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RESEARCH ARTICLE

Analysis of the explicit model predictive control for semi-active suspension

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Abstract

Explicit model predictive control (MPC) enhances application of MPC to areas where the fast online computation of the control signal is crucial, such as in aircraft or vehicle control. Explicit MPC controllers consist of several affine feedback gains, each of them valid over a polyhedral region of the state space. In this paper the optimal control of the quarter car semi-active suspension is studied. After a detailed theoretical introduction to the modeling, clipped LQ control and explicit MPC, the article demonstrates that there may exist regions where constrained MPC/explicit MPC has no feasible solution. To overcome this problem the use of soft constraints and combined clipped LQ/MPC methods are suggested. The paper also shows that the clipped optimal LQ solution equals to the MPC with horizon $N = 1$ for the whole union of explicit MPC regions. We study the explicit MPC of the semi-active suspension with actual discrete time observer connected to the explicit MPC in order to increase its practical applicability. The controller requires only measurement of the suspension deflection. Performance of the derived controller is evaluated through simulations where shock tests and white noise velocity disturbances are applied to a real quarter car vertical model. Comparing MPC and the clipped LQ approach, no essential improvement was detected in the control behavior.

Keywords

Explicit model predictive control · soft constraints · combined clipped LQ control/MPC · deterministic actual observer · passivity and saturation constraints · semi-active suspension · magneto-rheological (MR) damper.

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1 Introduction

The automotive suspension supports the vehicle body on the axles and provides good ride quality against the road disturbances while keeps good road holding. In the future cars the intelligent suspension is part of a vehicle dynamic control system [19]. In the newest, luxury cars one may change the vehicle characteristic by pushing a button. The drive feeling can be set to a comfort mode as in a limousine, to a sporty mode, or to automatic. The system influences the characteristic of gear-change, steering, motor and suspension.

The quarter car suspension model is adequate to analyze the car response to irregular road surface and design an approximately optimal suspension controller to increase the good drive feeling. The performance of the suspension in the time domain can be expressed by $\mathcal{L}_2$ norm. The suspension can be classified into three groups according to operation: passive, semi-active, active suspension. Passive suspension consists of only spring and dampers, the semi-active utilizes variable damper and in the active suspension hydraulic, air or electric actuator force is applied. The semi-active suspension has simpler mechanical structure than the active one, requires power only to change the dissipative force characteristic and it cannot become unstable because the semi-active suspension is a passive system. Due to its many advantageous properties the automotive industry builds the semi-active suspension often into the top vehicles. Besides the automotive industry, the semi-active dampers can also be used in buildings to compensate the oscillation during earthquakes and anywhere where the vibration is undesirable.

Recently, based on the analogy between the electrical and mechanical circuits, a new mechanical circuit element the inerter has been developed and applied to vehicle suspension with success. The first deployment of the inerter under the name J-damper happened in the McLaren Formula One Racing team, leading to significant performance gains in handling and grip [7]. Examples mentioned previously show that the research area of the controlled dampers is very active allowing new damper technology and new control methods.

Although lots of modern control methods exist, only little can treat constraints in efficient way. The main objective of the optimal control is to determine the solution of the infinite-horizon linear quadratic regulator problem with constraints (CLQR) which was studied by many researchers in the past few years [3, 6, 8, 9, 15, 35, 37]. The solution can be approximated by repeatedly solving constrained finite horizon optimal control problems in a receding horizon fashion which is also called model predictive control (MPC) and accepted mainly in the process industry. Unfortunately the online solution of the time consuming quadratic (QP) and linear (LP) programs limits the application of MPC mainly to processes with slow dynamics.

To overcome this limitation, the method of multi-parametric programming can be applied to "pre-calculate" the solution of the finite horizon CLQR problem in the explicit piecewise affine form. This technique enlarges the scope of applicability of MPC, allows insight into the controller structure and ensures to detect the reachable states and fault operations in advance. A serious drawback of the explicit solution of the MPC lies in the exponential growing of the number of control regions when the prediction horizon is increasing. New research directions study efficient searching algorithms to choose the feedback gains [10, 20, 38] and/or develop techniques to reduce the number of regions [22, 25]. A new approach to overcome of the above difficulty applies some approximation of the explicit MPC controller using polynomial approximation [24].

Disadvantage of the MPC comes from the fact that MPC is an optimization based control design method under constraints and there are situations where no solution exists, i.e. the controller cannot give any control action, which is forbidden in a real system. To treat this problem one can use non-MPC type of controllers in such control regions. Alternatively, the constraints can be softened. In the paper we will derive such controllers for the semi-active suspension. Furthermore the MPC, which has "larger complexity", will be compared to the clipped LQ controller which has "simpler complexity". Explicit MPC desires full state measurement which is usually not possible in practical applications. In order to overcome this problem the paper suggests a discrete time deterministic (actual) observer connected to the explicit MPC controller which requires only measurement of the suspension deflection. The results are presented through simulation of a real quarter car model.

The remainder of this paper is organized as follows. Section 2 introduces the model of the semi-active suspension and the passivity constraints. Section 3 summarizes the theoretical background of the mixed logical dynamical (MLD) systems. Explicit MPC and multi-parametric programming are discussed in Section 4. Section 5 presents the explicit MPC of the semi-active suspension and analyses some properties of the controller. In Section 6 the discrete time actual observer is derived and connected to the explicit MPC. In the last two sections simulation results show efficient working of the controller. Section 7 concludes the paper.

2 Quarter car model of the semi-active suspension and the optimal control problem

Motion equations of a two degree of freedom quarter car in Fig. 1 can be described as

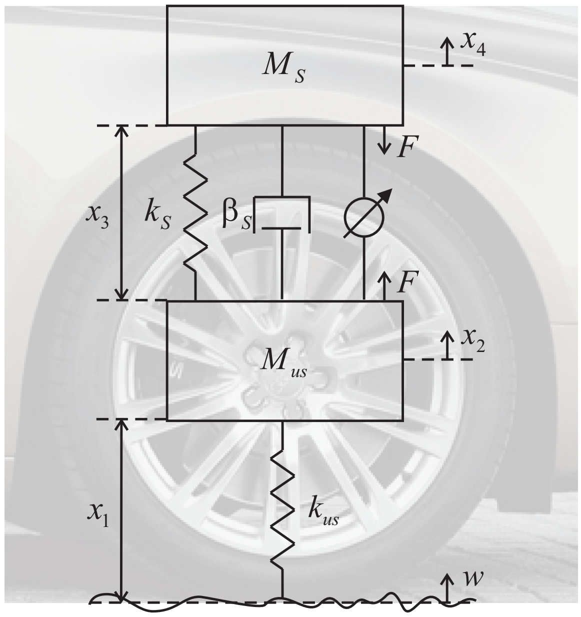

Fig. 1. Semi-active quarter car model

$$
\begin{aligned}
\dot{x}_1 &= x_2 - w \\
\dot{x}_2 &= \frac{1}{M_{us}} \left[k_s x_3 + \beta_s (x_4 - x_2) - k_{us} x_1 + F \right] \\
\dot{x}_3 &= x_4 - x_2 \\
\dot{x}_4 &= \frac{1}{M_s} \left[-k_s x_3 - \beta_s (x_4 - x_2) - F \right] \qquad (1)
\end{aligned}
$$

where M_s, M_{us} are the sprung and unsprung mass, respectively, k_s, k_{us} $[N/m]$ are the spring stiffness coefficients, β_s $[N/m/s]$ is the damping coefficient, x_1 $[m]$ is the tire deflection, x_2 $[m/s]$ is the unsprung mass velocity, x_3 $[m]$ is the suspension deflection, x_4 $[m/s]$ is sprung mass velocity, F $[N]$ is the adjustable force and w $[m/s]$ is the road velocity disturbance. Fig. 2 shows the nonlinear functions of the front suspension of a Renault Mégane Coupé [29] which are approximated in the model so that the spring force ($F_{k_s} = k_s x_3$) and the damping (dissipative) force ($F_{\beta_s} = \beta_s \dot{x}_3$) linearly depend on the deflection and the deflection speed, respectively. In the literature nonlinear approximations of the functions are also available for instance
$F_{k_s} = k_s x_3 + k_s^{nl} x_3^3$ and
$F_{\beta_s} = \beta_s \dot{x}_3 - \beta_s^{sym} |\dot{x}_3| + \beta_s^{nl} \sqrt{|\dot{x}_3|} sgn(\dot{x}_3)$ respectively [12]. Shaping the damping force in a passive suspension determines the dynamic and drive feeling of the car which is carried out through complex steps by the manufacturer.

The following normalized parameters will be introduced: sprung-to-unsprung mass ratio ρ, sprung mass-, wheel-hop natural frequencies ω_s, ω_{us} $[rad/s]$ and the normalized adjustable

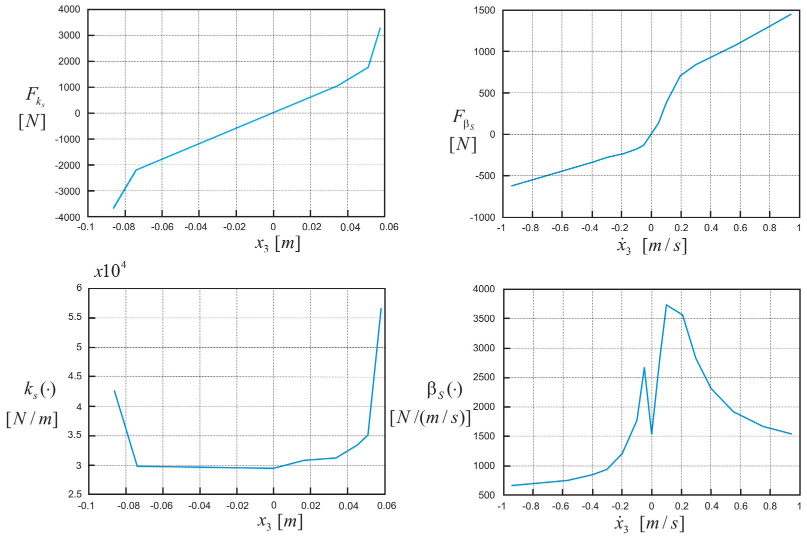

Fig. 2. Nonlinear spring- (left) and damping force (right) and their derivatives of a Renault Mégane Coupé [29]

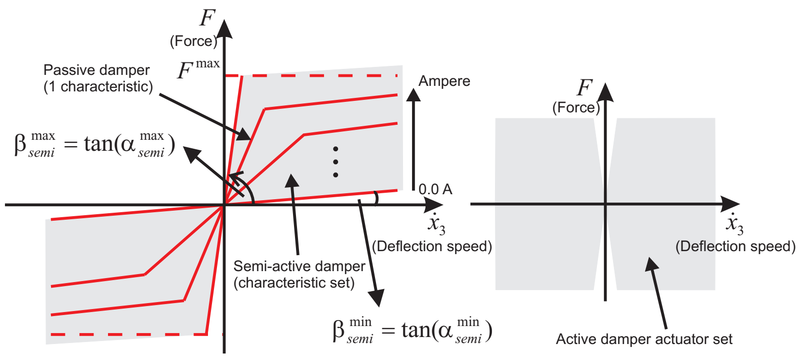

Fig. 3. Speed/Effort Rule (SER) of a passive, semi-active (left) and active (right) suspension system.

force $u\ [N/kg]$ which imply the normalized damping coefficient $\zeta = \beta_s/(2\sqrt{M_s k_s})$ to obtain numerically better conditioned state equations:

$$\dot{x}_2 = -\underbrace{\frac{k_{us}}{M_{us}}}_{\omega_{us}^2} x_1 - \underbrace{\frac{\beta_s}{M_{us}}}_{2\rho\zeta\omega_s} x_2 + \underbrace{\frac{k_s}{M_{us}}}_{\rho\omega_s^2} x_3 + \underbrace{\frac{\beta_s}{M_{us}}}_{2\rho\zeta\omega_s} x_4 + \underbrace{\frac{M_s}{M_{us}}}_{\rho} \underbrace{\frac{F}{M_s}}_{u}$$

$$\dot{x}_4 = \underbrace{\frac{\beta_s}{M_s}}_{2\zeta\omega_s} x_2 - \underbrace{\frac{k_s}{M_s}}_{\omega_s^2} x_3 - \underbrace{\frac{\beta_s}{M_s}}_{2\zeta\omega_s} x_4 - \underbrace{\frac{F}{M_s}}_{u}. \qquad (2)$$

According the Fig. 3, suspensions systems can be categorized into three groups. Passive suspension always dissipates energy through a fixed damping force characteristic. Also semi-active suspension can only dissipate energy but with varying damping force characteristic (left). Active suspension can both dissipate or generate energy using the almost total damping force plane depending on the actuator (right).

Due to their simple mechanical structure, low energy consumption, fast time response and low cost, the semi-active suspensions are more preferred in the industry than active ones when increasing the vehicle performance is required. The Magneto-Rheological (MR) (Fig. 4) damper is one of the most applied semi-active dampers which uses MR fluid (e.g. oil and ferro particles) whose viscosity, i.e. damping value β_{semi}, can be varied by applying magnetic field controlled by current. The

Fig. 4. Magneto-Rheological (MR) damper [29]

magnetic field orders the particles in such direction to increase the damping value. The damping characteristic can be controlled very accurately by changing the magnetic field. The semi-active suspension systems are passive systems, since the power consumption is required only for purposes of changing dissipative force characteristic in real-time, consequently they cannot become unstable. From another viewpoint, the semi-active suspension does not actively generate energy to the vibratory suspension system but only dissipates energy from it. Some researchers study the semi-active suspension system as bilinear system and the control input β_{semi} is used [30]. In this formulation the product of the states $(x_4 - x_2)$ and the control input β_{semi} appears in the model: $F = \beta_{semi}(x_4 - x_2)$ (see equations in (1)). The variable damper β_{semi} is constrained to

$$\beta_{semi}^{min} \leq \beta_{semi} \leq \beta_{semi}^{max}. \tag{3}$$

According to a recently applied more practical approach, the semi-active damper is simply modeled as a static map of the deflection speed-force, while the control input F has to satisfy the dissipativity and the saturation constraints (Fig. 3 (left)) [14, 29]. This interpretation provides linear state space model (1). In practice the inverse model of the real actuator is still needed to determine the current to the calculated force that finally modifies the damping coefficient.
Since the semi-active damper ensures stability, our aim is to achieve performance requirements. The model predictive control (MPC) results good performance while trying to satisfy the constraints. The name MPC includes the controller design technique, namely, one needs a model to predict the future behavior of the plant and the optimization is based on the predicted future of the plant. The semi-active suspension system can be modeled as

$$\begin{aligned}\dot{x} &= Ax + Bu + B_w w \\ y_{perf} &= \dot{x}_4 = C_{perf}x + D_{perf}u \\ y_{obs} &= x_3 = C_{obs}x\end{aligned} \tag{4}$$

where the state space matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\omega_{us}^2 & -2\rho\zeta\omega_s & \rho\omega_s^2 & 2\rho\zeta\omega_s \\ 0 & -1 & 0 & 1 \\ 0 & 2\zeta\omega_s & -\omega_s^2 & -2\zeta\omega_s \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \rho \\ 0 \\ -1 \end{bmatrix}, \quad B_w = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad D_{perf} = \begin{bmatrix} -1 \end{bmatrix},$$

$$C_{perf} = \begin{bmatrix} 0 & 2\zeta\omega_s & -\omega_s^2 & -2\zeta\omega_s \end{bmatrix},$$

$$C_{obs} = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}. \tag{5}$$

The output y_{perf} (sprung mass acceleration) is used for designing the MPC controller and the suspension deflection y_{obs} is the only measured (observed) output. The semi-active damper is modeled as a static map (Fig. 3) which determines the achievable forces (constraints). Dissipating power constraints are considered:

$$\begin{aligned}&\text{if } (\dot{x}_3 = x_4 - x_2) \geq 0 \\ &\quad \beta_{semi}^{min}(x_4 - x_2) \leq u \leq \beta_{semi}^{max}(x_4 - x_2) \\ &\text{if } (\dot{x}_3 = x_4 - x_2) \leq 0 \\ &\quad \beta_{semi}^{max}(x_4 - x_2) \leq u \leq \beta_{semi}^{min}(x_4 - x_2)\end{aligned} \tag{6}$$

The saturation constraints are:

$$u_{min} \leq u \leq u_{max} \tag{7}$$

Note that the constraints in Eq. (6) are state dependent. Consequently, the current control affects not only the future states of the system but it affects the future constraints of the force u through $(x_4 - x_2)$ as well. The range of the achievable control depends on the previous history of the control values. The performance index J contains a combination of the $y_{perf} = \dot{x}_4$ to reduce the vehicle body acceleration, x_1 to keep good road holding, and x_3 to hold the vehicle static weight [14]:

$$J = \int_0^\infty (q_1 x_1^2 + q_3 x_3^2 + \dot{x}_4^2)dt = \int_0^\infty (x^T Q_0 x + y_{perf}^2)dt, \tag{8}$$

where

$$Q_0 = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & q_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{9}$$

Substituting $\dot{x}_4^2$ from the state equations into integrand of the performance index (8), after some algebra we obtain the performance function in the usual form:

$$J = \int_0^\infty (x^T Q x + 2x^T N^T u + u^T R u)dt \tag{10}$$

where

$$Q = \begin{bmatrix} q_1 & 0 & 0 & 0 \\ \star & (2\zeta\omega_s)^2 & -2\zeta\omega_s^3 & -(2\zeta\omega_s)^2 \\ \star & \star & \omega_s^4 + q_3 & 2\zeta\omega_s^3 \\ \star & \star & \star & (2\zeta\omega_s)^2 \end{bmatrix}, \tag{11}$$

$$N^T = \begin{bmatrix} 0 \\ -2\zeta\omega_s \\ \omega_s^2 \\ 2\zeta\omega_s \end{bmatrix} = B_4 A_{(4,:)}^T \triangleq S_0^T, \quad [R = 1]. \tag{12}$$

The linearized real quarter-car semi-active suspension parameters are listed in Table 1.

Next theorem gives the solution of the LQ optimal control problem with constraints [40]:

Theorem 1 *Assume the full state measurement is available. Then the optimal control u for the system (4-5) with the passivity and saturation constraints (6-7) and the performance function defined in (10) can be obtained as*

$$\dot{P} = -PA(x, P) - A^T(x, P)P + PR(x, P)P - Q(x, P) \tag{13}$$

$$u_{opt} = sat[-K_{semi}(P(t))x] = sat[-(B^T P(t) + S_0)x] \tag{14}$$

$$J = x_0^T P(0)x_0. \tag{15}$$

Tab. 1. *The linearized semi-active suspension parameters* [14, 29]

Parameter	Value	Description
T_s	10 ms	Sampling time
M_s	315 kg	Sprung mass
M_{us}	37.5 kg	Unsprung mass
k_s	29500 N/m	Suspension stiffness
k_{us}	208000 N/m	Tire stiffness
β_s	0 N/(m/s)	Suspension damping
β_{semi}^{min}	700 N/(m/s)	Susp. damping lower slope
β_{semi}^{max}	4000 N/(m/s)	Susp. damping upper slope
F^{max}	4000 N	Sat. constraint
x_1	$[-0.05, 0.05]$ m	Tire deflection
x_2	$[-5, 5]$ m/s	Unsprung mass velocity
x_3	$[-0.2, 0.2]$ m	Suspension deflection,
x_4	$[-2, 2]$ m/s	Sprung mass velocity
q_1	1100	Weight on tire deflection
q_2	100	Weight on susp. deflection
A_{road}	$4.9 \cdot 10^{-6}$	Road constant
v	88 km/h	Car velocity

2.1 Clipped optimal

It is important to note, that the matrix Riccati differential equation in Theorem 1 cannot be simplified to an algebraic Riccati equation ($P(t) = P$) in spite of tending of the final time to infinity because the saturation causes switchings of matrices $A(x, P)$, $R(x, P)$ and $Q(x, P)$ along the trajectory. Therefore by taking constant matrix $P(t) = P$ and consequently $\dot{P} = 0$ and solving an algebraic Riccati equation, only a *sub-optimal* solution is obtained which is called *clipped optimal* LQ solution in the literature. The name refers to the situation when the desired semi-active force u is clipped according to (14) whenever it exceeds its passivity or actuator limitation constraints (6-7). Note that semi-active force in Eq. (14) consists of two parts: one part is the desirable total suspension force $-B^T P(t)x$ and the other part $u_p = -S_0 x = -(\omega_s^2 x_3 + 2\zeta\omega_s(x_4 - x_2))$ cancels the passive spring and damper forces.

Without the passivity constraints (6) for u, the active suspension is obtained. In this case, $P(t) = P$ and the matrix Riccati equation leads to the same algebraic Riccati equation as in the clipped optimal control.

The analysis of the semi-active performance index relating to optimal active or passive control leads to two suboptimal control laws. The following theorem considers the relation between the performance of the optimal semi-active suspension and that of the optimal active suspension [40]:

Theorem 2 *The cost of the semi-active suspension is always greater than that of the optimal active suspension and the relation can be quantified such as*

$$J_{semi} = \underbrace{x_0^T P_a x_0}_{J_{active,LQR}} + \int_0^\infty (u_a - u)^2 dt, \tag{16}$$

subject to constraints (6-7).

Since the first term in the integral is independent of the control signal, therefore only the second term (whole integral) minimization is needed, which is not trivial. An approximating solution can be derived by the minimization of only the integrand. This approach leads to the *clipped LQ* suboptimal semi-active control law:

$$\begin{aligned} \frac{d}{du}\{(u_a - u)^2\} &= -2(u_a - u) = 0 \\ \frac{d^2}{(du)^2}\{(u_a - u)^2\} &= 2 > 0 \longrightarrow minimum \\ &\Downarrow \\ u &= sat[u_a]. \end{aligned} \tag{17}$$

2.2 Steepest Gradient Method

Another possibility is to consider the relation of the semi-active performance index to the optimal passive one [40].

Theorem 3 *The cost of the semi-active suspension can be smaller than that of the optimal passive suspension. Consider the system matrix A_{opt} with the optimal damping $\beta_s = \beta_{s,opt}$ then the relation is*

$$J_{semi} = \underbrace{x_0^T P_{p,opt} x_0}_{J_{passive,LQR,opt}} - \int_0^\infty (2u(B^T P_{p,opt} x - u_{p,opt}) - u^2) dt \tag{18}$$

subject to constraints (6-7), where $P_{p,opt}$ is the solution of the Lyapunov equation

$$A_{opt}^T P_{p,opt} + P_{p,opt}^T A_{opt} = Q + A_{opt,(4,:)}^T A_{opt,(4,:)}. \tag{19}$$

$A_{opt,(4,:)}$ denotes fourth row of the matrix A_{opt}. The optimal damping $\beta_s = \beta_{s,opt}$ can be determined by optimization. According to the theorem, the semi-active performance index is smaller only if the integral is positive. The maximization of the integral is not trivial therefore an approach can be the maximization of the integrand. This leads to another semi-active suboptimal control law:

$$\begin{aligned} \frac{d}{du}\{\ldots\} &= -(2(B^T P_{p,opt} x - u_{p,opt}) - 2u) = 0 \\ \frac{d^2}{(du)^2}\{\ldots\} &= -2 < 0 \longrightarrow maximum \\ &\Downarrow \\ u &= sat[(B^T P_{p,opt} + S_{0,opt})x]. \end{aligned} \tag{20}$$

Since the control law reduces the performance index in every time instant with maximum rate the control law is called *steepest gradient method* (SGM).

Both the clipped optimal control and the steepest gradient method try to minimize the integrand of the additional term at every time instant, while the optimal semi-active control law minimizes the whole integral.

2.3 Model Predictive Control (MPC)

The optimal control problem can also be formulated in discrete time where the controller requires only measurement of the suspension deflection. The semi-active damper is modeled as

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + B_w w(k), \\ y_{perf}(k) &= x_4(k+1) = C_{perf}x(k) + D_{perf}u(k), \\ y_{obs}(k) &= x_3(k) = C_{obs}x(k) \end{aligned} \tag{21}$$

with the constraints (6-7) and $x(k) \in [x_{min}; x_{max}]$. If the matrix pair $(A, C_{obs}A)$ is observable then a deterministic actual discrete time state observer can be designed to the system

$$\hat{x}(0) = [0 \dots 0]^T_{n_x} \tag{22}$$

$$\begin{aligned} \hat{x}(k) &= F\hat{x}(k-1) + Gy_{obs}(k) + Hu(k-1), \\ F &= A - GC_{obs}A, \quad H = B - GC_{obs}B, \\ x_e(k) &= x(k) - \hat{x}(k), \\ x_e(k) &= Fx_e(k) \longrightarrow \text{stable, fast.} \end{aligned} \tag{23}$$

In the above formulas we used the same letters for the system matrices as earlier in the continuous case but from now they mean discrete time matrices. Discrete time implementation of the performance function can be obtained simply using the rectangular rule:

$$J = x^T(k)Q_N x(k) + \sum_{k=0}^{N-1} (x^T(k)Qx(k) + y^2(k))T_s, \tag{24}$$

where Q is defined as in (9) and T_s is the sample time. In Eq. (24) we approximate the discrete time infinite-horizon LQ regulator problem under constraints (CLQR) as a finite time optimal control problem (with "short" horizon), which is solved repeatedly in a receding horizon fashion. At each time instant an open-loop finite time optimal control problem is solved and only the first optimal control command is applied to the process. At the next time step the finite time optimal control is again solved over a shifted horizon based on the measured or estimated state. This type of the controller is called Receding Horizon Controller (RHC).

If the finite time optimal control law is calculated by solving online optimization at each time step, then the control method is also referred as online MPC. The CLQR with quadratic or linear (1-norm, ∞-norm) performance index implies quadratic (QP) or linear program (LP) that can be solved online by efficient tools based on active-set or interior point method.

For the solution of the infinite-horizon constrained LQR there exists no general method yet.

Several researchers recognized, that the constrained finite time optimal control (CFTOC) with the choice $Q_N = P_\infty$ where P_∞ is the solution of the unconstrained infinite-horizon LQ problem, sometimes also yields the solution of CLQR [3,6,8,9,15,35,37]. The set of initial conditions $x(0)$ for which the equivalence holds, depends on the length of the horizon N. There exist algorithms to compute the sufficiently long horizon N for any compact set of the initial states, that solves the infinite time CLQR, assuming the constraints are inactive for $k \geq N$ since the cost from N to ∞ can be calculated by $x(N)Q_N x(N)$, where Q_N equals the solution of the unconstrained infinite horizon Riccati equation ($Q_N = P_\infty$). These algorithms usually yield large horizon N therefore large optimization problem should be solved. The horizon N is initial state dependent therefore they cannot be applied for offline computation of explicit controllers.

The state space model (21) and the constraints (6-7) can be transformed to a hybrid dynamical system. We will derive an explicit MPC, where we assume the measurement states are available, and we compute separately the observer for the controller implementation.

3 Theoretical Background of the Mixed Logical Dynamical Systems (MLD)

Dynamical systems that are described by an interaction between continuous and discrete dynamics are called *hybrid dynamical systems* or *hybrid systems* shortly. The interest in hybrid systems is motivated by practical situations, for example, when nonlinear complex process is modeled as linear hybrid/switched system or the real system has hybrid properties.

Every hybrid system can be described as *Discrete Hybrid Automata* (DHA) [39] where the continuous dynamics is given by *switched affine system* (SAS, consisting of linear difference equations) and whose discrete dynamics is represented by *event generator* (EG), *finite state machine* (FSM), *mode selector* (MS, consisting of logic expressions), both synchronized by the same clock. DHA models are mathematical abstractions of domain specific hybrid modeling like mixed logical dynamical models (MLD) [4], piecewise affine systems (PWA) [36], linear complementary systems (LC) [18,31], extended linear complementary systems (ELC) [33] and max-min-plus-scaling systems (MMPS) [11,34]. DHA is formulated in discrete time even if the effects of the sampling time could be neglected to avoid the so called Zeno behavior.

In continuous time hybrid modeling the Zeno behavior means that switching times have finite accumulation point, that is, the system can make infinitely many switching if it approaches to this time, which can not be allowed in a physical system. Unfortunately in a complex hybrid system it is not an easy task to detect accumulation points that may have more than one location. Zeno behavior is not possible in discrete time.

The key idea of the MLD approach is that the constraints and the logical statements can be embedded into the state equations by a transformation and the hybrid system can be expressed by mixed integer linear inequalities [32]. Boolean variable can represent simple statements, e.g. $[X_i = true] \leftrightarrow \left[a^T x \leq b\right]$, where $x, a \in \mathbb{R}^n$, $b \in \mathbb{R}$. One can associate with a Boolean variable X_i a binary (logical) variable: $[X_i = true] \Leftrightarrow [\delta = 1]$ and $[X_i = false] \Leftrightarrow [\delta = 0]$. Boolean algebra defines logical operators (e.g. "and" ($\wedge$), "implies" ($\Rightarrow$), "if and only if" ($\Leftrightarrow$), and "exclusive or" ($\oplus$) etc.) to describe compound statements. The basic relations can be expressed simply by linear inequalities involving binary variables such as $X_1 \Rightarrow X_2$ is equivalent to $\delta_1 - \delta_2 \leq 0$ or $X_1 \Leftrightarrow X_2$ is equivalent to $\delta_1 - \delta_2 = 0$.

Hybrid modeling combines the continuous dynamics with logic rules. Consider $f : \mathbb{R}^n \to \mathbb{R}$ and $x \in \mathcal{X}$, where $\mathcal{X}$ is bounded. Define $M = \max_{x\in\mathcal{X}} f(x)$ and $m = \min_{x\in\mathcal{X}} f(x)$. It is easy to derive, for example, such equivalences:

$$[f(x) \leq 0] \Leftrightarrow [\delta = 1] \text{ true iff } \begin{cases} f(x) \leq M(1-\delta) \\ f(x) \geq \varepsilon + (m-\varepsilon)\delta, \end{cases} \tag{25}$$

where $\varepsilon > 0$ denotes the machine precision. The product $\delta_1\delta_2$ may be replaced by an auxiliary variable $\delta_3 = \delta_1\delta_2$ and then the product can be equivalently expressed by

$$\delta_3 = \delta_1\delta_2 \quad \text{is equivalent to} \quad \begin{cases} -\delta_1 + \delta_3 \leq 0 \\ -\delta_2 + \delta_3 \leq 0 \\ \delta_1 + \delta_2 - \delta_3 \leq 1. \end{cases} \tag{26}$$

Finally, introduce the auxiliary real variable $y = \delta f(x)$ that satisfies $[\delta = 0] \Rightarrow [y = 0]$, $[\delta = 1] \Rightarrow [y = f(x)]$ or equivalently

$$y = \delta f(x) \quad \text{is equivalent to} \quad \begin{cases} y \leq M\delta \\ y \geq m\delta \\ y \leq f(x) - m(1-\delta) \\ y \geq f(x) - M(1-\delta). \end{cases} \tag{27}$$

Any logic statement of the *finite state machine* and the *mode selector* $X_n \leftrightarrow f(X_1, X_2, \ldots, X_{n-1})$ has an equivalent conjunctive normal form (CNF):

$$\bigwedge_{j=1}^{m} \left(\left(\bigvee_{i\in P_j} X_i \right) \bigvee \left(\bigvee_{i\in N_j} \sim X_i \right) \right), \qquad N_j, P_j \in \{1, \ldots, n\}, \tag{28}$$

which can be translated into the following set of integer inequalities:

$$\begin{aligned} 1 &\leq \sum_{i\in P_1} \delta_i + \sum_{i\in N_1} (1-\delta_i) \\ &\vdots \\ 1 &\leq \sum_{i\in P_m} \delta_i + \sum_{i\in N_m} (1-\delta_i). \end{aligned} \tag{29}$$

The *event generator*: $\left[\delta_e^i(k) = 1\right] \leftrightarrow \left[H^i x_c(k) \leq W^i\right]$ can be expressed as:

$$\begin{aligned} H^i x_c(k) - W^i &\leq M^i(1-\delta_e^i) \\ H^i x_c(k) - W^i &\geq m^i \delta_e^i. \end{aligned} \tag{30}$$

Similarly, the *switched affine system*

$$\begin{aligned} &\text{If} \quad [\delta = 1] \quad \text{then} \quad z = a_1^T x + b_1^T u + f_1 \\ &\text{else} \quad z = a_2^T x + b_2^T u + f_2 \end{aligned} \tag{31}$$

can be translated into mixed linear inequalities:

$$\begin{aligned} (m_2 - M_1)\delta + z &\leq a_2^T x + b_2^T u + f_2 \\ (m_1 - M_2)\delta - z &\leq -a_2^T x - b_2^T u - f_2 \\ (m_1 - M_2)(1-\delta) + z &\leq a_1^T x + b_1^T u + f_1 \\ (m_2 - M_1)(1-\delta) - z &\leq -a_1^T x - b_1^T u - f_1. \end{aligned} \tag{32}$$

By collecting the equalities and inequalities the DHA i.e. every hybrid system can be described generally as Mixed Logical Dynamical (MLD) system:

$$\begin{aligned} x(k+1) &= Ax(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) &= Cx(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \\ E_2\delta(k) + E_3z(k) &\leq E_4x(k) + E_1u(k) + E_5, \\ x &= \begin{bmatrix} x_c(t) \\ x_b(t) \end{bmatrix}, \quad u = \begin{bmatrix} u_c(t) \\ u_b(t) \end{bmatrix}, \end{aligned} \tag{33}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_b}$ is the vector of continuous and binary states, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_b}$ are the inputs, $y \in \mathbb{R}^{p_c} \times \{0,1\}^{p_b}$ are the outputs and $\delta \in \{0,1\}^{r_b}$, $z \in \mathbb{R}^{r_c}$ are introduced as auxiliary binary and continuous variables, respectively, for transforming logic relations into mixed-integer linear inequalities. The matrices have the suitable dimensions.

The translation procedure is supported by the toolbox YALMIP and the tool HYSDEL (Hybrid Systems DEscription Language) included in Multi-Parametric Toolbox (MPT). By using the Multi-Parametric Toolbox (MPT) we can design, analyze and deploy optimal controllers for constrained linear, nonlinear and hybrid systems [23]. HYSDEL allows modeling a class of hybrid systems described by interconnections of linear dynamic systems, automata, if-then-else and propositional logic rules with high-level textual description [39]. YALMIP is a modeling language for advanced description and solution of convex and nonconvex optimization problems, allowing the user to concentrate on the high-level model. YALMIP takes care of the low-level modeling to obtain as efficient and numerically sound models as possible [27,28]. The next theorem gives equivalence of the different type hybrid models [5,17].

Theorem 4 *Consider $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ are sets of states, inputs, and outputs respectively, and assume $\mathcal{X}$ and $\mathcal{U}$ are bounded. Then DHA, PWA, MLD, LC, ELC, and MMPS well-posed hybrid models are equivalent to each other on $\mathcal{X}$, $\mathcal{U}$, $\mathcal{Y}$, where two hybrid systems are called equivalent if for all initial conditions and all input provide the same state and output trajectory.*

The following example is taken from [32] and illustrates the statement of the theorem.

Example Consider a piecewise affine system (PWA) with constraints

$$x(k+1) = \begin{cases} 0.8x(k) + u(k) & \text{if} \quad x(k) \geq 0 \\ -0.8x(k) + u(k) & \text{if} \quad x(k) < 0 \end{cases}$$

$$-10 \leq x(k) \leq 10, \quad -1 \leq u(k) \leq 1. \tag{34}$$

which can be rewritten in MLD form:

$$x(k+1) = 1.6 \underbrace{\delta(k)x(k)}_{z(k)} - 0.8x(k) + u(k)$$

$$\begin{aligned} x(k) &\geq m(1-\delta(k)) \\ x(k) &\leq -\varepsilon + (M+\varepsilon)\delta(k) \\ z(k) &\leq M\delta(k) \\ z(k) &\geq m\delta(k) \\ z(k) &\leq x(k) - m(1-\delta(k)) \\ z(k) &\geq x(k) - M(1-\delta(k)), \end{aligned} \tag{35}$$

where $\varepsilon > 0$ is the machine precision and $m = -10$, and $M = 10$. The first two inequalities ensure $[\delta(k) = 1] \Leftrightarrow [x(k) \geq 0]$ (25) and the other come from the identity (27). The MLD system can be rewritten in general form:

$$x(k+1) = 1.6z(k) - 0.8x(k) + u(k)$$

$$\begin{bmatrix} 10 \\ -10 \\ 10 \\ -10-\varepsilon \\ -10 \\ 10 \end{bmatrix} \delta + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix} z \leq \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} x + \begin{bmatrix} 10 \\ 0 \\ 10 \\ -\varepsilon \\ 0 \\ 10 \end{bmatrix} \delta \tag{36}$$

Max-Min-Plus-Scaling (MMPS) systems model such hybrid systems where operations maximization, minimization, addition and scalar multiplication are used. The MMPS equivalent form is:

$$x(k+1) = -0.8x(k) + 1.6\max(0, x(k)) + u(k)). \tag{37}$$

Several applications, for example in the area of electrical networks and mechanical systems, lead to linear complementarity (LC) hybrid model. Two vectors of variables are called complementarity if for all pair of variables v_i, w_i the complementarity condition $0 \leq v \perp w \geq 0$ is satisfied that is equivalent to the condition $\{v_i = 0\} \vee \{w_i = 0\}$ for all i. A simple example of the complementarity variables is the ideal diode where the voltage drop across the diode and the current through it are complementarity variables.
The PWA system can also be transformed to LC system:

$$x(k+1) = -0.8x(k) + u(k) + 1.6z(k)$$

$$0 \leq w(k) = -x(k) + z(k) \perp z(k) \geq 0 \tag{38}$$

Generalization of the LC hybrid models is the extended LC systems. The PWA system can also be translated to ELC model:

$$x(k+1) = -0.8x(k) + u(k) + 1.6d(k)$$

$$\begin{aligned} & -d(k) \leq 0, \quad x(k) - d(k) \leq 0, \\ & (x(k) - d(k))(-d(k)) = 0, \end{aligned} \tag{39}$$

where $d(k) \in \mathbb{R}^r$ is real-valued auxiliary variable.
Remark For modeling of semi-active problems YALMIP is chosen because it allows to define arbitrary constraints to MPC setup and to define custom objective function. HYSDEL, which cannot treat for example soft constraints, does not ensure such flexible modeling as YALMIP.

Now we are in the position to describe the semi-active MPC problem:

$$J(\xi, \hat{x}(t)) = \hat{x}^T(N)Q_N\hat{x}(N) + \sum_{k=1}^{N-1} (\hat{x}^T(k)Q\hat{x}(k) + y_{perf}^2(k))T_s$$

subject to:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + B_1u(k) + B_2\delta(k) + B_3z(k) + B_5 \\ y(k) &= C\hat{x}(k) + D_1u(k) + D_2\delta(k) + D_3z(k) + D_5 \end{aligned}$$

$$E_2\delta(k) + E_3z(k) \leq E_4\hat{x}(k) + E_1u(k) + E_5,$$

$$\hat{x}(k) = F\hat{x}(k-1) + Gy_{obs}(k) + Hu(k-1), \tag{40}$$

where $\xi \triangleq \left[u_0^T, \ldots, u_{N-1}^T, \delta_0^T, \ldots, \delta_{N-1}^T, z_0^T, \ldots, z_{N-1}^T\right]^T$. The Riccati solution matrix of the infinite time unconstrained LQ problem is chosen for the terminal weight $Q_N = P_\infty$. Using the Multi Parametric Toolbox (MPT) in MATLAB the mixed optimization problem with constraints (40) can be translated to mixed integer quadratic program (MIQP) and solved online or offline if all states can be measured ($\hat{x}(k) = x(k)$). To calculate the explicit controller the MPT toolbox has interfaces to efficient commercial solvers such as for example CPLEX, NAG. We used CPLEX 10.0 version to calculate the explicit solution. Since the solution of the optimization problem (40) including the discrete time actual observer is very complicated, and is not known yet, therefore we design the controller separately from the observer.

4 Explicit MPC and multi-parametric programming

This part of the article overviews shortly the derivation of the multi-parametric program in the LQ optimal control. First consider a discrete LTI system without constraints:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k, \qquad x(0) = x_0 \end{aligned} \tag{41}$$

and the performance function to be minimized:

$$J(U_N, x_0) = \frac{1}{2}x_N^T Q_N x_N + \frac{1}{2}\sum_{k=0}^{N-1} x_k^T Q x_k + u_k^T R u_k$$
$$Q_N = Q_N^T, Q = Q^T \geq 0, R = R^T > 0, \tag{42}$$

where the notation $U_N = [u_0^T, \ldots, u_{N-1}^T]^T$ is used. Note that predicted and real states are distinguished by x_k and $x(k)$, respectively. It is well known that the solution of this standard unconstrained finite time optimal control problem is a time varying state feedback control law

$$u^*(k) = K_k x(k), \qquad k = 0, \ldots, N-1$$
$$K_k = -(B^T P_{k+1} B + R)^{-1} B^T P_{k+1} A, \tag{43}$$

where the matrix $P_k = P_k^T \geq 0$ are obtained recursively by the algorithm

$$P_N = Q_N$$
$$P_k = A^T (P_{k+1} - P_{k+1} B (B^T P_{k+1} B + R)^{-1} B P_{k+1}) A + Q \tag{44}$$

and the optimal cost is given by $J(x(0)) = x^T(0) P_0 x(0)$. Another approach of the problem is obtained by substituting recursive expression of the state equation:

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^j B u_{k-1-j} \tag{45}$$

in the cost function that derives to Least Squares (LS) problem

$$J(x(0)) = \frac{1}{2}x^T(0) Y x(0) + \min_{U_N} \frac{1}{2} U_N^T H U_N + x^T(0) F U_N, \tag{46}$$

where $H = H^T > 0$. The matrices H, F, Y can be obtained from Q, R and (45). Note that Y does not depend on U_N. The LS solution is

$$U_N = -H^{-1} F^T x(0). \tag{47}$$

From the control command and $x(0)$ one can constitute state feedback control law at time $t = 0$: $u(0) = K(0)x(0)$. Consequently one can solve the unconstrained finite time horizon problem in two different ways:

- solve the Riccati difference equation (44) that yields closed loop solution $u(k) = K_k x(k)$
- solve number of N open loop control problems with LS method ($t = 0, \ldots, N-1$) and form a state feedback control for each of state $x(k)$ and the calculated control command.

Next consider the constrained finite time optimal control problem (CFTOC)

$$J^*(x(0)) = \min_{U_N} J(U_N, x(0)) \tag{48}$$

$$\text{subject to:} \quad E x_k + L u_k \leq M$$
$$x_{k+1} = A x_k + B u_k$$
$$x_0 = x(0), \quad k = 1, \ldots, N-1$$
$$x_N \in \mathcal{X}_f, \tag{49}$$

where $\mathcal{X}_f \subseteq \mathbb{R}^n$ is a terminal polyhedral region. Applying the train of thought mentioned previously we obtain:

$$J(x(0)) = \frac{1}{2}x^T(0) Y x(0) + \min_{U_N} \frac{1}{2} U_N^T H U_N + x^T(0) F U_N$$
$$\text{s. t.:} \quad G U_N \leq W + E x(0), \tag{50}$$

where $H = H^T > 0$ and the matrices H, F, Y, G, W, E can be obtained from P, Q, R. Note that Y does not depend on U_N. Introduce the vector $z = U_N + H^{-1} F^T x(0)$ and transform the problem by completing squares to the equivalent problem

$$J_z(x(0)) = \min_z \frac{1}{2} z^T H z$$
$$\text{s. t.:} \quad G z \leq W + S x(0), \tag{51}$$

where $S = E + G H^{-1} F^T$ and $J_z(x(0)) = J(x(0)) - \frac{1}{2}x(0)^T (Y - F H^{-1} F^T) x(0)$. The optimization problem is a quadratic program (QP) if $x(0)$ is fixed and can be solved by QP solver online. To obtain a feedback control law number of N open loop QP must be solved and constitute the feedback form $u(k) = K_k x(k)$. Riccati solution does not exist for the CFTOC problem. MPC problem is different from CFTOC only that at each step an N long horizon problem must be solved i.e. the length of the horizon does not decrease because of the RHC technique. Online solution of the MPC is very time and resource consuming which is suitable only for slow processes, while for fast embedded systems only limited resources are available. Explicit solution to the MPC can overcome this problem [2, 6, 26]. Consider the quadratic program (51) as a multi-parametric program where the initial condition x_0 yields the parameter and apply the Karush-Kuhn-Tucker (KKT) condition:

$$H z^* + G^T \lambda^* = 0$$
$$\lambda_i^* (G_i z^* - W_i - S_i x_0) = 0$$
$$\lambda_i^* \geq 0. \tag{52}$$

The $\lambda_i^* \geq 0$ gives the active constraints $(G_i z^* - W_i - S_i x_0) = 0$ and the $\lambda_i^* = 0$ determines the inactive constraints $(G_i z^* - W_i - S_i x_0) < 0$. One can pick some feasible x_0 and solve the QP to calculate the optimal z^*, λ^*. Substitute these solutions to the KKT conditions for the *active constraints*

$$H z^* + \hat{G}^T \hat{\lambda}^* = 0$$
$$\hat{G} z^* - \hat{W} - \hat{S} x_0 = 0 \tag{53}$$

and express z^*, λ^* from the equations:

$$\begin{aligned} z^* &= -H^{-1}\hat{G}^T\hat{\lambda}^* \\ \hat{\lambda}^* &= -(\hat{G}H^{-1}\hat{G}^T)^{-1}(\hat{W} + \hat{S}x_0). \end{aligned} \tag{54}$$

The optimal control is affine function of the initial condition x_0:

$$\begin{aligned} z^*(x_0) &= H^{-1}\hat{G}^T(\hat{G}H^{-1}\hat{G}^T)^{-1}(\hat{W} + \hat{S}x_0) \\ &= Kx_0 + L \\ \hat{\lambda}^*(x_0) &= -(\hat{G}H^{-1}\hat{G}^T)^{-1}(\hat{W} + \hat{S}x_0) \\ &= Mx_0 + N \end{aligned} \tag{55}$$

Generally speaking the optimal control is affine function of the initial condition x_0 in some neighborhood of the initial condition. The region can be calculated by substitution of z^*, λ^* into the inequalities:

$$\begin{aligned} G\underbrace{(Kx_0 + L)}_{z^*(x_0)} &\le W + Sx(0) \\ \underbrace{Mx_0 + N}_{\hat{\lambda}^*(x_0)} &\ge 0, \end{aligned} \tag{56}$$

which yield a polyhedral critical region (CR) $R = \{x_0 \mid Ax_0 \le b\}$ (see Fig. 5 (left)). A polyhedron is a convex set expressed as the intersection of a finite number of closed half-spaces. Next step a new x_0 is picked in a neighborhood direction and calculated the new polyhedron (Fig. 5 (right)).

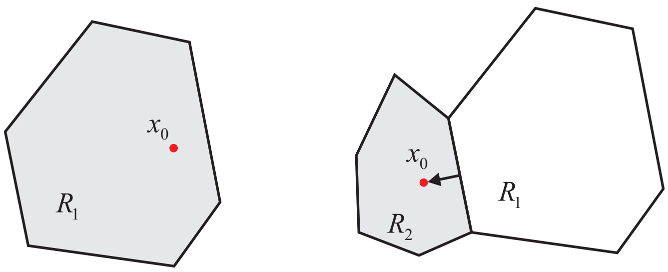

Fig. 5. Critical regions (CR) in multi-parametric programming

Apply the algorithm until the whole constrained state space is partitioned to polyhedra.

It can be shown that the *hybrid MLD optimal control problems*, such as for example the problem (40) without observer, can be translated into mixed integer quadratic program (MIQP) [4]. An explicit solution to such an MIQP can also be derived using multi-parametric programming, as shown in [6]. The notable difference being that several critical regions may overlap. In such a case the optimal control action is selected by taking the region in which the value of the performance objective is smallest.

The Multi-Parametric Toolbox (MPT) and HYSDEL allow to formulate and solve MPC problems for hybrid systems using a "high-level" approach. The MPT toolbox translates the problem into an MIQP form. Then, CPLEX can be used to calculate the optimal control action online. Alternatively, MPT allows to solve the MIQP using multi-parametric programming and to obtain the explicit representation of the feedback law, which can be implemented online easily as a look-up table. Evaluation of such a table boils down to identifying the controller region which contains the actual state measurements x_0.

The simplest searching algorithms are the sequential and binary tree approaches, respectively. The first method traverses the regions in a pre-determined order until the correct region is found. The second method constructs and evaluates a binary tree, which allows for faster region identification (Fig. 6 (left)). Unfortunately, the computation of explicit MPC controllers scales badly with increasing problem size. From a practical perspective, the procedure is applicable for systems with up to 4 state variables. Furthermore, we will see in the simulation that the another large drawback of the explicit (offline) control law is that the number of polyhedral regions grows dramatically with the prediction horizon and the number of constraints which decreases the practical applicability in embedded systems. For this reason a lot of efficient searching and storage algorithms have been developed [1, 10, 13, 16, 20, 21, 38, 41]. In [24] the key idea is that the optimal explicit piecewise affine controller is approximated by a single polynomial (Fig. 6 (right)), where number of the coefficients to be stored does not depend on the number of the regions. This type of controller does not require region storage and region identification. They prove the stability can be guaranteed and the constraints can also be satisfied.

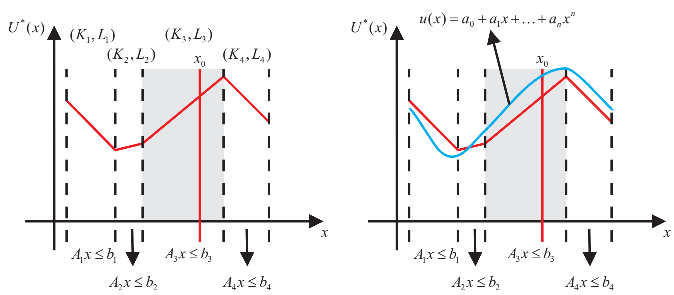

Fig. 6. Searching for the polyhedra containing of x_0 (left), polynomial approximation of the explicit control law (right)

5 Analysis of the MPC/Explicit MPC and the clipped optimal control for the semi-active suspension

This section illustrates the use of explicit MPC formulation for the semi-active suspension problem. It will also be shown that the clipped optimal LQ control is sub-optimal since it corresponds to the MPC for prediction horizon $N = 1$ [14, 40]. Furthermore, we analyze the explicit MPC, study some disadvantages of the MPC and the explicit MPC optimal control and give solutions how they can be treated. Fig. 7 shows the normalised dissipative and saturation constraints for the semi-active suspension which is nonconvex constraint but it can be described by the union of two polyhedral constraints:

Polyhedron $\mathcal{P}_1$:

$$\underbrace{\begin{bmatrix} 0 & 2\zeta_{max}\omega_s & 0 & -2\zeta_{max}\omega_s & 1 \\ 0 & -2\zeta_{min}\omega_s & 0 & 2\zeta_{min}\omega_s & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{H_1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ u_{max} \end{bmatrix}}_{K_1}$$

Polyhedron $\mathcal{P}_2$:

$$\underbrace{\begin{bmatrix} 0 & -2\zeta_{max}\omega_s & 0 & 2\zeta_{max}\omega_s & -1 \\ 0 & 2\zeta_{min}\omega_s & 0 & -2\zeta_{min}\omega_s & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{H_2} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ u \end{bmatrix} \leq \underbrace{\begin{bmatrix} 0 \\ 0 \\ -u_{min} \end{bmatrix}}_{K_2} \quad (57)$$

From physical setting up of the suspension the four states are also limited to $[x_{min}, x_{max}]$ i.e. to a state constraint hypercube. The explicit MPC requires also these bounds, otherwise the regions can be unbounded or the number of regions may be infinite. Consider the following state feedback MPC formulation to the semi-active suspension:

Objective/performance function:

$$J(x(0), U_N) = x_N^T Q_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + y_k^T y_k \quad (58)$$

subject to the constraints:

$$\begin{aligned} & x_{k+1} = Ax_k + Bu_k \rightarrow \text{linear prediction model} \\ & y_k = Cx_k + Du_k \\ & (x_k; u_k) \in (\mathcal{P}_1 \cup \mathcal{P}_2) \rightarrow \text{passivity and saturation} \\ & x_{min} \leq x_k \leq x_{max} \rightarrow \text{state hypercube} \\ & x_1 = x(0), \end{aligned} \quad (59)$$

where N is the (normalized) time horizon, we distinguish the current state $x(k)$ from the predicted state x_k and we denote U_N the open loop input sequence in the horizon. The MPC problem can be solved in online or offline (explicit) way, but naturally they yield the same result. The explicit MPC solution of the problem for $N = 1$ and the clipped LQ control from the initial condition $x_0 = \begin{bmatrix} 0 & 0 & 0.15 & -1.5 \end{bmatrix}^T$ are depicted in Fig. 8. The corresponding polyhedral partition of the state space is also shown. The continuous line trajectory corresponds to the MPC control while the trajectory marked with plus (+) gives the clipped LQ control. It can be seen well that the two trajectories are the same and the equivalence remains true also for initial conditions in such a region which is not in the neighborhood of the origin. Algebraic proof of the equivalence would be an interesting task. The statement was formulated in [14] without proof.

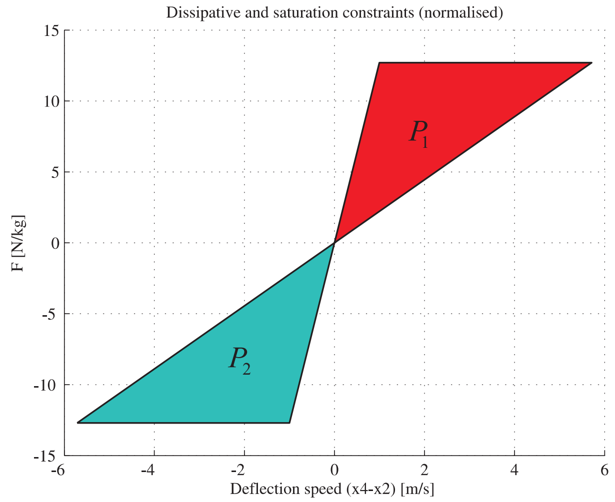

Fig. 7. Normalised dissipative and saturation constraints of the control signal

Theorem 5 *The clipped LQ controller is equivalent to the MPC with $N = 1$ and $Q_N = P_\infty$.*

The explicit MPC regions show in a descriptive way where the clipping will occur for the LQ control. By Fig. 8 eight regions were obtained which correspond to the 8 clipped LQ control states in Fig. 9:

$$u(x) = \begin{cases} 11.4220x_1 - 0.1753x_2 - 83.9268x_3 + 3.9330x_4 \\ = -K_{LQ}x \longrightarrow \text{regions \#1, \#5} \\ \\ 0x_1 - 2.2222x_2 + 0x_3 + 2.2222x_4 \\ = 2\zeta_{min}\omega_s(x_4 - x_2) \longrightarrow \text{regions \#2, \#6} \\ \\ 0x_1 - 12.6984x_2 + 0x_3 + 12.6984x_4 \\ = 2\zeta_{max}\omega_s(x_4 - x_2) \longrightarrow \text{regions \#3, \#7} \\ \\ -12.6984 = -F_{max}/M_s \longrightarrow \text{region \#4} \\ \\ 12.6984 = F_{max}/M_s \longrightarrow \text{region \#8} \end{cases} \quad (60)$$

It is important to note that the control laws have linear state feedback affine form.

Considering the explicit MPC calculated for $N = 3$, then the region marked with plus (+) in Fig. 10 shows one of the disadvantages of the MPC, namely, there are such regions in the space of the possible state vectors (state hypercube) where MPC can not give any control action, because no feasible solution of the MPC optimization problem exists. Such situation is not allowed in a real control system.

For example a disturbance may push the states outside the feasible region or the separately designed observer/filter may result such estimated states in the transient phase which may be outside the feasible region where no allowed control input exists.

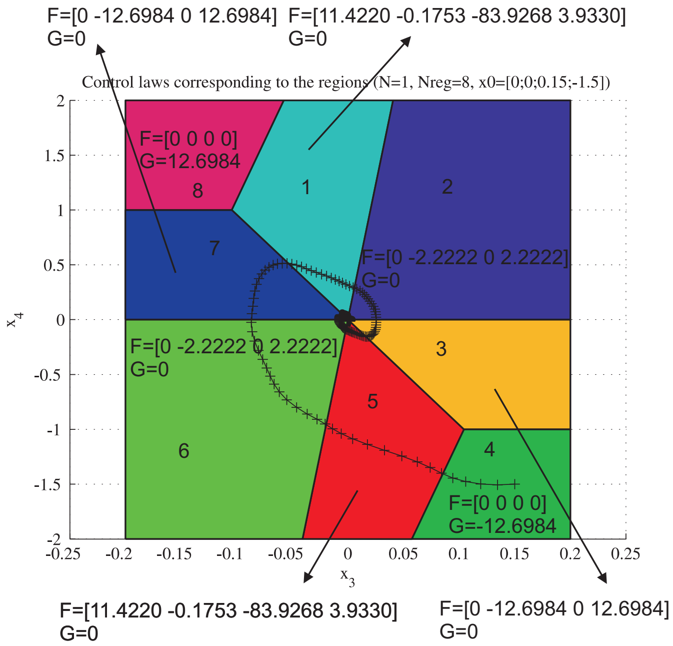

Fig. 8. Explicit MPC for N=1 and clipped LQ control state trajectory

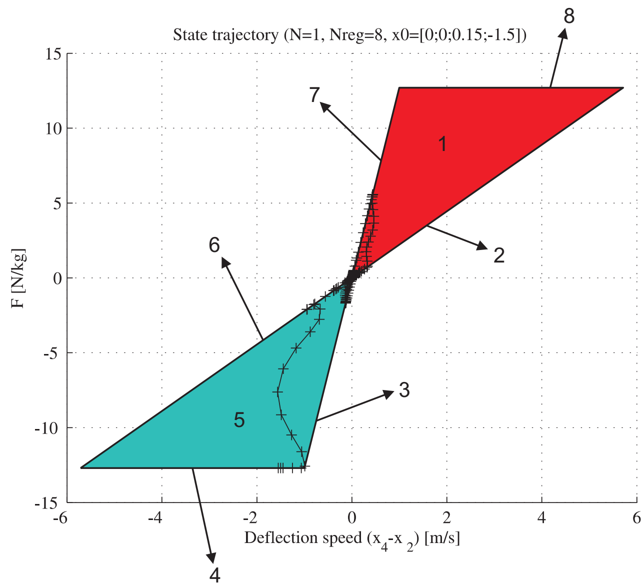

Fig. 9. Explicit MPC for N=1 and clipped LQ control state trajectory in the control polyhedra

Uncertainty in modeling of the semi-active suspension may also cause similar situations. Large dimensional complex shape of the state feasibility region inside of the state hypercube can not be treated easily with geometrical methods, but one can solve the problem in three other ways:

- One option is to use of so called soft constraints instead of hard constraints. Soft constraints mean that the respective constraint can be violated, but such a violation is penalized.
- Another possibility is to use another control strategy in these regions, for example, LQ control. This approach passes to our control scheme because the clipped LQ control corresponds

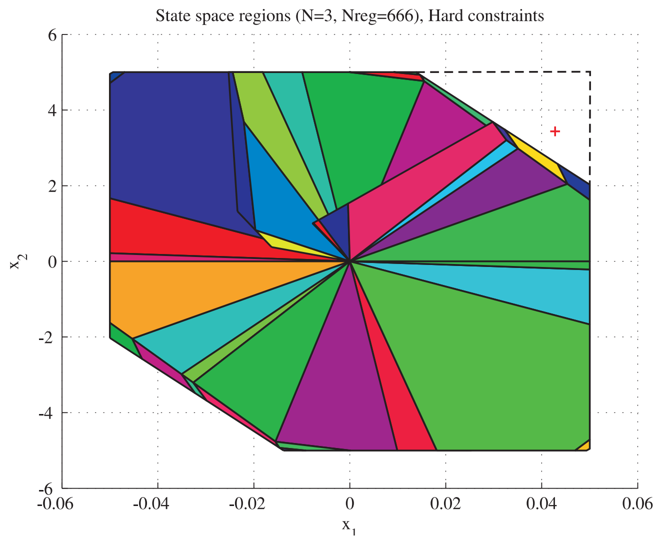

Fig. 10. MPC optimal control region is smaller than space (hypercube) of the possible states

to MPC for $N = 1$ in case of semi-active suspension system.

- Third possibility is to use enlarged state hypercube, e.g. $1.2x_{min} \leq x_k \leq 1.2x_{max}$ which does not really solve the problem but moves the problem in a larger hypercube and essentially other MPC problem will be solved.

In fact, one more method could be applied, namely, if we prescribe for the last predicted state $x_{end} = 0$ among the constraints. It can be proven that it yields stability guarantee for the linear MPC problems where the objective function becomes Lyapunov function which implies that the state will not leave the feasibility region. But it is a very strong constraint and usually there is no solution of the MPC problem. The semi-active suspension problem is of this type. Furthermore it does not solve the feasibility problem if the separately designed observer/filter is applied for the controller.
First, the state hypercube constraints are softened in Eq. (59) which implies modification in the objective function too:

Modified constraints:

$$\begin{aligned} & x_{min} - s_{x_k} \leq x_k \leq x_{max} + s_{x_k} \\ & 0 \leq s_{x_k} \leq \begin{bmatrix} 10 & 10 & 10 & 10 \end{bmatrix}^T \\ & x_{min} \leq x_1 \leq x_{max} \end{aligned} \tag{61}$$

Last constraint requires that at least the first predicted state must satisfy the hard constraint. To penalize the constraint violation, the objective function is completed by the term $s_{x_k}^T Q_s s_{x_k}$ where Q_s is a suitably large matrix.

Modified objective/performance function:

$$J(x(0), U_N) = x_N^T Q_N x_N + \sum_{k=0}^{N-1} x_k^T Q x_k + y_k^T y_k + s_{x_k}^T Q_s s_{x_k} \quad (62)$$

$$Q_s = 10^5 \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using soft constraints the state space hypercube will be filled out totally (see Fig. 11). This solution shows that the original regions remain the same, only they are completed with new regions at the corners. Fig. 12 demonstrates the working and the price of the soft constraints, namely, that the state trajectory started from the earlier infeasible regions will violate the hard constraints (state space hypercube). The continuous line shows the first open loop state trajectory of the MPC optimization problem (see Eqs. (59, 61, 62)). The closed loop MPC trajectory using soft constraints is depicted in Fig. 13 where the circle denotes the moment when the state violates the hypercube constraints. As point of special interest, approaching the trajectory to the origin is zoomed out to see the interesting loops in the trajectory.

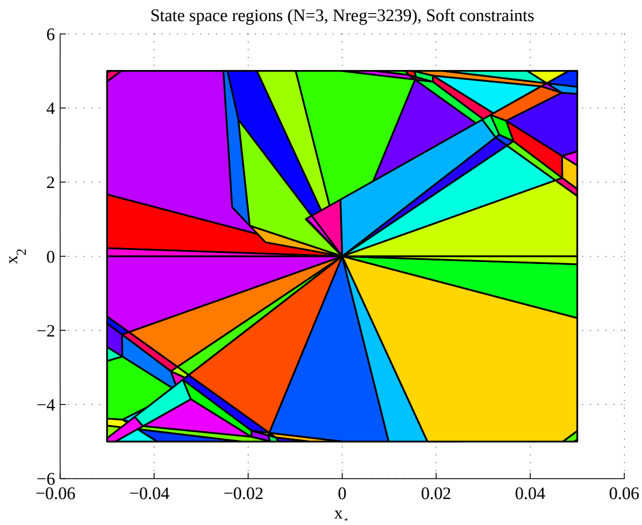

Fig. 11. State space regions using soft constraints

Another idea is to solve problem of the infeasibility regions if such controller is applied which yields control actions in these regions. The clipped LQ controller can be a good choice for us since it corresponds to the $N = 1$ MPC and always gives control action. The continuous trajectory in Fig. 14 depicts the combined clipped LQ and Model Predictive Control for the semi-active suspension. For comparison the dashed line gives the clipped LQ control.

Fig. 15 shows 6 slices of the explicit MPC ($N = 3$) using soft constraints. Despite of the soft constraints the fourth slice (right middle) does not fill out the state hypercube totally. This phenomenon is caused by the dissipative and saturation hard constraints (see Fig. 7). The difference compared to the above discussed case up to now is that, although the cutting shape is known, unfortunately it does not help in MPC because the optimization problem must be solved first to find where the cutting should be inserted. As in Fig. 15 is shown, there exists no control action for the state region marked with plus (+). Causes for reaching this region may be the same as mentioned earlier. For treatment of the problem one can also use another controller

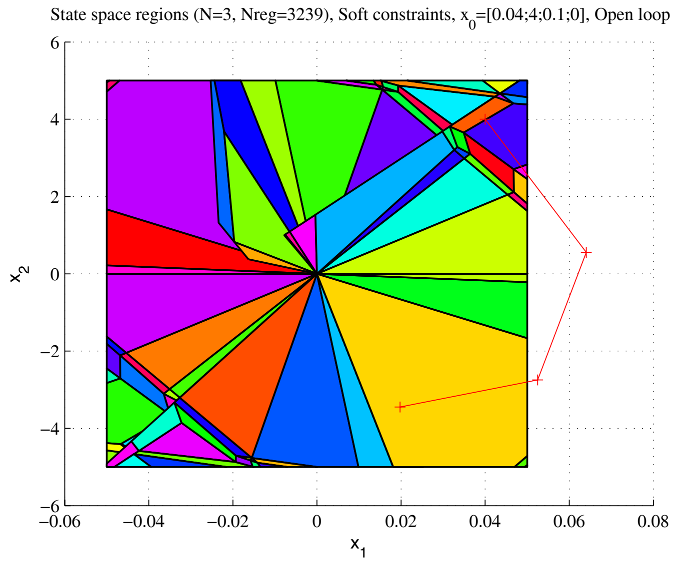

Fig. 12. Open loop trajectory using soft constraints

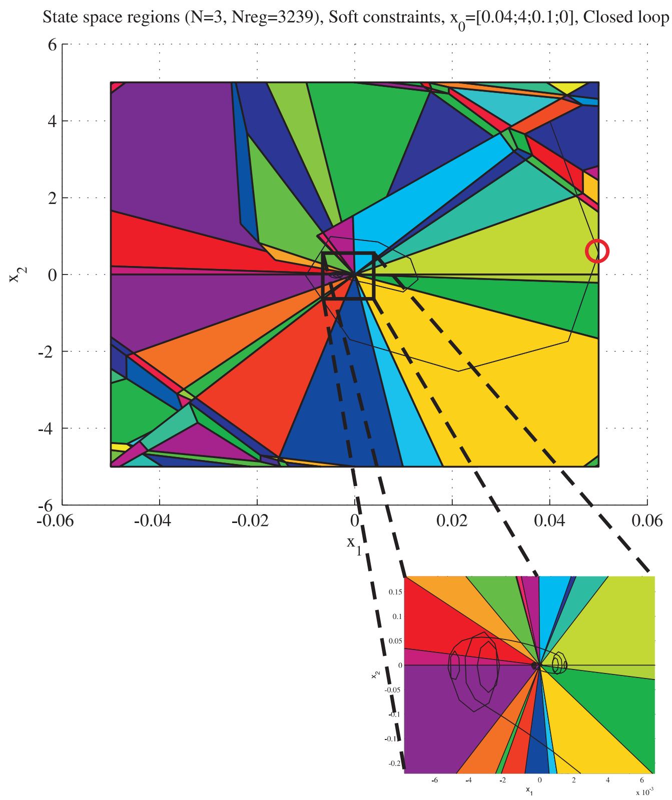

Fig. 13. Closed loop trajectory using soft constraints

type for instance clipped LQ in this region such as earlier.

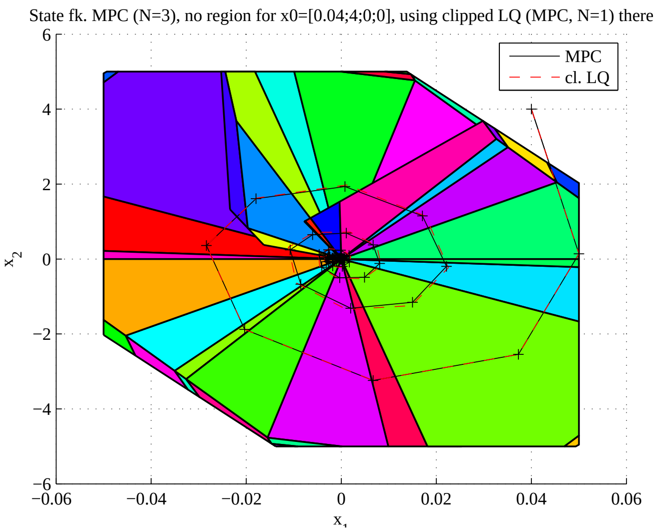

Fig. 14. Using clipped LQ control for the infeasible MPC regions

The main disadvantage of the explicit MPC is the exponential blow up of the number of regions when the prediction horizon is increasing. This property is inherent to the whole approach of parametric programming where the very central idea of explicit MPC is to enumerate all possible combination of active constraints. Since there can be exponentially many of them in prediction horizon (upper bound) therefore an exponential growth in the number of regions can be obtained. In

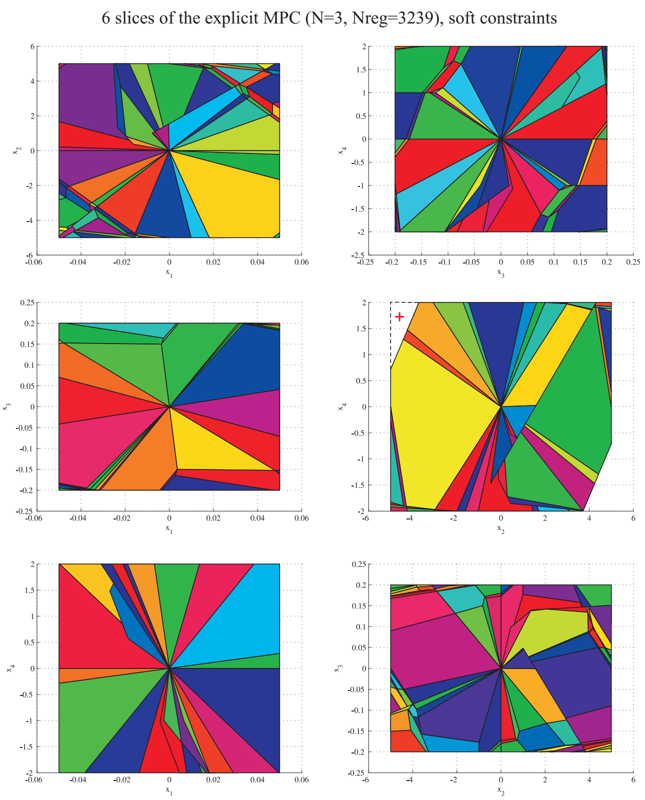

Fig. 15. Slices of the explicit MPC, $N = 3$

the worst case, the number of partitions (regions) equals to: $2^{(\textit{number of binary variables in the MPC problem})}$, where the (*number of binary variables in the MPC problem*) = (*prediction horizon*) $*$ (*number of binary variables in the MLD model*).

It can be proven that, using quadratic performance index, overlapping regions may arise. Without the overlapping the decomposition would contain not polyhedral regions too. In the overlapping areas that control action is chosen which has smaller performance value and so it will give the same result to online MPC.

Table 2 and Fig. 16 show the exponential blow up in our case. It can be seen that the numbers of the hard constraints are one order of magnitude smaller than soft constraints. The "CPLEX error" means that the CPLEX solver crashed due to numerical problems. Calculation of the explicit MPC regions using soft constraints and prediction horizon $N = 5$ consumed about 1 hour (on Acer Notebook with parameters: AMD Athlon(tm) 64 Processor 3000+ 1.8 Ghz, 2.00 GB RAM). The enormous number of the regions in the explicit MPC decreases the applicability for real systems since the online searching among the regions can take long time. Techniques for reducing the number of regions are currently under research [22, 25].

Tab. 2. *Number of explicit MPC regions with respect to the prediction horizon*

N	Hard constraints	Soft constraints
1	8	8
2	92	370
3	666	3239
4	3008	13320
5	11024	43266
6	35006	CPLEX error

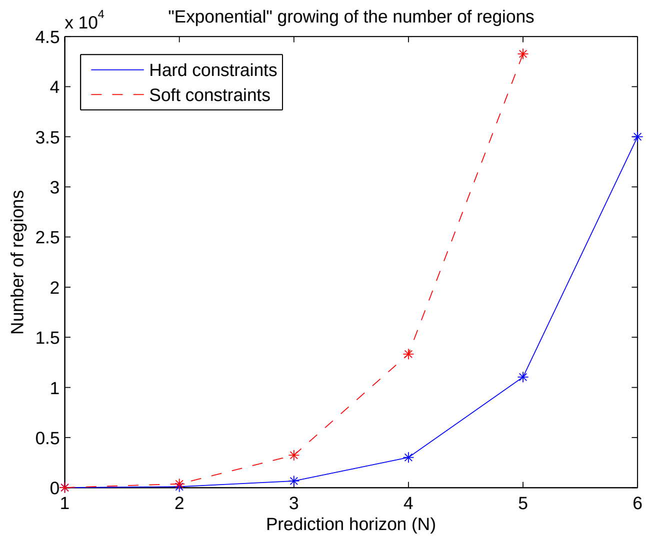

Fig. 16. Exponential blow up number of regions w.r.t prediction horizon in case of hard- and soft constraints

End of this section, MPC for the semi-active suspension with control horizons $N = 15$ and $N = 30$ are compared to the clipped LQ control (which is equivalent to MPC with $N = 1$). We present results for shock tests (i.e. nonzero initial conditions) with no road disturbances and "white noise" road velocity disturbance for zero initial condition. The road velocity disturbance w is modeled as discrete-time normal distribution with mean zero and the following standard deviation [14]:

$$w_{RMS} = \sqrt{\frac{2 \cdot \pi \cdot v \cdot A_{road}}{T_s}},$$

where $A_{road} = 4.9 \cdot 10^{-6}$, $v = 88\ km/h$ and $T_s = 10\ ms$. This simulations were performed with online MPC computation using hard constraints. From the Figs. 17, 18 one can recognize

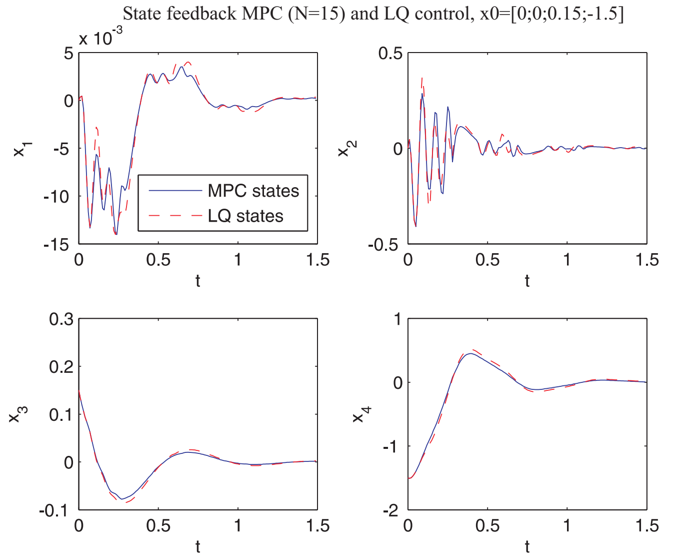

Fig. 17. State transients using MPC with $N = 15$ and clipped LQ control of the semi-active suspension

that there is no essential difference between the trajectories in the shock test case. Saturation of the control signal can be observed at the beginning of the transient. Fig. 19 and Fig. 20 also show simulation results with MPC and clipped LQ semi-active suspension for road disturbance. Both control methods yield very similar results, nevertheless the MPC "foresees" $N = 30$ steps and lot of states are located on the boundary of the allowed control region as shown in Fig. 20.

6 Output feedback explicit mpc with deterministic actual observer

The control scheme for the output feedback MPC control of the semi-active suspension is given in Fig. 21. The measured output of the semi-active suspension is the suspension deflection ($y = x_3 = C_{obs}x$). Note that y in Fig. 21 denotes the observed output now and not the performance output used in (58). Estimated states are bounded by the state hypercube and, after calculation of the control signal, the dissipative and saturation constraints are applied, and finally the actuator based on the input control force give the corresponding current to the MR

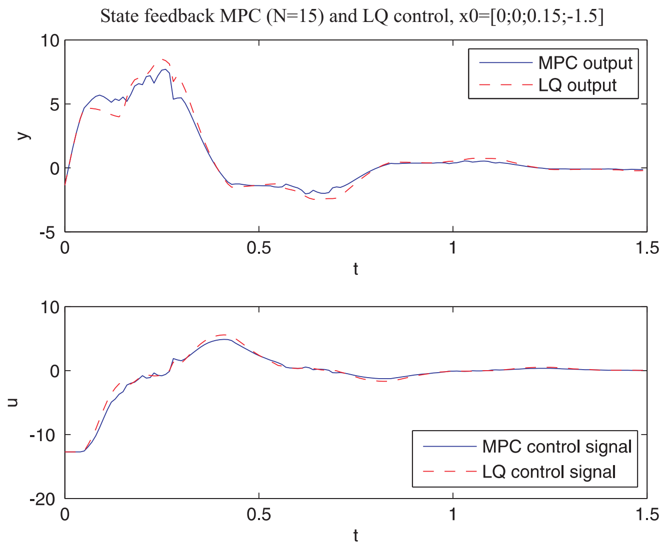

Fig. 18. Output and control signal transients using MPC with $N = 15$ and clipped LQ control of the semi-active suspension

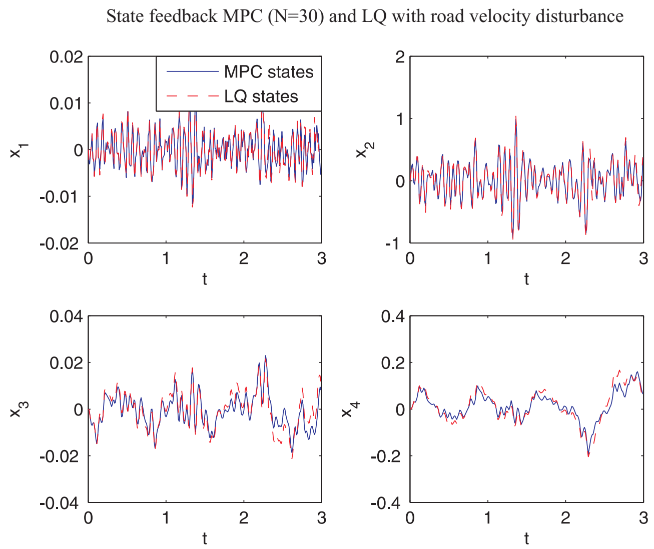

Fig. 19. State transients using MPC with $N = 30$ and clipped LQ control of the semi-active suspension using road velocity disturbance

damper in the suspension system. It can easily be checked that the matrix pair $(A, C_{obs}A)$ is observable where A, C_{obs} denote discrete time state matrices. Hence actual, deterministic discrete time observer was designed to the suspension system according to (23). The fast poles $p_z = \begin{bmatrix} 0.06 & 0.07 & 0.08 & 0.09 \end{bmatrix}^T$ were chosen for the observer.

Fig. 22 shows the output feedback MPC control with $N = 1$ to a shock test $x_0 = \begin{bmatrix} 0.03 & 2 & -0.1 & -1 \end{bmatrix}^T$. The observed states converge to the real states after an initial transient. Since the observer is separately designed for the controller, the estimated states can reach such regions in map of the explicit MPC regions where no control action exists. In order to treat this problem some techniques were presented in the previous section. Fig. 23 shows the closed loop output feedback MPC simulation results

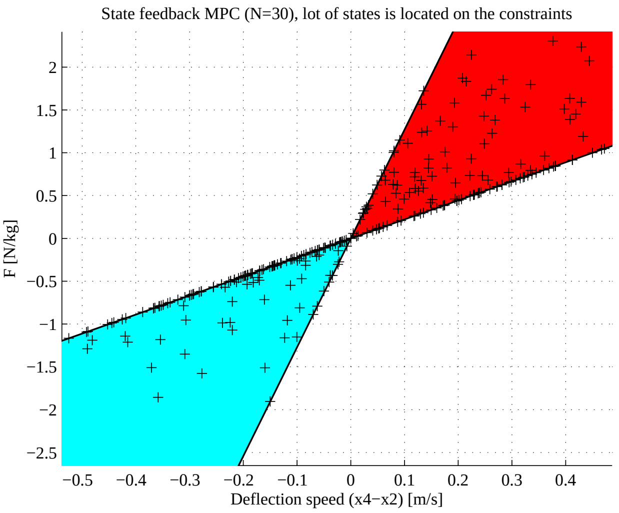

Fig. 20. Location of the states in the control region defined by dissipative and saturation constraints using MPC with $N = 30$ and exciting with road velocity disturbance

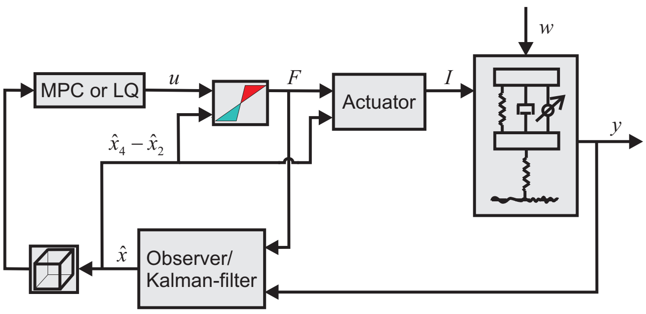

Fig. 21. Output feedback control scheme

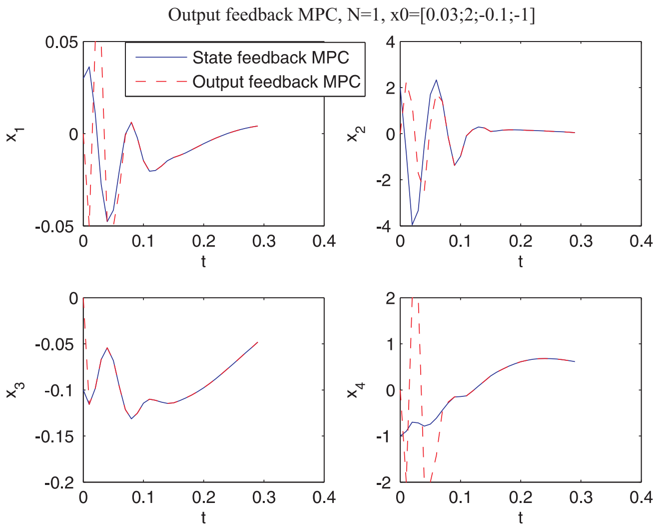

Fig. 22. Output and state feedback MPC

with random noise disturbance input and zero initial condition. It can be seen that the estimated state x_3 is the measured state, therefore its estimation is perfect. In the estimation of the states x_1 and x_2 one can still discover the original states but estimation of the velocity x_4 is wrong. The goal with the deterministic observer was to examine only what problems we face when an observer is applied in the explicit MPC framework (see previous section). Finally the road disturbance rejection is presented

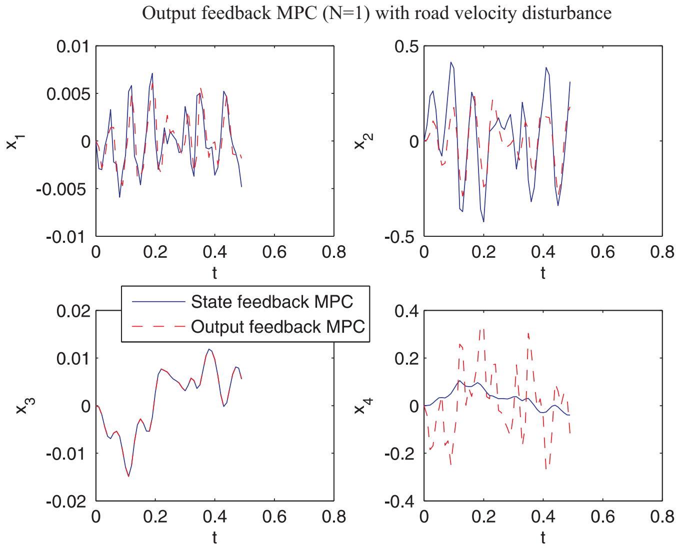

Fig. 23. Closed loop output feedback MPC result with random noise disturbance input and zero initial condition

in Figs. 24, 25 for the state feedback and for the output feedback using deterministic observer MPC case. The state feedback MPC proves the efficient working of the controller in the semi-active suspension system while the output feedback MPC does not work well because of the not appropriate observer.

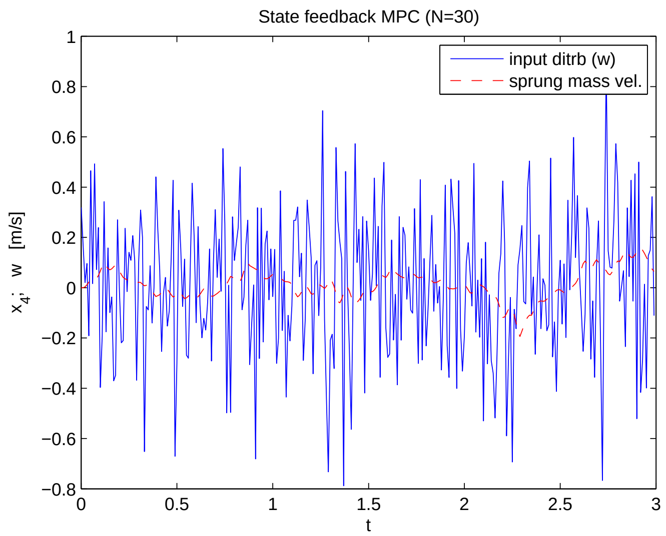

Fig. 24. Disturbance rejection using state feedback MPC for the semi-active suspension

7 Conclusions

To analyze the optimal explicit MPC approach, the quarter car semi-active suspension model was chosen. We have shown that

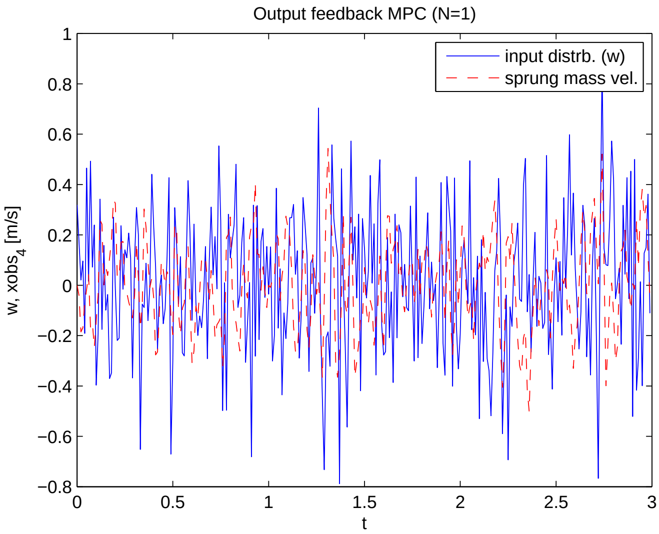

Fig. 25. Disturbance rejection using output feedback MPC with deterministic observer for the semi-active suspension

the explicit MPC is a promising method to increase the practical applicability of the MPC to such real systems where the time consuming online optimization is not allowed because fast control action is required. After a detailed theoretical summary of the explicit/hybrid MPC the practical questions of the control method have been analyzed. Through the explicit MPC we have shown that the optimal MPC control does have a linear state feedback form. Two main disadvantages of the explicit MPC are the exponential blow-up of the number of regions with increasing the prediction horizon and the requirements of the full state measurement. In explicit MPC we pointed out that regions may exist where no optimal solution exists which is not allowed in a real system. We can reach such regions in many cases such as: independently designed observer from the controller, disturbance input or modeling uncertainty. In order to treat this problem soft constraints and combined clipped LQ/MPC have been suggested.

It was shown that the clipped LQ corresponds to the MPC with $N = 1$ not only around the origin but in the whole space. MPC with greater control horizon $N = 15$ and 30 do not yield essential improvement for the system. Our examination has been included shock tests and road disturbance excitation. Finally, deterministic actual observer was designed to the semi active suspension system which requires only the measurement of the suspension deflection where we have not expected extremely good estimation for velocity. The goal with the deterministic observer was to examine what problems we face when an observer is applied in the explicit MPC framework. For modeling and analyzing the semi active MPC problem the MPT and YALMIP toolboxes were applied.

Technique for reducing the number of regions for real time applications are currently under research.

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