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A NEW ANALYSIS OF THE INTRODUCTION OF THE MAGNETIC FIELD STATE QUANTITIES IN SOLIDS

Andrei NICOLAIDE

Transylvania University of Brasov
Bd. Eroilor, Nr. 29
500036 Brasov, Romania
e-mail: nicolaide@leda.unitbv.ro

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Abstract

The introduction of the primitive magnetic field state quantities on a macroscopic and a microscopic scale is achieved by thought experiments. A comparison is made between the introduction of the macroscopic quantities via a direct procedure and via the average of the microscopic values. The average of the microscopic quantities is performed, in the known works, by the calculation of the average of the microscopic electric current density in the right-hand side of the law of magnetic circuit or of the Ampère theorem, and then the average of the vector curl of the left-hand side is expressed. This indirect derivation involves certain difficulties. In the present work, a direct deduction is used by calculating directly the average of the primitive magnetic field state quantity. For this purpose, two physical models have been proposed: the elementary magnetic dipole model and the elementary solenoid model. It is shown that the average for the former model gives the macroscopic field strength, whereas the average for the latter gives a quantity proportional to the magnetic induction.

Keywords: macroscopic and microscopic magnetic field quantities.

Nomenclature

$\boldsymbol{B}$	– macroscopic magnetic induction;
$\boldsymbol{B}_{\text{micro}}$	– microscopic magnetic induction for the elementary solenoid model;
$\boldsymbol{E}$, $\boldsymbol{E}_{\text{micro}}$	– electric field strength, macroscopic and microscopic values, respectively;
$\boldsymbol{F}$	– force acting upon a point-like particle with electric or magnetic charge;
g	– thickness of the wall of the elementary solenoid;
$\boldsymbol{H}$	– macroscopic magnetic field strength;
$\boldsymbol{H}_{\text{micro}}$	– microscopic magnetic field strength for the elementary solenoid model;
$\boldsymbol{H}_{\text{micro dip}}$	– microscopic magnetic field strength for the elementary magnetic dipole model;
h_S	– height (length) of an elementary solenoid;
i_0	– current intensity of an elementary (called also molecular, Amperian, microscopic) electric current loop or of an elementary solenoid;

J_{0L} – linear density of the electric current sheet of an elementary solenoid;
$\boldsymbol{M}$ – magnetization;
$\boldsymbol{M}_j = \mu_0 \boldsymbol{M}$ – magnetic polarization;
$\boldsymbol{m} = \boldsymbol{m}_0$ – denotes the Amperian magnetic moment of an elementary (called also molecular, Amperian, microscopic) electric current loop or of an elementary (called also microscopic) magnetic dipole, as well as the Amperian magnetic moment of an elementary solenoid;
$\boldsymbol{m}_j = \mu_0 \boldsymbol{m}$ – denotes the Coulombian magnetic moment for the cases above;
$\boldsymbol{n}$ – unit vector of the normal to an elementary (molecular, Amperian, microscopic) current loop surface or unit vector of the axis of an elementary solenoid;
$\boldsymbol{n}_i$ – unit vector of the axis of the elementary solenoid with the ordinal number i;
n_S – number of elementary solenoids of a physically infinitesimal space domain;
n_0 – volume numerical concentration of elementary (Amperian, molecular, microscopic) current loops or of elementary solenoid centres;
q – electric charge of a particle;
q_m – fictitious Coulombian magnetic charge of a particle;
$\boldsymbol{r}$ – position vector;
r_0 – internal radius of the elementary solenoid;
S_0 – area of the surface of an elementary current loop or cross-section area of an elementary solenoid;
$\boldsymbol{T}_M$ – torque acting on an elementary current loop or on an elementary magnetic dipole;
V_m, V_{micro} – magnetic potential at a point, macroscopic, and microscopic values;
$V_{\text{micro dip}}$ – magnetic potential produced at a point by an elementary (microscopic) magnetic dipole;
$\boldsymbol{v}$ – velocity of an electrically charged particle;
Δv – physically infinitesimal domain of space (volume);
μ_0 – magnetic permeability of vacuum (magnetic constant);
ρ_{mp} – volume density of the fictitious polarization magnetic charge.

Indices

cavit – index denoting a quantity related to a cavity;
micro dip – index denoting microscopic quantities in the case of magnetic dipoles.

The SI system of units is used.

1. Introduction

The quantities, which characterize the macroscopic state of the magnetic field, are the quantities $\boldsymbol{B}$ and $\boldsymbol{H}$. These quantities may characterize the macroscopic state whereas only one of the two quantities $\boldsymbol{B}_{\mathrm{micro}}$ or $\boldsymbol{H}_{\mathrm{micro}}$ mentioned above suffices to characterize the microscopic magnetic state [1]–[13].

For the sake of clearness, we shall denote the quantities which occur according to the symbols given in nomenclature.

The manners to introduce the macroscopic or microscopic quantities are based on a thought experiment, sometimes also practically possible but only approximately.

The macroscopic quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ can be introduced in two manners; the macroscopic mode, based on direct determinations, and the microscopic mode, the latter being based on the computation of the mean value (average) of the microscopic values.

The two modes, macroscopic and microscopic, as shown below, have several inconveniences, although the latter represents an important progress in comparison with the former one. However, also in this mode, in the known works, a difficulty subsists for the following reason. The average of the microscopic quantities is performed, in the known works [12], [13], [17], by the calculation of the average of the microscopic electric current density in the right-hand side of the law of magnetic circuit or of the Ampère theorem. Then, the average of the vector of the curl operator of the left-hand side is expressed. Thus, an indirect derivation is used. In these circumstances, the potential component vector is difficult to be expressed because of the existence of the curl operator acting on that vector. At the same time, no satisfactory mention exists for making evident, that, if starting from various models, it is possible to obtain by calculating the averages both macroscopic values $\boldsymbol{B}$ and $\boldsymbol{H}$ according to the used model.

The aim of the paper is to try to obtain both quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ directly from the calculation of the mean values (averages) of the microscopic values using various models. Before presenting this new approach, we consider that it is necessary to recall the essential sequences of the two modes mentioned above in order to compare their properties more easily.

A comparative analysis of the various manners of the direct introduction of the macroscopic quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ is not given in literature either. For this reason, this subject will be further examined.

In order to simplify the analysis, the macroscopic conduction, convection and displacement currents will not be taken into consideration. The presence of these currents can modify the values of the different magnetic state quantities, but cannot modify the relation between them.

2. The Macroscopic Mode of Introducing the Macroscopic Quantities $\boldsymbol{B}$ and $\boldsymbol{H}$

According to the known papers, the macroscopic quantities $\boldsymbol{B}$ and $\boldsymbol{H}$ have been introduced in a systematic manner, essentially in the treatise of MAXWELL [1, Vol. 2, Arts. 395–400]. We shall briefly recall the sequence. The starting quantity is the magnetic potential considered as being produced by a fictitious (imagined) magnetic charge distributed in the volume or on the surface of the magnetized bodies.

The magnetic potential may be expressed by the same mathematical relations irrespective of the fact whether the point at which the potential has to be determined is outside or inside the magnetized body. The quotient obtained by dividing the force acting on a point-like body charged with the fictitious magnetic charge by the value of this fictitious charge, can be got from the relations:

$$\boldsymbol{F} = q_m \boldsymbol{H}; \tag{1a}$$

$$\boldsymbol{H} = -\text{grad } V_m. \tag{1b}$$

The state quantities of the magnetic field are introduced by the general mode used in physics for introducing a kind of physical quantity: one defines the measuring proceeding and the unit of measure. The thought experiment, in the case of this model, can be explained as follows.

For defining the force acting on a fictitious magnetic charge at a point *in vacuo*, it is assumed that at this point a small or point-like body (test body) charged with fictitious magnetic charge is brought, and it is placed at rest at the considered point. From the expression of the force, also in a reference system at rest, we get, like in Electrostatics, the magnetic field strength.

If the point, at which the force acting on the magnetically charged body has to be determined, is outside the substance, generally, no difficulty can arise. If this point is inside the substance, then, one begins by removing the magnetized substance from a small domain surrounding and containing the point. Thus, a cavity is achieved, in which the small body charged with the fictitious magnetic substance will be brought. It is assumed also that in the whole domain from which the substance is removed, the magnetization of the substance is homogeneous.

The force acting on the small body depends, in general, on the form of the cavity, and on the inclination of the walls of the cavity with respect to the direction of the magnetization $\boldsymbol{M}$ at the considered point.

In the case in which the form and the position of the cavity are given, the magnetically charged point-like body placed in the cavity may be considered as being *in vacuo*, and we come back to the initial case, when the magnetically charged point-like body is *in vacuo*. In work [1], the cavity is considered as having the form of a right circular cylinder with the axis parallel to the direction of the magnetization. For a similar purpose, according to other works [5, p. 167, 290], [8, vol. I, p. 298], [9, tome III, p. 244], a cavity having the form of an ellipsoid of revolution may be used. However, the derivations for the two types of the cavity are, to some extent, different. In the case of the cylindrical cavity, the charged point-like body must be placed at the centre of the cavity. In the case of the ellipsoidal cavity, this body may be placed at any point inside the cavity. The quantities determined in the cavity will be denoted by the index *cavit*.

Two cases will be considered according to work [1].

Case I

The diameter of the cylinder is very small compared to its length. Because the axis of the cylinder is parallel to the magnetization, the magnetic polarization charge density on the lateral surface of the cylinder will be zero, and it will be different from zero on the basis of the cylinder. In this case, one obtains:

$$\boldsymbol{F}_{\text{cavit}} = q_m \boldsymbol{H}_{\text{cavit}}; \tag{2a}$$

$$\boldsymbol{H}_{\text{cavit}} = \boldsymbol{H}. \tag{2b}$$

Case II

The length of the cylinder is negligible compared to its diameter.

The quotient obtained by dividing the force acting on the magnetically charged point-like body by its fictitious magnetic charge will be denoted by $\frac{1}{\mu_0}\boldsymbol{B}$. In this case, one obtains:

$$\boldsymbol{F}_{\text{cavit}} = q_m \boldsymbol{H}_{\text{cavit}}; \tag{3a}$$

$$\frac{1}{\mu_0}\boldsymbol{B}_{\text{cavit}} = \frac{1}{\mu_0}\boldsymbol{B} = \boldsymbol{H} + \boldsymbol{M}. \tag{3b}$$

This mode of introducing the macroscopic quantities $\boldsymbol{H}$ and $\boldsymbol{B}$, given in the treatise of Maxwell, was subsequently analysed and developed in other papers, among which, especially, by RADULET [2] and BOBBIO [17, p. 323].

This mode, as mentioned above, has the advantage of being relatively simple, but has the following inconveniences:

1. It is assumed that the magnetization was homogeneous before achieving the cavity and, although not usually mentioned, is considered to remain homogeneous far enough from the cavity.
2. Another objection to the method refers to the fictitious magnetic charge, but this circumstance may be avoided. Hence, it is possible to consider instead of the magnetically charged point-like body (test body, proof body) a small magnet. Then, instead of the force acting on the body, one can consider the torque acting on the magnet. This remark has been made for a long time in the work of BLOCH [4, p. 145]. For determining the magnetic field strength, it is necessary to introduce the small magnet at the considered point, under a

certain angle, so that the torque should have the maximum value; otherwise the determination of the vector $\boldsymbol{H}$ is not unique according to the relation:

$$\boldsymbol{T}_M = \boldsymbol{m}_j \times \boldsymbol{H}. \tag{4}$$

3. The Microscopic Mode of Introducing the Macroscopic Quantities *B* and *H*

For several reasons, among them the first inconvenience mentioned above, in many approaches one obtains the macroscopic magnetic state quantities by calculating the mean values (averages) of the microscopic magnetic state quantities. At the ground of this mode lies the theory of LORENTZ [3, Chapter IV, Nos. 111–114]. According to this theory, the introduction of the quantities $\boldsymbol{B}_{\text{micro}}$ and $\boldsymbol{H}_{\text{micro}}$ results in introducing only $\boldsymbol{H}_{\text{micro}}$ or $\boldsymbol{B}_{\text{micro}}$ because all charges are considered *in vacuo*, and *in vacuo* $\boldsymbol{B}_{\text{micro}} = \mu_0 \boldsymbol{H}_{\text{micro}}$. The thought experiment for introducing the magnetic field state quantities, in the case of this model can be explained as follows.

The macroscopic quantity $\boldsymbol{B}$ or $\boldsymbol{H}$ can be introduced from the expression of the force acting on an electrically charged particle (charge carrier) with the electric charge q, moving in vacuum, according to the expression of Lorentz:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \mu_0 \boldsymbol{H}) = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \tag{5}$$

or in order to emphasize that, in general, microscopic quantities are considered:

$$\boldsymbol{F} = q(\boldsymbol{E}_{\text{micro}} + \boldsymbol{v} \times \mu_0 \boldsymbol{H}_{\text{micro}}) = q(\boldsymbol{E}_{\text{micro}} + \boldsymbol{v} \times \boldsymbol{B}_{\text{micro}}). \tag{6}$$

The starting point of the theory of Lorentz is that the equations of Maxwell expressed in a suitable form are available for the exterior, as well as for the interior of an electrically charged particle [3, Chapter I, No. 7].

Starting from the quantity $\boldsymbol{B}_{\text{micro}}$ introduced according to relation (6), the macroscopic quantity is obtained by calculating the mean value of the microscopic quantity for physical infinitesimals of space and time. Lorentz introduced the concept of mean value (average) in space and time, related to the notions (concepts) which have been called physical infinitesimals of space and time, respectively. These notions have been developed subsequently in several papers, among them, minutely, in works [4, p. 408], [5, p. 455], [6, p. 404] and [11, p. 47, 49]. The principle of the calculation of the mean value (average) of a state quantity was introduced by LORENTZ [3, Chapter IV, Nos. 111–114]. A detailed development concerning the conditions and the procedure for performing the calculation has been given in literature, in the work of FOURNET [5, pp. 455–461].

The average of the microscopic quantities is performed in the known works [12], [13], [17, p. 228], by the calculation of the average of the microscopic electric current density in the right-hand side of the law of magnetic circuit or of the Ampère theorem. Then, the average of the vector curl of the left-hand side is expressed. Thus, an indirect derivation is used. This indirect derivation involves certain inconveniences as mentioned in Introduction.

A very detailed analysis of the subject is given in works [12] and [13, Section 6.7]. In these works, for obtaining the averages of the magnetic field state quantities above, the calculation of the average of the electric current density microscopic value $\boldsymbol{J}_{\text{micro}}$ of the right-hand side of the law of magnetic circuit has been performed. In that approach, the electric charges are considered to be point-like charges, and certain calculations have been carried out as follows. The statistic average value has been calculated using certain distribution functions, the delta function, and a Taylor series expansion in terms of the distances from the point-like charges above, to the centres of mass of the molecules. Only the average with respect to space has been carried out, considering that the average with respect to time will be no more necessary. The replacement of charge carriers by point-like charges is quite different from the Lorentz theory. The other calculations are similar to those above, hence an indirect derivation.

It is worth noting that the resulting equations, in all approaches above, are the same.

4. The Proposed Direct Derivation of the Macroscopic Quantities $\boldsymbol{B}$, $\boldsymbol{H}$ and $\boldsymbol{M}$, and of Their Relationship, Starting from the Microscopic Quantities

We shall accept that all magnetic effects are produced by elementary (molecular, Amperian, microscopic) current loops, as mentioned in the works of ELLIOTT [6, p. 404], IVÁNYI [15, p. 7] and VALLÉE [14, p. 67]. For computing magnetic fields, the elementary current loops can be replaced by equivalent elementary (microscopic) magnetic dipoles [5, p. 227], [6, pp. 404–408], [8, vol. II, p. 225, 230], [15, pp. 7–8], [16].

If the elementary currents are considered to be constant, hence, the mean values with respect to time are considered, and then, for obtaining the macroscopic values of the magnetic field quantities, only the average values with respect to space have to be calculated.

For introducing the macroscopic magnetic field state quantities, we shall present two approaches based on two physical models utilized for describing the state of the magnetized substance [16].

The first approach is based on the *microscopic (elementary) magnetic dipole model* [5, p. 227], [6, pp. 404–408], [8, vol. II, p. 230], [15, p. 7], in the modified form of the *elementary magnetic dipole model*; the second approach is based on the *microscopic (elementary, molecular, Amperian) current loop model*, in the modified form of the *elementary solenoid model*. Each of the proposed models permits to obtain results somewhat different but related to one another. The replacement of the point-like charges by non-point-like ones is advantageous. For instance, it is difficult to attribute to a point-like electric charge a magnetic spin moment. A corresponding remark can be made for fictitious magnetic charges.

For the sake of simplicity, we shall make the following assumptions:

1. All the elementary solenoids are circular and of the same sizes and value of current.

2. The physical infinitesimal of time is great enough, so that the moving charge of the elementary solenoid could be replaced, at the considered time, by a constant current of intensity i_0.

4.1. The Approach to the Microscopic Magnetic Dipole Model in the Modified Form of the Elementary Magnetic Dipole Model

A domain containing a magnetized substance composed of a set of microscopic current loops placed *in vacuo* will be considered. We shall consider the microscopic magnetic dipoles equivalent to the microscopic current loops. Further on, this dipole will be referred to as *elementary magnetic dipole*.

If, as above, the microscopic currents are considered to be constant, the fictitious magnetic charges will also be constant, hence, for obtaining the macroscopic values of the magnetic field quantities, only the average values with respect to space have to be calculated.

The thought experiment for introducing the magnetic field state quantity, in the case of this model, is similar to that of the macroscopic approach.

Every elementary magnetic dipole placed at any point P produces at any observation point N situated at a distance large enough a potential that may be expressed apart from a constant by:

$$V_{\text{micro dip}} = \frac{\boldsymbol{m}_j \cdot \boldsymbol{r}}{4\pi \mu_0 r^3}, \tag{7}$$

where $\boldsymbol{r}$ denotes the position vector of the observation point with respect to the centre of the dipole.

The magnetic field strength at the same point N will be:

$$\boldsymbol{H}_{\text{micro dip}} = -\text{grad}_N \left(\frac{\boldsymbol{m}_j \cdot \boldsymbol{r}}{4\pi \mu_0 r^3} \right). \tag{8}$$

The value of $\boldsymbol{H}_{\text{micro dip}}$ at the observation point is assumed to be measured by the torque exerted on a small magnet placed at rest at this point.

By summing up the components produced by all elementary magnetic dipoles at any point N, the resultant quantity $\boldsymbol{H}_{\text{micro dip}}$ at that point will be obtained. From the form of the used expressions, follows that the quantity $\boldsymbol{H}_{\text{micro dip}}$ will have only a potential component.

We shall consider a physically infinitesimal domain of volume Δv surrounding and containing any point N. The macroscopic value, at the same point N, may be obtained after calculating the mean value (average) of the microscopic quantity above, over the physically infinitesimal domain (volume) Δv. It follows that the macroscopic value at the same point N will be:

$$\boldsymbol{H}(N) = \langle \boldsymbol{H}_{\text{micro dip}} \rangle. \tag{9}$$

We shall admit that only magnetic polarization charges exist. In this case, like in Electrostatics,

$$-\text{div}\ \boldsymbol{M}_j = \rho_{mp}. \tag{10a}$$

In the same case, like in Electrostatics, one can write:

$$\text{div}\ \boldsymbol{H} = \frac{1}{\mu_0}\rho_{mp}. \tag{10b}$$

For deriving the expressions (10a) and (10b), taking into account formula (9), it is possible to start from the integral form of the flux of the vector $\boldsymbol{H}_{\text{micro dip}}$ through a closed surface [8, vol. I, p. 59, 102, vol. II, p. 308].

It is useful to note that in the case of the used model, the expression obtained for $\boldsymbol{M}_j$ is similar to expression (18) obtained below but in the case of another model.

From relations (10a) and (10b), it follows:

$$\text{div}\ (\mu_0 \boldsymbol{H} + \boldsymbol{M}_j) = \text{div}\ \boldsymbol{B} = 0. \tag{11}$$

We denote:

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}_j. \tag{12}$$

In this approach, the macroscopic quantity $\boldsymbol{H}$ is the average (mean value) of $\boldsymbol{H}_{\text{micro dip}}$, and the quantity $\boldsymbol{B}$ that results from relation (12) can denote only a macroscopic value.

Taking into account the potential character of the quantity $\boldsymbol{H}_{\text{micro dip}}$, it follows that:

$$\text{curl}\ \boldsymbol{H} = 0. \tag{13}$$

Taking into account also the equations of the magnetic field that have to be satisfied by the magnetic state quantities $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{M}_j$, it follows that the vector quantity $\boldsymbol{B}$ is determined apart from an additional component, namely a curl (rotational) vector. The same component has to be added to the vector quantity $\boldsymbol{M}_j$, since the vector quantity $\boldsymbol{H}$ must remain unchanged being directly determined. By convention, the simplest solution has been chosen, namely the additional component has been adopted equal to zero.

4.2. The Approach to the Microscopic Current Loop Model in the Modified form of the Elementary Solenoid Model

We shall consider an electrically charged particle (electric charge carrier) *in vacuo* moving with respect to a reference system of the substance with the velocity v.

The thought experiment for introducing the quantity $\boldsymbol{B}_{\text{micro}}$ in the case of this model is similar to that in the macroscopic approach.

The quantity $\boldsymbol{B}_{\text{micro}}$ may result from the force acting on the electrically charged particle *in vacuo*, according to the formula (6) of Lorentz. We can consider any moving charged particle as moving in vacuum.

We shall examine which is the magnetic induction acting on a moving electrically charged particle at a point N. At this point, the magnetic induction $\boldsymbol{B}_{\text{micro}}$ will be produced by all the microscopic electric currents (the other types of currents being not taken into consideration as mentioned at the beginning).

Let us assume the model of a microscopic current loop in the form of a very small solenoid that is more suitable for calculation than a current loop. This solenoid is formed by an electric current having the form of a right circular cylinder the height of which can have any value (generally small) compared to its radius, and the thickness of its wall is very small. It follows that $g \ll r_0$. Further on, we shall refer to this solenoid as *elementary solenoid*. The linear density of the current shell round the solenoid will be denoted by J_{0L} [A/m].

We shall consider the physically infinitesimal domain of space (volume) Δv surrounding and containing any point N. A representation of the considered configuration is given in *Fig. 1*. We shall also assume that the physically infinitesimal volume Δv contains n_S elementary solenoids. We shall calculate the mean value (average) of the microscopic magnetic field strength, in this case, over the volume Δv.

Each elementary solenoid can be regarded as belonging to an infinitely long solenoid of the same radius. Therefore, we can consider that the infinitely long solenoid consists of three parts: the elementary solenoid (solenoid of small length, bounded by two circular bases) and other two semi-infinite solenoids, each of them situated on one part of the elementary solenoid, in order to build-up the whole infinitely long solenoid.

To calculate the magnetic field strength at a point inside the elementary solenoid, the superposition principle will be used.

The distance ρ from the axis of the solenoid to any interior point satisfies the relation $\rho \in [0, r_0)$, whereas the distance from the axis to any exterior point satisfies the relation $\rho > r_0 + g$.

We shall calculate first the magnetic field strength at a point in the interior of an elementary solenoid. The magnetic field strength at a point in the interior of this solenoid will have components of three types (*Fig. 1*):

1. A component of type *a*, produced by the infinitely long solenoid given by the known relation;
2. A component of type *d* produced by the two solenoids, each of them situated on one side of the elementary solenoid, taken with changed sign, because it has to be subtracted from the previous one;
3. A component of type *b* produced by all other elementary solenoids from the interior as well as from the exterior of the domain Δv excepting the considered elementary solenoid.

   It is to be added that at points that do not belong to the domain of an elementary solenoid, only the following component occurs:
4. A component of type *c* produced by all solenoids situated *inside* as well as *outside* the physically infinitesimal volume.

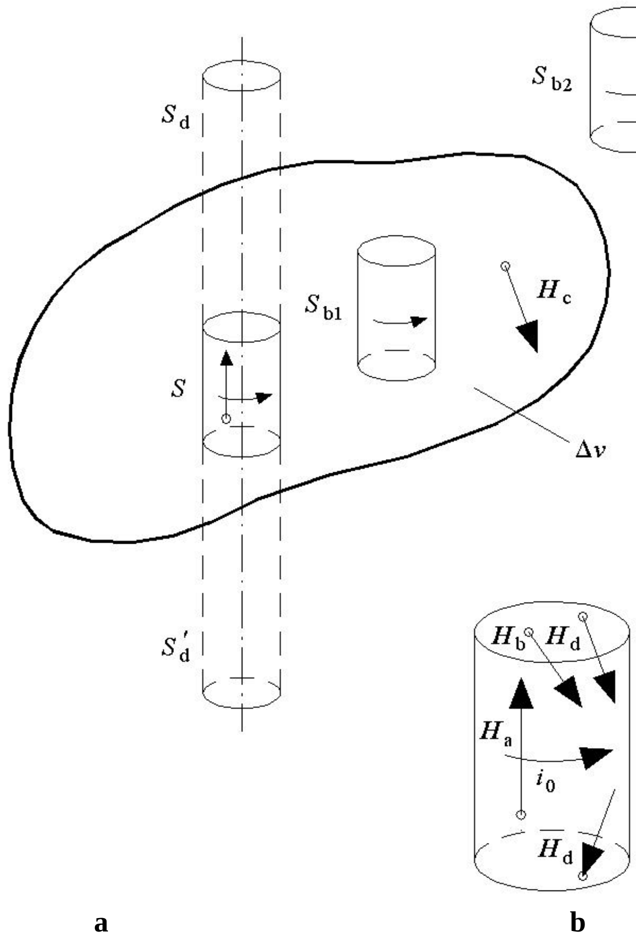

Fig. 1. Infinitesimal domain (volume) and components of the magnetic field strength: **a** – The infinitesimal volume Δv containing the elementary solenoids like S, $S_{\rm b1}$ (inside the domain), $S_{\rm b2}$ (outside the domain), and the semi-infinite solenoids $S_{\rm d}$ and $S'_{\rm d}$, the magnetic field strength $\boldsymbol{H}_{\rm c}$. **b** – The twice magnified solenoid S and the components of various types of the magnetic field strength inside it: $\boldsymbol{H}_{\rm a}$, component produced by the infinite solenoid $S_\infty = S \cup S_{\rm d} \cup S'_{\rm d}$; $\boldsymbol{H}_{\rm d}$, component produced by both semi-infinite solenoids $S_{\rm d}$ and $S'_{\rm d}$, at two points (up and down) and taken with changed sign; $\boldsymbol{H}_{\rm b}$, component produced by all elementary solenoids (like $S_{\rm b1}$ or $S_{\rm b2}$), excepting S. The components are given at points marked by a small circle. The suffix *micro* of the magnetic field strength has been omitted in the figure, for the sake of simplicity.

The various component vectors may be of a potential or a curl nature. A vector will be considered of potential nature, if it belongs to a field of vectors, which derives from a potential *at any point of the considered domain*. A vector will be considered of curl (rotational) nature, if it belongs to a field of vectors the curl of which is different from zero at least *at one point of the considered domain*.

We shall calculate the mean value of the component of type a in this case.

The expression of the component of type a, i.e., $\boldsymbol{H}_{\text{micro a}}$, is at any point in the interior of one elementary solenoid:

$$\boldsymbol{H}_{\text{micro a}} = \frac{i_0}{h_S}\boldsymbol{n} = J_{0L}\boldsymbol{n}, \tag{14}$$

where h_S means the height (length) of the elementary solenoid, whereas the other symbols are the same as previously. For a more detailed analysis of the magnetic field of the solenoid, we may assume that the wall of the solenoid has a certain thickness g, very small compared to its radius. At the same time, we can assume that the current density has a certain distribution along this thickness. The integral of this density along the thickness g will give the linear current density J_{0L} above. Further on, the influence of the wall thickness will be disregarded. Indeed, it can be shown that the smaller the ratio g/r_0, the smaller that influence will be. It is to be noted that the curl of the mentioned component is not zero at any point of the domain occupied by the solenoid, because the electric current density is different from zero within the solenoid wall.

The number of elementary solenoids contained by the domain Δv can be calculated by the relation:

$$n_S = n_0 \Delta v. \tag{15}$$

The sum of this component over the volume Δv will be:

$$\sum_{i=1}^{i=n_S} (\boldsymbol{H}_{\text{micro a}})_i = \sum_{i=1}^{i=n_S} J_{0L}\boldsymbol{n}_i. \tag{16}$$

The mean value (average) of this component over the volume Δv will be:

$$\langle \boldsymbol{H}_{\text{micro a}} \rangle = \frac{1}{\Delta v} \sum_{i=1}^{i=n_S} J_{0L} S_0 h_S \boldsymbol{n}_i = \frac{1}{\Delta v} \sum_{i=1}^{i=n_S} \boldsymbol{n}_i i_0 S_0 = \frac{1}{\Delta v} \sum_{i=1}^{i=n_S} \boldsymbol{m}_i. \tag{17}$$

It is to be noted that the value given by relation (17) is the macroscopic value at a point N (the co-ordinates of which remain constant during the calculation of the mean value) and not a volume distribution. Only the value of the average $\boldsymbol{H}$, in terms of the co-ordinates of the point N, represents the volume distribution.

In the particular case in which all magnetic moments are parallel to the same direction, we get:

$$\langle \boldsymbol{H}_{\text{micro a}} \rangle = \boldsymbol{n} n_0 m_0. \tag{17a}$$

From relation (17) follows that the mean value (average) of the components of type a, namely $H_{\text{micro a}}$, over the volume Δv will be:

$$\boldsymbol{H}_{\text{a}} = \langle \boldsymbol{H}_{\text{micro a}} \rangle = \boldsymbol{M} = \frac{1}{\mu_0} \boldsymbol{M}_j . \quad (18)$$

The vector $\boldsymbol{M}$ of relation (18) denotes the magnetization by definition. We shall return to this remark after relation (26b). In the hypothetical case in which the considered substance filled the whole space and the solenoids of above were infinitely long, only the component of type a would exist. We recall that the magnetic field of an infinitely long solenoid is zero outside. In all real cases, i.e. when the length of any solenoid of above is finite, the components of the other types will also exist.

The mean value (average) of components of types d, i.e. $\boldsymbol{H}_{\text{micro d}}$ and b, i.e. $\boldsymbol{H}_{\text{micro b}}$ over the volume Δv will be:

$$\boldsymbol{H}_{\text{db}} = \langle \boldsymbol{H}_{\text{micro db}} \rangle , \quad (19)$$

and will be a potential component (i.e. its curl will be zero). Indeed, the curl of the vector components of types d and b (microscopic quantities) is zero at any point within the domain of an elementary solenoid, hence where these components are considered, since the currents that produce these components are zero at every point of that domain, though they are different from zero at other points.

The magnetic field strength at any point that does not belong to the volume of an elementary solenoid will be referred to as component of type c, i.e. $\boldsymbol{H}_{\text{micro c}}$; this component is produced, as mentioned, by all solenoids situated *inside* as well as *outside* the physically infinitesimal volume.

It is to be noted that the superposition principle can be used for calculating the magnetic field strength at a point inside as well as outside the elementary solenoid.

The component of type c, i.e., $\boldsymbol{H}_{\text{micro c}}$, at any point outside an elementary solenoid, is also given by a sum of components of types a and d, like above, with the remark that the component of type a outside the solenoid lengthened to infinity is zero.

However, the magnetic field strength produced by an elementary solenoid at any point in its exterior may also be calculated by the Biot-Savart-Laplace law, and it is a potential component.

The mean value (average) of components of types c, i.e. $\boldsymbol{H}_{\text{micro c}}$, over the volume Δv will be:

$$\boldsymbol{H}_{\text{c}} = \langle \boldsymbol{H}_{\text{micro c}} \rangle , \quad (20)$$

and will be a potential component.

The mean value (average) of components of types d, b and c over the physically infinitesimal volume Δv will be obtained by summing up the corresponding averages (mean values) above, and will be $\boldsymbol{H}$ or $\mu_0 \boldsymbol{H}$, respectively:

$$\mu_0 \boldsymbol{H} = \langle \mu_0 \boldsymbol{H}_{\text{micro dbc}} \rangle = \mu_0 \left(\boldsymbol{H}_{\text{db}} + \boldsymbol{H}_{\text{c}} \right) . \quad (21)$$

It follows that:

$$\langle \mu_0 \boldsymbol{H}_{\text{micro}} \rangle = \langle \mu_0 \boldsymbol{H}_{\text{micro a}} \rangle + \langle \mu_0 \boldsymbol{H}_{\text{micro dbc}} \rangle\,; \tag{22}$$

$$\langle \mu_0 \boldsymbol{H}_{\text{micro}} \rangle = \mu_0 \boldsymbol{M} + \mu_0 \boldsymbol{H} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}; \tag{23}$$

and

$$\boldsymbol{H}_{\text{micro}} = \boldsymbol{H}_{\text{micro adbc}}. \tag{23a}$$

Consequently, the resultant macroscopic induction at any point N will be, as above:

$$\boldsymbol{B} = \mu_0 \boldsymbol{H} + \boldsymbol{M}_j. \tag{24}$$

Taking into account that the quantity $\boldsymbol{H}_{\text{micro}}$ is produced only by currents with a volume distribution, it follows that it yields a solenoidal field, thus:

$$\text{div}\ \boldsymbol{H}_{\text{micro}} = 0, \tag{25}$$

and

$$\text{div}\ \boldsymbol{B} = 0. \tag{26a}$$

Having in view that the quantity $\boldsymbol{H}$ corresponds to the components of types d, b and c which have a potential character, it follows that:

$$\text{curl}\ \boldsymbol{H} = 0. \tag{26b}$$

Taking into account also the equations of the magnetic field that have to be satisfied by the magnetic state quantities $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{M}$, it follows that the vector quantity $\boldsymbol{H}$ is determined apart from an additional component, namely a gradient vector. The same component has to be subtracted from the vector quantity $\boldsymbol{M}$, since the vector $\boldsymbol{B}$ must remain unchanged, being directly determined. By convention, the simplest solution has been chosen, namely the additional component has been adopted equal to zero.

Under this condition, the vector $\boldsymbol{M}$ defined in the framework of both models has the same expression.

To the quantity $\boldsymbol{H}$ of formula (23), a component that will be called of type e, produced by an external magnetic field, if any, has to be added. This quantity could have both components, potential and curl ones.

Therefore, the proposed mode permits to obtain directly the macroscopic quantities $\boldsymbol{H}$ and $\boldsymbol{B}$, and their relationship with $\boldsymbol{M}$.

4.3. Comparison of the Calculations When Utilizing the Two Microscopic Models

For computing the magnetic field strength, two models have been examined: the elementary magnetic dipole model and the elementary solenoid model.

4.3.1. The Case of the Elementary Magnetic Dipole Model

In this case, the microscopic magnetic field strength at any point inside the substance where no magnetic dipole exists has a certain value $\boldsymbol{H}_{\text{micro dip}}$. The value is produced by all dipoles of the magnetized substance.

As mentioned above, the value of $\boldsymbol{H}_{\text{micro dip}}$ at the observation point is assumed to be measured by the torque exerted on a small magnet placed at rest at this point. The considered quantities $\boldsymbol{H}_{\text{micro dip}}$ at any point where no dipole exists correspond to the formula of the magnetic field strength at a relatively distant point. The quantity at a point within the domain assumed to be a dipole is considered, taking into account the contribution of the dipole under consideration. The quantity $\boldsymbol{H}_{\text{micro dip}}$ has a potential character.

4.3.2. The Case of the Elementary Solenoid Model

In this case, the microscopic magnetic field strength at any point inside the substance, where no elementary current loop exists, has a certain value $\boldsymbol{H}_{\text{micro}}$. This value is produced by all elementary solenoids of the magnetized substance. This value, at the observation point where no elementary solenoid exists, is assumed to be measured by the force exerted on a small moving electric charge carrier passing through the point.

The magnetic field strength at a point within an elementary solenoid is produced by all elementary solenoids, the solenoid under consideration included.

If the observation point is outside the domain of an elementary solenoid or a dipole, respectively, the values $\boldsymbol{H}_{\text{micro}}$ and $\boldsymbol{H}_{\text{micro dip}}$ coincide. If the observation point is inside an elementary solenoid or a dipole, respectively, the values $\boldsymbol{H}_{\text{micro}}$ and $\boldsymbol{H}_{\text{micro dip}}$ differ. This difference is due, as shown above, to the definition and measuring principle, different in the two cases. Consequently, the mean values of the microscopic magnetic field strengths in the two cases are:

$$\left\langle \boldsymbol{H}_{\text{micro dip}} \right\rangle = \boldsymbol{H}, \tag{27a}$$

$$\langle \boldsymbol{H}_{\text{micro}} \rangle = \frac{1}{\mu_0} \boldsymbol{B}. \tag{27b}$$

It is to be noted that one can also write the following relations:

$$\boldsymbol{B}_{\text{micro}} = \mu_0 \boldsymbol{H}_{\text{micro}}, \tag{28a}$$

$$\langle \boldsymbol{B}_{\text{micro}} \rangle = \boldsymbol{B}. \tag{28b}$$

The reason of this difference consists, as shown, in the fact that the microscopic magnetic field strength has been defined in different manners for the two models.

In another expression form, the reason is that at microscopic scale, in the case of the elementary magnetic dipole model, only a potential component of the magnetic field strength occurs, whereas in the case of the elementary solenoid model, both a potential and a curl component are present.

Since the macroscopic quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ satisfy relations (11) and (13) in the case of the first model, and in the case of the second model they satisfy relations (26a) and (26b), the quantities $\boldsymbol{H}$ and $\boldsymbol{B}$ should coincide in the cases of both models.

5. Conclusion

There are two ways for introducing the macroscopic magnetic field state quantities, namely the magnetic field strength and the magnetic induction at a point. These ways are the macroscopic mode and the microscopic mode. The second mode is advantageous compared to the first, but the usual mode of application does not permit to obtain directly the two magnetic field state quantities and their relationship with the magnetization.

In this paper it has been emphasized that two microscopic models may be used for applying the second method: the elementary magnetic dipole model and the elementary solenoid model.

1. The microscopic magnetic field strength for the elementary magnetic dipole model and the elementary solenoid model has been introduced (defined) in the present paper in a different manner for each model.
2. The mean value (average) of the microscopic magnetic field strength for the elementary dipole model represents the macroscopic magnetic field strength.
3. The mean value (average) of the microscopic magnetic field strength for the elementary solenoid model represents the magnetic induction divided by the permeability of vacuum.
4. The analysis carried out permits to obtain the mean values (averages) directly for both models, the relation between the macroscopic values of the magnetic field strength, magnetization and magnetic induction.

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Appendix

Influence of the Magnetic Field Strength in the Wall of the Elementary Solenoid on the Mean Value (Average) of the Magnetic Field Strength

Let us assume that the wall of an elementary solenoid has a certain thickness g, small compared to the internal radius r_0 of the solenoid. Also, we shall assume a linear variation of the current density $\boldsymbol{J}$, inside the wall, along the radius. We shall calculate the magnetic field strength inside the wall under certain simplifications.

We shall assume that the magnetic field strength at any point inside the wall varies only with the radius, hence it is depending on a single variable. Therefore, $\boldsymbol{H} = \boldsymbol{k}H_z$. We shall use the cylindrical system of co-ordinates denoted by ρ, φ, z.

With the adopted symbols, and the assumption above, we found from the curl expression:

$$(\text{curl } \boldsymbol{H})_\varphi = -\frac{\partial H_z}{\partial \rho} = J\,(\rho)\,. \tag{A1}$$

By integrating the equation, we get:

$$\begin{aligned} H_z &= -J\,(u)\,u + J_{0L} = -\frac{J_{0L}}{g}u + J_{0L}; \\ \rho &= r_0 + u. \end{aligned} \tag{A2}$$

By integrating over the wall domain (volume) and taking into account that $g \ll r_0$, we obtain:

$$2\pi r_0 \int_0^g H_z h_S \,\mathrm{d}u = 2\pi r_0 \int_0^g \left(-\frac{J_{0L}}{g}u + J_{0L}\right) h_S \,\mathrm{d}u = \frac{1}{2}\, J_{0L} g 2\pi r_0 h_S. \quad \text{(A3)}$$

When calculating the mean value of the magnetic field strength, the terms of type (A3) have to be divided by the volume Δv, which contains a certain number of elementary solenoids. But, we can remark that, even with respect to the volume of a single elementary solenoid $\pi {r_0}^2 h_S$, the value of type (A3) is negligible if $g \ll r_0$. Hence, under the condition expressed by the last inequality, the influence of the wall can be neglected.