac{s}{v} = r_0 - \frac{c}{v} s .$$

By derivation[2]

[1] A is assumed to be in the plane of the movement.

[2] Equation (1) expresses the fact that the tangent of the path must intersect the radius vector at a constant angle depending on $\frac{c}{v}$.

$$\frac{dr}{d\varphi} = -\frac{c}{v}\frac{ds}{d\varphi} = -\frac{c}{v}\sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2}\,, \tag{1}$$

whenee

$$(v^2 - c^2)\left(\frac{dr}{d\varphi}\right)^2 = c^2 r^2\,.$$

This equation is meaningless unless $v > c$, i. e. the speed of the airplane exceeds

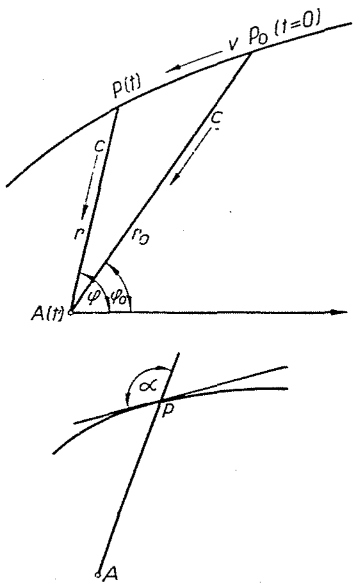

Fig. 1

the velocity of the sound. — The differential equation (1) is of the simplest type and has the general solution

$$r = k\,e^{\pm\frac{c}{\sqrt{v^2-c^2}}\varphi}\,,$$

(k is an arbitrary positive constant) and thus we come to the conclusion that : in order to cause a sound detonation the airplane must fly on a logarithmical spiral around the point A and the tangent of the path must intersect the radius vector at an angle α for which

$$\operatorname{tg}\alpha = -\frac{\sqrt{v^2 - c^2}}{c} \quad \text{or} \quad \cos\alpha = -\frac{c}{v}\,. \tag{2}$$

The airplane must travel along this track toward point A.

An alternative proof[3] may be based on the fact that during a time dt the ways covered by the airplane resp. the sound are $v \cdot dt$ resp. $c \cdot dt$ and that thereafter the sound has the same ways $P'A = QA$ to cover. Hence

$$\cos(\pi - \alpha) = -\cos \alpha = \frac{c}{v}$$

and — as known — a curve intersecting all the radius vectors at a constant angle can be nothing else than a logarithmical spiral. Provided $v = c, \alpha = 0$ or π.

Assuming $\alpha = 0$ the jet plane will move away from A in a straight line and no sound-detonation occurs.

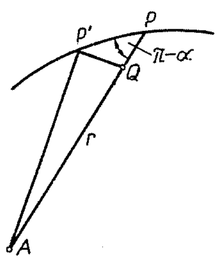

Fig. 2

Supposing $\alpha = \pi$, the airplane approches to point A at the velocity of the sound and causes a sound detonation when reaching A.

Finally, if the airplane is travelling toward point A at a speed $v > c$, no sound-detonation will occur, but it is interesting to note that the sound-signs sent out by the airplane arrive at A in reversed order.

The orbit tends to be a circular one provided $v \to \infty$.

Let the sound energy emitted per second by the airplane be denoted by J. Then the sound energy arriving simultaneously in the immediate neighbourhood of A may be easily computed:

$$E = \frac{J}{4\pi} \int_0^t \frac{dt}{r^2} = \frac{J}{4\pi v} \int_0^s \frac{ds}{r^2} = \frac{J}{4\pi v} \int_{\varphi_0}^{\varphi} \frac{1}{r^2} \frac{ds}{d\varphi} d\varphi$$

and making use of (1)

$$E = \frac{J}{4\pi v} \frac{v}{c} \int_{\varphi_0}^{\varphi} \left(\frac{-\frac{dr}{d\varphi}}{r^2} \right) d\varphi = \frac{J}{4\pi c} \left(\frac{1}{r} - \frac{1}{r_0} \right).$$

[3] Suggested by P. SONKOLY.

This energy can increase arbitrarily by the decrease of r but a very small r cannot be realised. What is the effect caused by the momentary appearance of the above energy at A? At all events a terrifying one, but the authors know nothing else about it.

Assuming the movement to take place in space let us denote the radius vector of P by $\bar{r} = \bar{r}(p)$ and its derivative according to the parameter p by $\dot{\bar{r}}$, thus we obtain from (1)

$$\frac{d|\bar{r}|}{dp} = -\frac{c}{v}|\dot{\bar{r}}| \tag{3}$$

and we are faced with the problem, wether there are or not — on an arbitrarily prescribed surface — paths satisfying (3) and what should the surface, the constant velocity v and the path be in order to produce a sound detonation.

Assuming the equation of the surface by means of two parameters p, q in the form

$$x = x(p, q), \quad y = y(p, q), \quad z = z(p, q) \text{ [4]}$$

and equation of the path to be $q = q(p)$ we have

$$|\bar{r}| = \sqrt{x^2 + y^2 + z^2}, \quad |\dot{\bar{r}}| = \sqrt{E + 2F\dot{q} + G\dot{q}^2}$$

where

$$E = x_p^2 + y_p^2 + z_p^2, \quad F = x_p x_q + y_p y_q + z_p z_q, \quad G = x_q^2 + y_q^2 + z_q^2$$

and substituting this in (3)

$$\frac{x(x_p + x_q\dot{q}) + y(y_p + y_q\dot{q}) + z(z_p + z_q\dot{q})}{\sqrt{x^2 + y^2 + z^2}} = -\frac{c}{v}\sqrt{E + 2F\dot{q} + G\dot{q}^2}\,.$$

This equation is an ordinary differential equation of the first order for $q = q(p)$ in the form of $F(p, q, \dot{q}) = 0$ and always has a solution either real or complex, since it is for $\dot{q}$ an algebraic equation of the second degree.
This may be written

$$(v^2H^2 - c^2r^2G)\dot{q}^2 + 2(v^2HI - c^2r^2F)\dot{q} + (v^2I^2 - r^2c^2E) = 0, \tag{4}$$

where

$$H = xx_q + yy_q + zz_q, \quad I = xx_p + yy_p + zz_p.$$

Equation (4) has real solutions provided

$$D = c^2r^2(F^2 - EG) + v^2(H^2E - 2HIF + I^2G) \geq 0.$$

[4] Assumed to be continuously derivable in p, q.

Therefore, assuming $D \geqq 0$ in a neighbourhood of P_0 (e. g. $F^2 - EG \geqq 0$, $H^2E - 2\,HIF + I^2G \geqq 0$), this point will be crossed by two (in limit case one) paths existing in the same region of P_0. Flying along these paths the airplane will produce a sound detonation in the origin.

Let us now examine the intuitive geometrical side of the problem.[5] On an arbitrary surface a curve is to be constructed starting from P_0 and intersecting all the radius vectors at an angle α for which $\cos\alpha = \dfrac{c}{v}$. Supposing that the normal vector $\bar{n}$ of the surface forms an angle β with the radius vector,

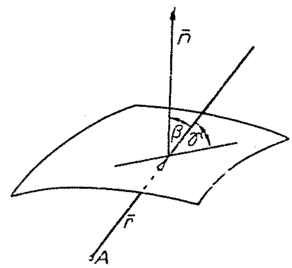

Fig. 3

all the tangents of the surface in P_0 form with the radius vector an angle γ for which $90° - \beta \leqq \gamma \leqq 90° + \beta$. Since P_0 has a neighbourhood where the condition $90° - \beta < \gamma < 90° + \beta$ is satisfied, we can construct step by step a polygonal path and from this by a limit passage a path starting from P_0 and intersecting all the radius vectors at the angle α, provided $90° - \beta < \alpha <$ $< 90° + \beta$.

An example. Let us determine the resp. paths on a circular cone. Let the axis of the cone be the axis z and the cylindrical coordinates of P be p, q and z. Since

$$x = q\cos p\,,\; y = q\sin p\,,\; z = z(p)\,,\; q = q(p)\,,\; |\bar{r}| = \sqrt{q^2 + z^2}\,,\; |\dot{r}| = \sqrt{q^2 + \dot{q}^2 + \dot{z}^2}$$

we obtain from (3)

$$\frac{q\dot{q} + z\dot{z}}{\sqrt{q^2 + z^2}} = -\frac{c}{v}\sqrt{q^2 + \dot{q}^2 + \dot{z}^2} \quad \text{or} \quad v^2(q\dot{q} + z\dot{z})^2 = c^2(q^2 + \dot{q}^2 + \dot{z}^2)(q^2 + z^2).$$

The surface will be a cone with its vertex in the origin, provided $z = \lambda q$, where λ is a constant. Then from the last equation

$$(\lambda^2 + 1)(v^2 - c^2)\,\dot{q}^2 = c^2 q^2\,,$$

[5] Suggested by E. MAKAI.

whence again $v > c$, and

$$q = k\, e^{\mu p} \quad \left(\mu = \frac{1}{\sqrt{\lambda^2+1}}\ \frac{c}{\sqrt{v^2-c^2}} \quad \text{and} \quad k > 0\right).$$

The equation of the orbit is

$$x = k\, e^{\mu p} \cos p\,,\ y = k\, e^{\mu p} \sin p\,,\ z = \lambda\, k\, e^{\mu p}\,,$$

i. e. the path is a helix on the cone and the tangent of the path intersects the radiusvector at a constant angle. Reversely, given this angle the cone will be determined. The only place of the sound detonation can be the vertex of the cone.

Case of variable velocity

I. *Movement in a plane*

1. Let the velocity be given a function of the way covered (i. e. of the length of the arc of the path) : $v = v(s)$. Then from equation

$$dr = -c\,\frac{ds}{v(s)} = -c\,dt$$

we have

$$r - r_0 = -c \int_0^s \frac{ds}{v(s)} = F(s) = -c\,t\,, \tag{1}$$

hence $s = \Phi(t)$. Since

$$\sqrt{v^2-c^2}\,\frac{dr}{dt} = c\,r\,\frac{d\varphi}{dt} \quad \text{and} \quad \frac{dr}{dt} = -c$$

we get

$$\frac{d\varphi}{dt} = -\frac{\sqrt{v^2-c^2}}{r} = -\frac{\sqrt{\bar v^2-c^2}}{r_0-c\,t}\,,$$

where

$$\bar v = \bar v(t) = v(\Phi(t))\,.$$

Finally,

$$\varphi - \varphi_0 = -\int_0^t \frac{\sqrt{\bar{v}^2 - c^2}}{r_0 - c\,t}\,d\,t \qquad \left(t < \frac{r_0}{c}\right). \tag{2}$$

Equations (1) and (2) yield φ and r as functions of t. Eliminating t we obtain the polar equation of the path. r depends always linearly on t.

2. Let the velocity be a prescribed function of the time: $v = v\,(t)$. Then we obtain in the same manner

$$r - r_0 = -\,c\,t \tag{1'}$$

$$\varphi - \varphi_0 = -\int_0^t \frac{\sqrt{v^2 - c^2}}{r_0 - c\,t}\,d\,t. \qquad \left(t < \frac{r_0}{c}\right) \tag{2'}$$

Let us prove the following statement:

In order to cause a sound detonation in the origin an arbitrary path may be chosen.

This can be seen very easily.

Let the path be given in the form of $r = r(\varphi)$. From equations

$$\frac{d\,r}{d\,\varphi} = -\frac{c}{v\,(\varphi)}\,\frac{d\,s}{d\,\varphi}, \quad s = s\,(\varphi), \quad \varphi = S\,(s)$$

$v\,(s)$ may be determined since $\frac{d\,r}{d\,\varphi}$, $\frac{d\,s}{d\,\varphi} = \sqrt{r^2 + \left(\frac{d\,r}{d\,\varphi}\right)^2}$ are known quantities. Then by $s = \Phi\,(t)$

$$v = v\,(s) = v\,(\Phi\,(t)) = \bar{v}\,(t)\,.$$

A sound detonation will occur provided the path will be run at this speed. Giving the path by a parameter τ in the form

$$r = r\,(\tau)\,, \quad \varphi = \varphi\,(\tau)$$

we obtain

$$r\,(\tau) = r\,(t) = r_0 - c\,t$$

whence

$$\tau = f\,(t) \quad \text{and} \quad \varphi = \varphi\,(f\,(t)) = \psi\,(t)$$

and comparing this to (2')

$$\varphi = \varphi_0 - \int_0^t \frac{\sqrt{v^2 - c^2}}{r_0 - c\,t}\,d\,t = \psi\,(t)$$

and by derivation

$$-\frac{\sqrt{v^2-c^2}}{r_0-c\,t}=\psi'(t)\,,$$

whence we find

$$v=v(t)=\sqrt{c^2+(r_0-c\,t)^2\,\psi'(t)^2}\,.$$

II. *Movement in the space*

1. Let us prescribe a surface and the velocity for the movement in all points :

$$x=x(p,q)\,,\ y=y(p,q)\,,\ z=z(p,q)\,,\quad v=v(x,y,z)=\bar{v}(p,q)\,,$$

then for the determination of the path on the surface we obtain the same differential equation as in case of a constant velocity. Only v should be replaced by $v(p, q(p))$. The equation will be

$$\dot{q}^2\,(v^2H^2-c^2r^2G)+2\,\dot{q}\,(H\,I\,v^2-c^2r^2F)+(I^2v^2-c^2r^2E)=0\,. \tag{3}$$

Every point P_0 of the surface will be crossed by two (in limit case by one) paths provided

$$D=c^2r^2(F^2-G\,E)+v^2(H^2E-2\,H\,I\,F+G\,I^2)\geqq 0 \tag{4}$$

in some neighbourhood of P_0.

Inversely, given the curve $q=q(p)$ on the surface, $v=v(p)$ may be computed from (3) provided $D\geqq 0$ in some region of P_0.

2. Let us take now an arbitrary curve of the space :

Prescribing the speed arbitrarily no sound detonation will occur. In order to produce the phenomenon, equation

$$\frac{d\,|\bar{r}|}{d\,\tau}=-\frac{c}{v(\tau)}\,|\dot{\bar{r}}|\qquad\left(\dot{\bar{r}}=\frac{d\,\bar{r}}{d\,\tau}\right)$$

must be satisfied. $v(\tau)$ may be determined herefrom since $\frac{d\,|\bar{r}|}{d\,\tau}$ and $|\dot{\bar{r}}|$ are known quantities. By means of the equations

$$t=\int_{\tau_0}^{\tau}\frac{1}{v(\tau)}\,\frac{d\,s}{d\,\tau}\,d\,\tau=\int_{\tau_0}^{\tau}\frac{1}{v(\tau)}\,|\dot{\bar{r}}|\,d\,\tau=F(\tau)\,,\ \ \tau=\Phi(t)$$

we obtain

$$v = v(\tau) = v(\Phi(t)) = \bar{v}(t)$$

and thus

$$x = x(\Phi(t)) = \bar{x}(t)\,.$$

$y = \bar{y}(t)$, $z = \bar{z}(t)$ may be computed similarly.

Summary

This paper gives the explanation of the so called „sound detonation". It is proved here that a jet engine can produce a sound detonation covering an arbitrary path in the space and the speed depending on the path and necessary for this end is determined too.

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