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RESEARCH ARTICLE

# Uncertainty remodeling for robust control of linear time-invariant plants[1]

*Gábor* Rödönyi / *József* Bokor

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***Abstract***

*The paper proposes a measure of robust performance based on frequency domain experimental data that allows non-conservative modeling of uncertainty. Given the nominal model of the plant and closed-loop performance specifications the iterative control design and remodeling of model uncertainty based on that measure leads to a controller with improved robust performance. The structured dynamic uncertainty is allowed to act on the nominal model in a linear fractional transformation (LFT) form. The proposed method is a modification of the structured singular value with implicit constraints on model consistency. The usefulness of the method is demonstrated on a vehicle control simulation example.*

***Keywords***

*structured dynamic uncertainty · uncerting modeling · robust control*

***Acknowledgement***

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**Gábor Rödönyi**

Systems and Control Laboratory of the Computer and Automation Research Institute, MTA, P.O. Box 63, H-1518 Budapest, Hungary
e-mail: rodonyi@sztaki.hu

**József Bokor**

Systems and Control Laboratory of the Computer and Automation Research Institute, MTA, P.O. Box 63, H-1518 Budapes, Hungary

## 1 Introduction

In robust control theory the model uncertainty in system dynamics is treated as a model set. In the $\mathcal{H}_\infty/\mu$ framework this set is described by linear fractional transformation (LFT) of the nominal model by structured (block-diagonal), norm-bounded perturbations, $\Delta$, which is otherwise unspecified. Robust stability (stability of each system in the model set) is analyzed by the structured singular value $\mu$. For the analysis of robust performance the nominal plant is augmented by outputs $z$ – that should be small – measuring performance and normalized inputs. Some of these inputs denoted by $r$ are known (for example reference signal) while others, denoted by $d$, model disturbances on the system and belong to an unknown but norm-bounded signal set. It is known that robust performance is equivalent to a robust stability problem where the performance output is fed back to these inputs through a fictive perturbation block $\Delta_p$. The uncertain closed-loop system is represented in the so called $\Delta$-$P$-$K$ structure depicted in Fig. 1(a). See [1] and [31] for more details on robust control theory.

In the robust control framework much effort has been taken in order to decrease conservatism of uncertainty set descriptions of physical systems. More and more sophisticated structures of perturbations including dynamic, time-varying, and real parametric uncertainty have been developed and analyzed [5, 8], however it bears the price of increased computational complexity. Hence only lower and upper bounds of the structured singular value are calculated [9, 10, 20, 30], that are not necessarily tight [8].

This paper follows a different direction of decreasing conservatism instead of further detailing the uncertainty model or trying to tighten the upper bound of $\mu$. We exploit that the different kinds of sources of uncertainty in the real usually cover only a subset of a unit ball of a signal (disturbances) or system (perturbations) space and may have hidden relations, interactions between them. This means that the effects of the uncertainty sources like neglected dynamical components and disturbance

[1] This paper revises and extends the results of the paper 'Uncertainty remodeling for robust control of linear time-invariant plants' presented in the Mediterranean Control Conference, 2008, Ajaccio, Corsica.

sources may be *counteracting*. The steady inclusive counteractions allow us to decrease the assumed sizes of the individual uncertainties, albeit they are larger in the real. On the other hand the several sources of uncertainty might be *interchanged*. For example every modeling error can be described as effects of disturbances resulting in the classical $\mathcal{H}_\infty$ problem. Thus this paper focuses a modified $\Delta$-$P$-$K$ structure of Fig. 1(b), where weighting functions of perturbations and disturbances are pulled out from $P$ to emphasize the variable part of the uncertainty model. Then $P_0$ is fixed in our problem.

These thoughts lead us to the fields of *model (in)validation* and *uncertainty modeling*. The goal of invalidation is to check the consistency of the model set with the available input-output (IO) measurement data. The uncertain model is said to be consistent if there exist elements of the allowed perturbation set and disturbance signal set that satisfy the assumed norm bounds and could have produced the data. In this paper we focus on frequency domain methods, so the problem reduces to separate constant matrix problems over finite set of frequencies what relaxes the computational complexity. The invalidation test in general corresponds to a search with equality and inequality constraints. In many papers the equality constraint with the IO data is included in the generalized plant and a modified (or skewed) $\mu$ calculation gives conditions on consistency [7, 16, 23, 27]. The frequency domain results are based on proving the existence of a stable, causal and bounded perturbation $\Delta$ satisfying the constraints by tangential Nevanlinna-Pick interpolation [4, 6, 29]. Model validation problems no longer assume physical meaning of the uncertainty. The uncertain model is a mathematical tool to describe the deviation of data from nominal model. Therefore model validation is strongly related to *uncertainty modeling* where for a given nominal model and fixed structure of uncertainty consistent uncertainty model set is created by determining norm-bounds of disturbance signals and perturbations. The resulted sets are normalized usually by frequency dependent weighting functions. It is no problem to find one consistent model set allowing appropriately large bounds for example on an additive disturbance; the main question is how to chose between all consistent model sets and how to determine the trade-off between perturbation- and disturbance channels. For example in [21] and [18] the trade-off is fixed based on some a priori information and the norm-bounds are minimized simultaneously; in [14, 19] additive unstructured uncertainty is minimized by identification of the nominal model; the size of disturbance is fixed in [12] and the $\nu$-metric of a co-prime factor uncertainty is minimized by identifying the nominal model; in [2] the disturbance have predefined statistical properties and the resulted bounds for the perturbation have some statistical confidence. Other references in stochastic or time domain frameworks are [3, 11, 15, 17, 24, 25, 28].

The contribution of present paper in uncertainty modeling is that the uncertainty model structure is the general LFT form with structured dynamic perturbations and the criterion of optimizing in all consistent models is the robust performance level ($\mu$) - precisely the same criterion as for the control design. This criterion including consistency constraints and new variables defines a new measure of robust performance. In this context the uncertainty model is purely mathematical without physical meaning. One criticism against this approach can be its exaggerated optimism when data is not enough informative. In practical applications, when not enough experiments can be taken or they are too expensive, lower limits of the uncertainty norm-bounds can be given based on a priori physical knowledge. Since the general problem leads to bilinear matrix inequalities (BMIs), which are NP-hard to solve, also the application of the method for structured, additive uncertainties is presented. In this case the problem is still one of BMIs, but can efficiently be solved by a series of convex programs.

The goals of the paper are formulated in section 3. The main results are presented in section 4 and the usefulness of the proposed measure is demonstrated in section 5.

## 2 Notations

The dimension of a vector $x$ is denoted by $n_x$. Let $x^T$ stand for transpose and $x^*$ for conjugate transpose of $x$. $I_x = I_{n_x}$ and $I$ denote identity matrices, the subscript $_x$ implies correspondent dimension with vector $x$.

A bounded-energy signal $d$ belongs to the set $\mathcal{L}_2 \triangleq \{d : \|d\|_2^2 = \int_0^\infty d(t)^T d(t)dt < \infty\}$ in the continuous time-domain. A subset of this with unity norm is denoted by $\mathbf{B}\mathcal{L}_2$. The set of all proper and real rational stable transfer matrices is denoted by $\mathcal{RH}_\infty$. A bounded set of this is $\mathbf{B}\mathcal{H}_\infty \triangleq \{\Delta \in \mathcal{RH}_\infty : \|\Delta\|_\infty = \sup_\omega \bar{\sigma}(\Delta(j\omega)) < 1\}$.

Upper and lower linear fractional transformations of two systems, say $A$ and $B$, are denoted by $\mathcal{F}_U(A, B)$ and $\mathcal{F}_L(A, B)$, respectively.

In this paper the perturbation set $S_\Delta$ is defined as

$$\begin{aligned} S_\Delta \;\; = \;\; & \{\Delta \in \mathcal{RH}_\infty^{n_\xi \times n_\xi} : \Delta = \mathrm{diag}\{\Delta_1, \ldots, \Delta_\tau\}, \\ & \Delta_i(j\omega) \in \mathbb{C}^{n_{\xi_i} \times n_{\xi_i}},\ i = 1, \ldots, \tau\} \end{aligned}$$

The normalized subset of this is $BS_\Delta$, where $\Delta \in \mathbf{B}\mathcal{H}_\infty$ is also satisfied. The signals $\xi = [\xi_1^T, \ldots, \xi_\tau^T]^T$ and $\eta = [\eta_1^T, \ldots, \eta_\tau^T]^T$ are partitioned according to the block structure of $\Delta$ in Fig. 1.

For a signal or system $x$ in the subscripting $x_{lki}$ $l$ stands for indexing experiments, $k$ for frequency $\omega_k$ in a grid and $i$ for the $i$th element of the signal vector $x$ or $i$th block of the block-diagonal system matrix $x$, respectively. Some of the indexes may miss. If index $k$ is present then $x$ is a frequency domain operator or signal. The $Im\{M\}$ and $ker\{M\}$ denote the image space and kernel space, respectively, of the matrix $M$.

## 3 Problem formulation

The problem of identification of a consistent uncertainty model that minimizes robust performance level $\mu$ for a given closed-loop system can be divided into two subproblems: A)

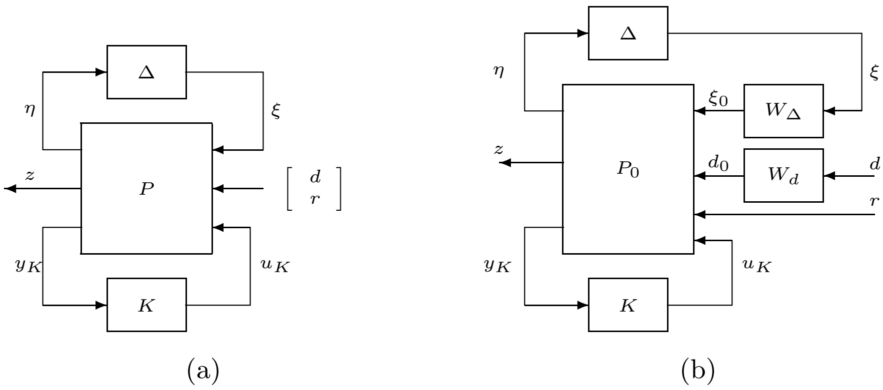



**Fig. 1.** (a) $\Delta$-$P$-$K$: the closed loop uncertain system in robust control. The unmodeled dynamics $\Delta$ (block-diagonal) and the disturbance/noise $d$ represent the model uncertainty, $r$ stands for known signals (e.g. reference), $z$ for error signals that should be small. $P$ is the generalized plant and $K$ denotes the controller. (b) two weighting functions associated with uncertainty are pulled out from $P$.

characterizing all consistent uncertainty models by parametrization and B) optimization in the parameter space defined in A).

### 3.1 Uncertainty characterization problem

For the sake of simplifying notations a part of $\Delta$-$P$-$K$ structure of Fig. 1(b) relevant to the model validation problem is emphasized in Fig. 2. Without loss in generality $u$ denotes any measured or known signal containing $u_K$ and possibly $r$, and $y$ denotes the measurable output signals (not necessarily the same as $y_K$). System $G$ defines the LFT structure of the uncertainty model. Note $G_{23}$ is the nominal model of the system. In this interconnection $G$ is fixed in advance and we look for appropriate weighting functions $W_\Delta$ and $W_d$ solving Problems 1 and 2.

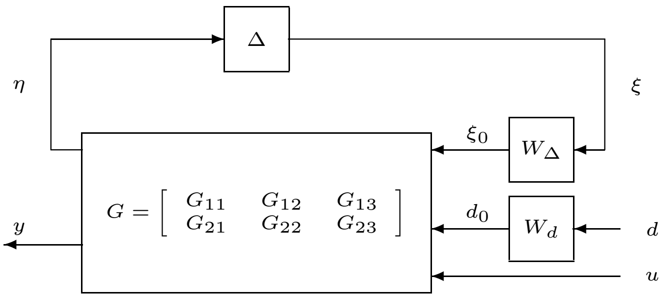



**Fig. 2.** Uncertain system in the model validation problem. The LTI system $G$ is part of $P_0$ in Fig. 1(b). $y$ and $u$ are known or measurable signals.

**Problem 1** *Assume there are open- and closed-loop input-output measurements in $\mathcal{L}_2$ available in the frequency domain. The data set is denoted as $S_{yu} = \{(y_{lk}, u_{lk}) : y_{lk} \in \mathbb{C}^{n_y}, u_{lk} \in \mathbb{C}^{n_u}, l = 1, ..., N, k = 1, ..., n_\omega\}$, where $n_\omega$ is the number of frequency samples and $N$ denotes the number of experiments. Characterize all the diagonal weighting functions $W_\Delta = \mathrm{diag}\{w_{\Delta_1} I_{\xi_1}, \ldots, w_{\Delta_\tau} I_{\xi_\tau}\} \in \mathcal{RH}_\infty^{n_\xi \times n_\xi}$ and $W_d = \mathrm{diag}\{w_1, \ldots, w_{n_d}\} \in \mathcal{RH}_\infty^{n_d \times n_d}$ of Fig. 2 such that there exist for every experiments $l = 1, ..., N$ a perturbation $\Delta_l \in BS_\Delta$ and a disturbance $d_l \in \mathbf{B}\mathcal{L}_2^{n_d}$ that satisfy*

$$y_{lk} = \mathcal{F}_U(G_k, \Delta_{lk} W_{\Delta,k}) \begin{bmatrix} W_{d,k} d_{lk} \\ u_{lk} \end{bmatrix} \qquad (1)$$

*for $k = 1, ..., n_\omega$, $l = 1, ..., N$. Note the index $k$ refers to the complex matrix or vector at frequency $\omega_k$.*

### 3.2 Optimization problem: search for the uncertainty model

In standard $\mathcal{H}_\infty/\mu$ control [31] the robust performance level

$$\mu_{\Delta_a}(M) = \frac{1}{\max_{\Delta_a}\{\bar{\sigma}(\Delta_a) : \det(I - M\Delta_a) = 0\}}$$

is defined as the reciprocal of the $\mathcal{H}_\infty$-norm of the minimum destabilizing structured perturbation $\Delta_a = \mathrm{diag}\{\Delta, \Delta_p\}$, where $\Delta_p$ is the fictive perturbation on the performance channel $z \mapsto [d^T, r^T]^T$. Let $M = \mathcal{F}_L(P, K)$. Then for all $\Delta \in S_\Delta$ with $\|\Delta\|_\infty < \frac{1}{\beta}$ the loop $\mathcal{F}_U(M, \Delta)$ is well-posed, internally stable and $\|\mathcal{F}_U(M, \Delta)\|_\infty \le \beta$ if and only if $\mu_{\Delta_a}(M) \le \beta$ [31, Theorem 11.9]. The goal with structuring perturbations instead of using unstructured perturbations was decreasing conservatism of the uncertainty description, since $\mu_{\Delta_a}(M) \le \|M\|_\infty$. The goal in this paper is the same: decreasing conservatism by tuning of the weighting functions based on measurement data. Therefore a modified robust performance criterion is given as follows.

Let $W = \mathrm{diag}\{W_\Delta, W_d\}$, $W_I = \mathrm{diag}\{W_\Delta, W_d, I_r\}$ and introduce the notation $W(\Theta)$ symbolizing all solutions $W$ for Problem 1. The free parameter $\Theta$ is symbolic at this moment and refers to the space of all consistent uncertainty models. Let $M_0 = \mathcal{F}_L(P_0, K)$, so $M = M_0 W_I$. The new measure is defined as

$$\mu_{\Delta_a,\Theta}(M_0) = \inf_{\Theta} \mu_{\Delta_a}(M_0 W_I(\Theta)) \qquad (2)$$

Clearly $\mu_{\Delta_a,\Theta}(M_0) \le \mu_{\Delta_a}(M)$ if the uncertainty model in $M$ is consistent.

**Problem 2** *Given a controller $K$ and data set $S_{yu}$. Find weighting functions that solve Problem 1 and the minimization in* (2).

## 4 Main results

### 4.1 Parametrization of uncertainty models

The signal parametrization result of [27] is borrowed here. The uncertain model is consistent with data if and only if there exist $\xi_0$ and $d_0$ solving

$$e_l = y_l - G_{22}u_l = \begin{bmatrix} G_{21} & G_{23} \end{bmatrix} \begin{bmatrix} \xi_{0l} \\ d_{0l} \end{bmatrix}.$$

For solvability assume that $e_{lk} \in \mathrm{Im}\{[G_{21,k}\ G_{23,k}]\}$ for all $k$ and to avoid trivial solutions assume $\dim(\ker[G_{21,k}\ G_{23,k}]) > 0$ for all $k$. In this case let a particular solution for $[\xi_0^T, d_0^T]^T$ be denoted by $a_0 = [a_\xi^T, a_d^T]^T$, then all solutions can be formalized as $\begin{bmatrix} \xi_{0lk} \\ d_{0lk} \end{bmatrix} = \begin{bmatrix} a_{\xi,lk} \\ a_{d,lk} \end{bmatrix} + \begin{bmatrix} B_{\xi,k} \\ B_{d,k} \end{bmatrix} \theta_{lk}$, where $\begin{bmatrix} B_{\xi,k} \\ B_{d,k} \end{bmatrix}$ is a basis for the kernel of $\begin{bmatrix} G_{21,k} & G_{23,k} \end{bmatrix}$ and $\theta_{lk} \in \mathbb{C}^{n_\xi + n_d - n_y}$ is any free parameter at frequency $\omega_k$ for the experiment $l$. Note that $\xi_{0lk}$, $d_{0lk}$ and $\eta_{lk}$ are all affine functions of $\theta_{lk}$.

The parametrization of weighting functions in $W(\Theta)$ can be given indirectly through the $\theta_{lk}$ parameters and some inequality constraints as follows.

**Theorem 1** *Given stable and stable invertible weighting functions $W_\Delta$ and $W_d$ and given the set $S_{yu}$ of measurement data, then for every experiment $l = 1, ..., N$ there exists a perturbation $\Delta_l \in BS_\Delta$ and a disturbance $d_l \in \mathbf{B}\mathcal{L}_2^{n_d}$ that satisfy* (1) *for all $k$ and $l$, if and only if there exist $\theta_{lk}$ $k = 1, ..., n_\omega$ and $l = 1, ..., N$ such that*

$$|w_{\Delta,i}(j\omega_k)| \geq \frac{|\xi_{0lk,i}(\theta_{lk})|}{|\eta_{lk,i}(\theta_{lk})|}, \quad i = 1, \ldots, \tau \tag{3}$$

$$|w_{d,i}(j\omega_k)| \geq |d_{0lk,i}(\theta_{lk})| \quad i = 1, \ldots, n_d, \tag{4}$$

*for all $l$ and $k$.*

*Proof.* The proof is very straightforward in the frequency domain for the constant matrix case. To prove the existence of a causal stable and bounded transfer function matrix $\Delta_l(j\omega)$ that matches some matrices $\bar{\Delta}_{lk}$ on a frequency grid the tangential Nevanlinna-Pick interpolation theorem can be applied [4], [6].

The whole solution space $W(\Theta)$ of Problem 1 is characterized by the set $\{\theta_{lk} \in \mathbb{C}^{n_\xi + n_d - n_y},\ l = 1, ..., N,\ k = 1, ..., n_\omega\}$ and the constraints (3) and (4).

### 4.2 Solution of the optimization problem

The computation of the standard $\mu_{\Delta_a}(M)$ is NP-hard in general, therefore one used to calculate lower and upper bounds which are tight for practical systems in case of only complex blocks. In order to guarantee robust performance we are interested in the upper bound which is calculated by convex optimization. To this end a scaling matrix $D \in \mathcal{RH}_\infty$ is introduced with $\Delta D = D\Delta$, then $\mu_{\Delta_a}(M) \leq \inf_D \bar{\sigma}(DMD^{-1})$, where square $M$ was assumed for notational brevity. One approach to find the infimum is to consider constant matrix problems on a frequency grid with real $D$ variables at each frequenciy and then fit minimum phase transfer functions on the solutions. Accordingly the scaling matrix $D \in S_D$ is defined with $S_D = \{D = \mathrm{diag}(x_1 I_{\xi_1}, ..., x_\tau I_{\xi_\tau}),\ 0 < x_j \in \mathbb{R}\}$ and the standard $\mu$ upper-bound calculation reveals $\mu_{\Delta_a}(M) \leq \max_\omega \inf_{D_\omega \in S_D} \bar{\sigma}(D_\omega M(j\omega) D_\omega^{-1})$.

This constant matrix approach fits well to the frequency domain modeling problem. The upper bound of (2) can be written as

$$\max_\omega \inf_\Theta \inf_{D_\omega \in S_D} \bar{\sigma}(D_\omega M_0(j\omega) W_I(\Theta, j\omega) D_\omega^{-1})$$

where the minimization in $\Theta$ is subject to (3) and (4). For general system $G$ (3) and (4) define nonlinear, non-convex constraints. However in the frequently used special case, when the perturbation and disturbance are additive to the nominal model, the optimization in $\Theta$ can be solved using linear matrix inequalities (LMI) for which efficient numerical algorithm exists [22]. Additive perturbation and disturbance involve that $\eta$ is independent of $\theta$ and can be calculated in advance from $u$, i.e. $G_{11} = 0, G_{12} = 0$. This case includes input/output multiplicative perturbations as well.

In the following (3) and (4) is rewritten in a compact LMI form. Define real diagonal matrices $V_k = \mathrm{diag}\{|W_\Delta(j\omega_k)|^2, |W_d(j\omega_k)|^2\} \in \mathbb{R}^{n_\xi + n_d \times n_\xi + n_d}$, $V_{I,k} = \mathrm{diag}\{V_k, I_r\}$; diagonal of complex vectors $A_{lk} = \mathrm{diag}\left\{\frac{a_{\xi,lk,1}}{|\eta_{lk,1}|}, ..., \frac{a_{\xi,lk,\tau}}{|\eta_{lk,\tau}|}, a_{d,lk,1}, ..., a_{d,lk,n_d}\right\} \in \mathbb{C}^{n_\xi + n_d \times n_\xi + n_d}$; and diagonal of complex matrices $B_{lk} = \mathrm{diag}\left\{\frac{B_{\xi,k,1}}{|\eta_{lk,1}|}, ..., \frac{B_{\xi,k,\tau}}{|\eta_{lk,\tau}|}, B_{d,k,1}, ..., B_{d,k,n_d}\right\} \in \mathbb{C}^{n_\xi + n_d \times n_\theta(\tau + n_d)}$; and let the complex column vector $\Theta_{lk} = \mathrm{columnvec}\{\theta_{lk}, ..., \theta_{lk}\} \in \mathbb{C}^{n_\theta(\tau + n_d)}$. Then (3) and (4) reappear as

$$\begin{bmatrix} V_k & (A_{lk} + B_{lk}\Theta_{lk})^* \\ A_{lk} + B_{lk}\Theta_{lk} & I_{\xi + n_d} \end{bmatrix} \geq 0 \tag{5}$$

Define $D_{R,k} = \mathrm{diag}\{D_k^2, I_{n_d + n_r}\}$ and $D_{L,k} = \mathrm{diag}\{D_k^2, I_z\}$. The following main result that solves Problem 2 can be proved by simple algebra.

**Theorem 2** *Let $G_{11} = 0, G_{12} = 0$. The uncertain system of Fig. 1(b) is not invalidated by the data set $S_{yu}$ and robust $\mathcal{H}_\infty$- performance is satisfied at level $\gamma$, i.e. $\|\Delta\|_\infty \leq \gamma^{-1}$ and $\|\mathcal{F}_U(M, \Delta)\|_\infty < \gamma$ if there exist $D_k$, $V_k$ and $\theta_{lk}$ for all $k = 1, ..., n_\omega$, $l = 1, ..., N$ that solve*

$$\begin{bmatrix} \gamma^2 D_{L,k} & D_{L,k} M_{0,k} V_{I,k} \\ V_{I,k} M_{0,k}^* D_{L,k} & V_{I,k} D_{R,k} \end{bmatrix} > 0 \tag{6}$$

*and* (5).

*Proof.* Using Schur complement (5) is equivalent to $|w_{\Delta_i}|^2 - \frac{|a_{\xi,lk,i} + B_{\xi,k,i}\theta_{lk}|^2}{|\eta_{lk,i}|^2} \geq 0$, $i = 1, ..., \tau$ and $|w_{d_i}|^2 - |a_{d,lk,i} + B_{d,k,i}\theta_{lk}|^2 \geq 0$, $i = 1, ..., n_d$ which are the consistency conditions by Theorem 1. Equivalently to [31, Theorem 11.9] it can be stated that $\|\Delta\|_\infty \leq \gamma^{-1}$ and $\|\mathcal{F}_U(M, \Delta)\|_\infty < \gamma$ if and only if $\mu_{\Delta_a}(M) < \gamma$. This is satisfied if there exist

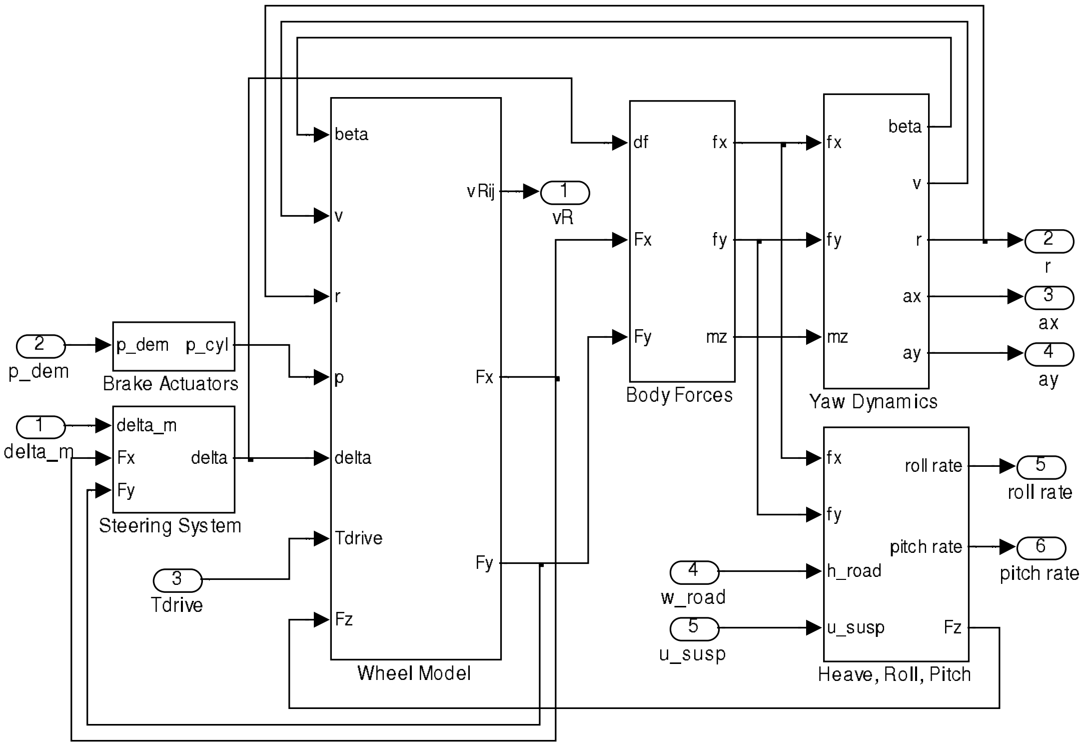



**Fig. 3.** Simulink block structure of the vehicle model. The input signals to the model are the followings: brake pressures to the four wheels (P_dem, zero for the rear wheels), driver torque to the steering wheel (Tdriver=0), driving moments to the four wheels (Tdrive, velocity feedback controller to keep constant speed), pitch (pitch_R=0) and roll (roll_R) angles of the road, vertical road disturbances (w_road=0), suspension actuators (u_susp=0).

$D_k \in S_D$ such that $\gamma^2 - D_L^{\frac{1}{2}} M D_R^{-\frac{1}{2}} \left( D_L^{\frac{1}{2}} M D_R^{-\frac{1}{2}} \right)^* > 0$ or equivalently $\gamma^2 D_L - D_L M D_R^{-1} M^* D_L > 0$. Using Schur complement $\begin{bmatrix} \gamma^2 D_L & D_L M \\ M^* D_L & D_R \end{bmatrix} > 0$. Note that $M$ can be written as $M = M_0 V_I^{\frac{1}{2}} \Phi$, where $\Phi$ is a diagonal matrix of repeated scalar frequency functions with unit amplitudes and with the angles of the $w_{\Delta_i}$ and $w_{d,i}$ weighting functions. Apply a congruent transformation for the last inequality by the diagonal matrix $\text{diag}\{1, V_I^{\frac{1}{2}} \Phi^{-1}\}$ to get (6) on each frequency $\omega_k$ Thus the proof is complete.

The $\gamma$ must be minimized subject to the matrix inequality constraints. Then weighting functions $W_\Delta$ and $W_d$ of Problem 2 are constructed by over-bounding the elements of $\sqrt{V_k}$, $k = 1, ..., n_\omega$ in magnitude via the method of [26]. The minimization subject to the inequalities (5) and (6) can be performed separately for each $k$ via LMIs if alternately $D_k$ or $V_k$ are fixed.

Note that in frequency domain model validation problems in [27], [4], [6] and in this paper stable perturbation was assumed, however this condition can be relaxed as Gu has shown in [13]. A certain winding number condition must be checked [13, Lemma 3] before applying the method in this paper.

## 5 Application example

The lateral dynamics of a heavy truck is to control in emergency situations using front-wheel brakes. Without any intervention of the driver the vehicle can be steered by applying brakes to either the left or the right side front wheel. A 17-degree of freedom nonlinear Matlab/Simulink simulation model (see Fig. 3) serves as the real plant for acquiring identification data and testing closed loop performance. This simulation model contains the dynamics of suspension, yaw, roll, pitch, heave motions, steering systems, wheel and brake actuator dynamics and a road model. The road model pretends effects of lateral road slope as well.

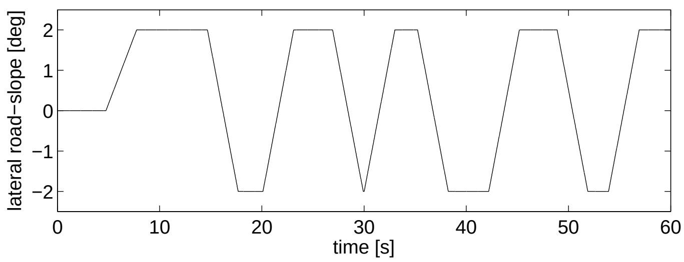



**Fig. 4.** Example for the roll angle $\varphi_R$ of the road

The goal of the control is to track a yaw-rate reference signal defined by some higher level control algorithm. The controller uses yaw-rate $\dot{\psi}$ and steering wheel angle $\delta_m$ measurements and acts on the brake-pressures of the front wheels. Good performance means small yaw-rate tracking error and control energy.

The nominal model, denoted by $G_n$ and used for control design, assumes a flat vehicle model neglecting all vertical dynamics and wheel dynamics. It simplifies yaw dynamics and the steering system. Furthermore the system is linearized. The forward vehicle speed is 12m/s. The nominal model $G_n$ in the state-space looks like $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $x = [\dot{\psi}, \delta, \dot{\delta}]^T$, $u = \Delta p$, where $\dot{\psi}$ denotes yaw-rate, $\delta$ denotes steering angle and $\Delta p$ is the brake pressure difference applied to the front wheels and

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|c} -10.5589 & 22.92 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 22.7567 & -66.43 & -3.255 & -0.3603 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$$

The model uncertainty comes from the neglected dynamics and outer disturbance. During the experiments the roll angle $\varphi_R$ of the road varies (an example of $\varphi_R$ is plotted in Fig. 4) causing the vehicle to skid sidewards and turn round the vertical axis. The reason for this cornering is the acting of different side-forces at the front and rear owing to the different wheel loads. The evolving yaw moment also turns the steering system, thus amplifying the cornering. This disturbing effect increases with steering angle and decreases with velocity. All the uncertainty effects are modeled by an input-multiplicative perturbation and additive disturbances on $y$. The closed-loop system with performance outputs are shown in Fig. 5 (compare with Fig. 1). The uncertain model structure $G$ of Fig. 2 is $G = \begin{bmatrix} 0 & 0 & 1 \\ G_n & I_2 & G_n \end{bmatrix}$.

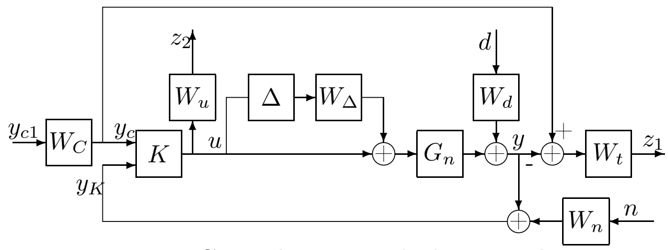



**Fig. 5.** The $\Delta$-$P$-$K$ structure. $G_n$ is the nominal plant, $y_c$ the yaw-rate reference, $y_{c1}$ the normalized yaw-rate reference, $n$ denotes sensor noise, $z_1, z_2$ performance signals, $W_C$, $W_u$, $W_t$, $W_\Delta$, $W_d$ and $W_n$ are weighting functions, $K$ is the controller

In order to illustrate the results of the paper a robust controller is designed by $\mu$-synthesis based on engineering judgement on weighting function selection. Engineering judgement says: "Since a large dynamics is neglected and the effect of lateral road slope (which is less then 3 degree) is related to the control input, pick up $W_\Delta$ and $W_d$ so that the nominal model error $y - G_n u$ be mainly covered by input-multiplicative perturbation. Additive disturbance is assumed to a minimal extent required to have consistent uncertainty model". The weighting functions $W_\Delta$ and $W_d$ can be seen in Fig. 7 plotted by dashed lines. For good tracking in steady state, a high gain of $W_t$ is required on low frequency, therefore $W_t = 0.25\frac{(s+1)^2}{(s+50)^2}$. The control input is penalized by $W_u = 0.0008\frac{(s+50)^2}{(s+0.2)^2}$ beyond 1 rad/s in order to avoid high-frequency dynamics of the controller. Further judgements on performance weighting functions are beyond question in this paper.

The generalized plant $P$ is defined by the mapping

$$\left[\begin{array}{c} \eta \\ \hline z \\ \hline y_K \end{array}\right] = \left[\begin{array}{c|c|cc|c} 0 & 0 & 0 & 0 & 1 \\ \hline -W_t G_{n,r} & -W_t W_{d,r} & 0 & W_C & -W_t G_{n,r} \\ 0 & 0 & 0 & 0 & W_u \\ \hline 0 & 0 & 0 & W_C & 0 \\ G_n & W_d & W_n & 0 & G_n \end{array}\right] \left[\begin{array}{c} \xi \\ \hline d \\ n \\ \hline y_{c1} \\ \hline u \end{array}\right]$$

where $G_{n,r}$ denotes the nominal transfer function to the yaw rate. Having specified the $\Delta$-$P$-$K$ structure the controller is designed by $\mu$-synthesis. A peak $\mu$ value of 2.049 is achieved which means robust performance is not fulfilled. In this case the following questions arise. Should the nominal model be re-identified with higher order? Should the uncertainty model be refined with more perturbation blocks with the price of increasing controller order? Should the performance requirements be weakened?

In simulation environment the effect of road slope (disturbance) and unmodeled dynamics can be separated. We found that the effect of disturbance is much larger than that of neglected dynamics, see Fig. 6, where disturbance and perturbation contributions in the nominal model are compared. Considerable perturbation is present at low frequencies below 0.6 $rad/s$.

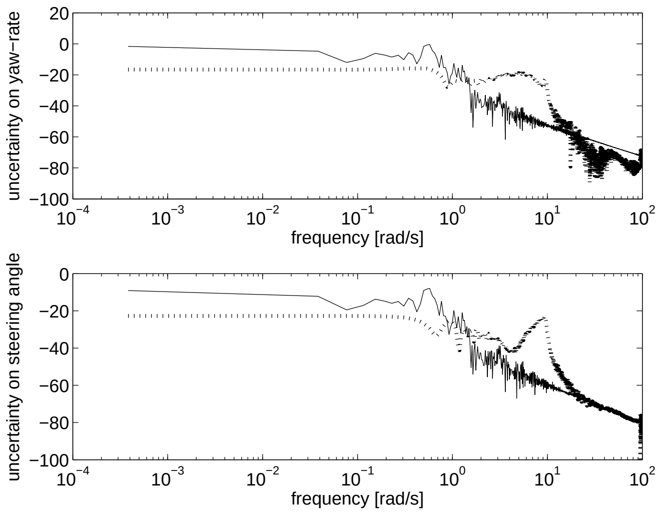
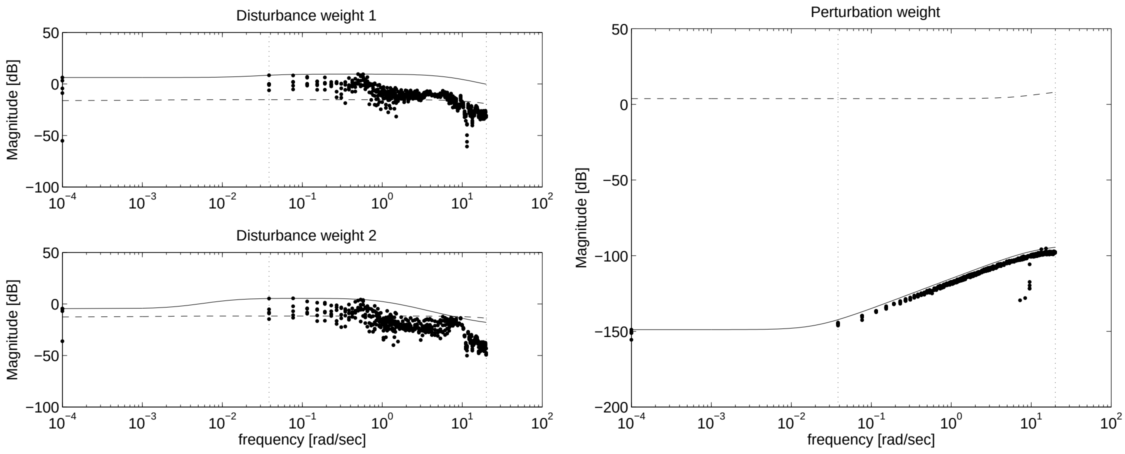



**Fig. 6.** Perturbation (dotted) and disturbance (solid) contributions in the nominal model error. For yaw-rate (top) and steering angle (bottom)

The engineering hypothesis was false. But in the real we usually cannot make such an analysis. The method proposed in the previous sections is applied to remodel the uncertainty. The controller is given, the data used for identifying the nominal model is given, Theorem 2 can be applied. After over-bounding $\sqrt{V_k}$




(a) Weight for the additive disturbances (b) Weight for the input-multiplicative perturbation

**Fig. 7.** Weight functions of the two additive disturbances and the input-multiplicative perturbation. The initial (dashed), the result after tuning of the model (solid).

magnitude points by $W_\Delta$ and $W_d$ functions and fitting a $D(j\omega)$ scaling matrix on the $D_k$ magnitude points a new controller is designed by simple $\mathcal{H}_\infty$ method (only the K-step in D-K iteration of the $\mu$-synthesis). The proposed uncertainty remodeling method and controller design is repeated a few times as shown in Fig. 7, where the initial weights are plotted with dashed lines, the final with solid lines and the $\sqrt{V_k}$ points in the intermediate iteration steps with dots. It can be seen that weights of disturbances increase and the weight of perturbation decreases. The achieved peak $\mu$ value is 0.843, so the robust performance is achieved. We can trust in the resulted controller provided the data were well informative i.e. no future experiment with the controller will invalidate the model. If it is invalidated, the new data must be added to the set $S_{yu}$ and redesign of the model and controller is needed.

One might think that we got back the real distribution of the uncertainty. It is emphasized that the information of the real distribution is not contained in the data. The resulting weight functions fit the robust performance criterion and the data set is only a constraint for having an unfalsified uncertainty model, which have questionable physical meaning.

The initial and resulted controllers are compared. The speed is kept 12m/s in the nonlinear simulator model by PID control of driving torque on the rear axles. The road slope is varying. No extra sensor noise is defined. In Fig. 8 from top to down the control input in a transient, the yaw-rate reference tracking in a transient and the yaw-rate reference tracking in steady state can be seen. It is shown that control input became moderated and yaw-rate tracking improved in both steady state and transient.

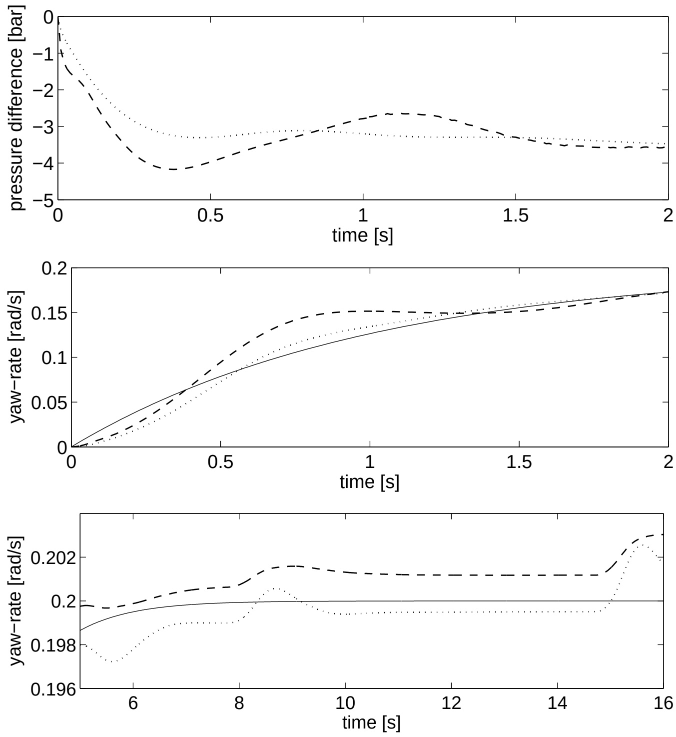



**Fig. 8.** Closed-loop simulation at $v = 12m/s$. Solid line: yaw-rate reference $r_{ref}$, dashed: initial, dotted: tuned.

## 6 Conclusions

A new robust performance measure is defined which implicitly contains model consistency conditions. The proposed measure answers the following question. Given a controller, is robust performance satisfied for any model set consistent with the available measurements? If no (the measure is greater than one), the controller is falsified. If the answer is yes, then does this fact gives confidence in the controller? It depends on the data whether it represents well the model uncertainty. Therefore the measure is applied in controller synthesis problem with iterative redesign of the controller (just like $\mu$). To increase our confidence at any iteration step new measurement data performed by the new controller can be added. The usefulness of the iteration is proved through a vehicle control problem.

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