PERIODICA POLYTECHNICA SER. EL. ENG. VOL. 48, NO. 3–4, PP. 119–132 (2004)

SPECIAL PROPERTIES OF THE FERMAT-PROBLEM APPLIED TO LOCAL TOPOLOGY OPTIMIZATION

István GÓDOR

Department of Telecommunications and Mediainformatics
Budapest University of Technology and Economics
H-1117 Budapest, Magyar Tudósok Körútja 2., Hungary
Phone: +36-1-437 7237, Fax: +36-1-437 7767
e-mail: godor@tmit.bme.hu

Received: May 27, 2004

Abstract

In the field of network planning, local optimization techniques are frequently applied to improve the topology of the network by determining between which nodes a connection should exist. In many cases, some links can be merged at extra nodes (Steiner points) in order to save some costs. Finding these extra points belongs to the *weighted Fermat–Weber-problem*. In this paper, a new representation and construction of the solution to the Fermat-problem is proposed. General conditions of the technological applicability are presented. Furthermore, upper bounds are given to the achievable cost saving in advance without the construction of the Steiner points.

Keywords: weighted Fermat-problem, local topology optimization, Steiner-point.

1. Introduction

In the field of network planning, local optimization techniques are frequently applied. The local optimization can be the building-block of each step of a planning algorithm or an independent final phase of a given method. In the case of topology planning, local optimization means the improvement of the quality of the link structure between the nodes, and many times it is restricted to small parts of the entire network. In these circumstances, not only the determination is possible between which nodes a connection should exist, but also links can be merged at extra nodes (Steiner points, see [7]) in order to save cost.

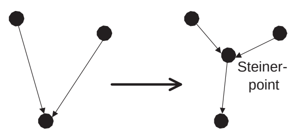

Fig. 1. Application of Fermat-problem

In the case of wireline links, cost saving mainly comes from the decrease of the trace length (less cable canal is needed 'under' the roads and pavements). In the case of wireless links, the number of repeaters (relay stations) can be decreased. Of course, the efficiency of this kind of topology optimization rises with reduction of angles between the links to be merged.

There are both exact and heuristic algorithms to find these extra "merging" points, however, the general formulation of the solutions (the optimal position of the extra points) is rather complex. (The planning task belongs to the *weighted Fermat-problem* in the case of merging two links [2, 3] (see *Fig. 1*) and to the *weighted Weber-problem* in the case of merging several links [1, 5].) Moreover, the problem of deciding in advance whether or not the application of such extra points results in cost saving and how much the gain will be is an open question. The existing solutions first determine the optimal position of the extra point (referred to as point P in the following) and then calculate the improvement.

In this paper, the following topics connected to the weighted Fermat-problem will be investigated. First in Section 2, some general properties of point P are given and a new coordinate system is presented to describe P. Furthermore, the barycentric co-ordinates of P are shown for a special symmetrical case. In Section3, for the case when the merging of two links decreases the cost, a lower bound is given for the multiplexing gain (capacity gain of merging links, denoted by M) and an upper bound is given for the angle (denoted by γ) between the two links to be merged. Then the connection of M and γ is analysed. After that, the amount of cost saving is analysed in function of M and γ. Finally, the paper is closed by a short conclusion.

2. New Properties of the Solution to the Fermat-Problem

DEFINITION 2.1 *The weighted Fermat-problem can be formulated as follows. Let $\triangle ABC$ be a given triangle with positive weights w_A, w_B and w_C associated with the three vertices. For any point X in the plane, let $|AX|$, $|BX|$ and $|CX|$ be the Euclidean distances between X and A, B, C. Then the weighted Fermat-problem is to find a point P such that $F(P) = \min(F(X) \in \mathcal{R}^2)$, where $F(X) = w_A|AX| + w_B|BX| + w_C|CX|$.*

An important question is how the weights of the nodes determine the position of P. Without the loss of generality, we can assume that $\triangle ABC$ is labelled such that $w_C \geq w_B \geq w_A$. Technological considerations focus the analysis on the case when the weights of the nodes are positive. The case when any of the weights can be equal to or less than 0 is out of the scope of the paper. (A possible solution to this latter case can be found in [2].)

CLAIM 2.1 If $w_C \geq w_B + w_A$, then $P = C$ is the solution. (Note that if $w_C = w_B + w_A$ and the nodes of the triangle are collinear as $C - B - A$ or $C - A - B$, then any point of the section between node C and the node in the middle can also be a solution.)

CLAIM 2.2 If $w_C < w_B + w_A$ and the nodes of the triangle are collinear, then the node in the middle is the solution.

CLAIM 2.3 If $w_C < w_B + w_A$ and A, B, C are not collinear, then P can be determined by Krarup's construction [2].

In the literature, there are several techniques to find/construct P (see e.g. [4, 6, 2]). Most of the techniques are based on geometrical construction by applying the so-called weight triangles (the ratio of the sides of these triangles corresponds to the weights) and Simpson lines (connecting a node of the weight triangle to the corresponding node of the original triangle, e.g. AA_1 in *Fig. 2*). However, the general description of P is practically out of the scope of literature.

For the case described by Claim 2.3 KRARUP showed that point P was in the intersection of three circles. These circles are the circumscribing circles of the weight triangles constructed outward the original triangle (see ABC_1, CBA_1, ACB_1 in *Fig. 2* and Theorem 1 in [2] for details).

DEFINITION 2.2 *In Fig. 2, the angles with index w denote the angles of the weight triangles and the angles with hat denote the viewing angle of the sides of $\triangle ABC$. In the following, these angles with hat are referred to as* Fermat-angles.

Since PAC_1B, PBA_1C and PCB_1A are cyclic quadrilaterals, the following equations hold true for the Fermat-angles.

$$\begin{aligned} \hat{\alpha} &= \pi - \alpha_w \\ \hat{\beta} &= \pi - \beta_w \\ \hat{\gamma} &= \pi - \gamma_w \\ \hat{\alpha} + \hat{\beta} + \hat{\gamma} &= 2\pi \end{aligned} \tag{1}$$

In the following, the intersection of the three viewing circles according to the Fermat-angles (around the sides of the triangle) is considered as a way to find P in the case of non-collinear triangles, and this technique is referred to as *Angle-technique*. In the rest of the section, the conditions are shown in which the *Angle-technique* is applicable.

THEOREM 2.1 *If the angles of $\triangle ABC$ are less or equal to the corresponding Fermat-angles (i.e. $\alpha \le \hat{\alpha}$ and $\beta \le \hat{\beta}$ and $\gamma \le \hat{\gamma}$), then P will be either an interior point in $\triangle ABC$ or a vertex of it. Furthermore, only one of the angles can be equal to its corresponding Fermat-angle, otherwise Claim 2.1 and 2.2 determine P.*

Proof. There are four cases to be investigated.

a) If $\alpha < \hat{\alpha}$ and $\beta < \hat{\beta}$ and $\gamma < \hat{\gamma}$, then any two of the viewing circles intersect each other at an interior point of the triangle and at the node, which is common for the corresponding two sides of the triangle. Because P is unique [2], thus P must be an interior point of the triangle, since it is the common intersection of the circles.

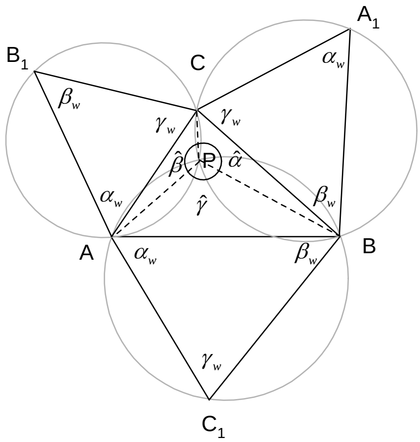

Fig. 2. Fermat-angles

b) Without the loss of generality, we assume that $\gamma = \hat{\gamma}$ and $\alpha < \hat{\alpha}$ and $\beta < \hat{\beta}$. Then the viewing circle according to $\hat{\gamma}$ is equal to the circumscribing circle of $\triangle ABC$ and the two viewing circles intersect each other at an interior point of $\triangle ABC$ and at node C. Thus node C gives the solution.

c) Without the loss of generality, we assume that $\gamma = \hat{\gamma}$ and $\beta = \hat{\beta}$ and $\alpha < \hat{\alpha}$. Then $\hat{\alpha} = \pi + \alpha$ holds. Since $\hat{\alpha}$ is a viewing angle, $\hat{\alpha} \leq \pi$ also holds. Thus $\alpha \leq 0$. Since $\alpha \geq 0$ is also true in $\triangle ABC$, α must be equal to 0, which means that $\triangle ABC$ is collinear and Claims 2.1 and 2.2 determine the solution and this construction cannot be applied.

d) The last case, when $\alpha = \hat{\alpha}$ and $\beta = \hat{\beta}$ and $\gamma = \hat{\gamma}$ is impossible according to Definition 2.2. □

THEOREM 2.2 *If one angle of $\triangle ABC$ is greater than its corresponding Fermat-angle, then P will be the corresponding vertex of $\triangle ABC$ (e. g. if $\alpha > \hat{\alpha}$, then $P = A$). If there are several such angles, then Claim 2.1 would determine P.*

Proof. There are three cases to be investigated.

a) Without the loss of generality, we assume that $\gamma > \hat{\gamma}$ and $\alpha \leq \hat{\alpha}$ and $\beta \leq \hat{\beta}$. According to Definition 2.2, $\gamma > \hat{\gamma} = \pi - \gamma_w$, so $\gamma + \gamma_w > \pi$. MARTELLI [3] proved that in this case $P = C$ was the solution.

b) Without the loss of generality, we assume that $\gamma > \hat{\gamma}$ and $\beta > \hat{\beta}$ and $\alpha \leq \hat{\alpha}$. Then $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 2\pi$ can be written as $\hat{\alpha} + \beta + \gamma > 2\pi = \pi + \alpha + \beta + \gamma$, followed by $\hat{\alpha} > \pi + \alpha$. According to Definition 2.2, $\pi - \alpha_w > \pi + \alpha$, i.e. $0 > \alpha_w + \alpha$. In a non-collinear $\triangle ABC$, $\alpha > 0$, thus α_w is negative, what contradicts to Claim 2.3 and Claim 2.1 gives P.

c) According to Definition 2.2, all angles of $\triangle ABC$ cannot be greater than the corresponding Fermat-angles. □

The next theorem gives a general description of P in a special coordinate system.

THEOREM 2.3 *Independently from $\triangle ABC$, point P is uniquely determined by the Fermat-angles, so P can be given in a* Fermat-angle coordinate system *as $P = P(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ and these angles can be expressed by the weights as follows.*

$$\hat{\alpha} = 2\text{arccot}\sqrt{\frac{w_A^2 - (w_B - w_C)^2}{(w_B + w_C)^2 - w_A^2}} \tag{2a}$$

$$\hat{\beta} = 2\text{arccot}\sqrt{\frac{w_B^2 - (w_A - w_C)^2}{(w_A + w_C)^2 - w_B^2}} \tag{2b}$$

$$\hat{\gamma} = 2\text{arccot}\sqrt{\frac{w_C^2 - (w_A - w_B)^2}{(w_A + w_B)^2 - w_C^2}} \tag{2c}$$

Proof. Theorem 2.1 and 2.2 prove the applicability of the *Angle-technique* and the uniqueness of the solution comes from Definition 2.2. In order to prove the above formulae, see weight triangle CBA_1 in *Fig.* 2. The cosine law gives that

$$\alpha_w = \arccos\frac{(|BC|w_B)^2 + (|BC|w_C)^2 - (|BC|w_A)^2}{2(|BC|w_B)(|BC|w_C)} = \arccos\frac{w_B^2 + w_C^2 - w_A^2}{2w_B w_C}.$$

Using Definition 2.2, one gets that

$$\hat{\alpha} = \pi - \arccos\frac{w_B^2 + w_C^2 - w_A^2}{2w_B w_C}.$$

By taking the cosine of the above equation and Equation (2a), the following equation has to be proved (note that $\pi/2 - \arctan x = \text{arccot}\, x$, $x \geq 0$)

$$\frac{w_B^2 + w_C^2 - w_A^2}{2w_B w_C} = \cos\left(2\arctan\sqrt{\frac{w_A^2 - (w_B - w_C)^2}{(w_B + w_C)^2 - w_A^2}}\right).$$

First, the arc tangent expression is changed to arc cosine, then the double angle formula is applied and finally some basic simplifications are made.

$$\cos\left(2\arctan\sqrt{\frac{w_A^2-(w_B-w_C)^2}{(w_B+w_C)^2-w_A^2}}\right)=\cos\left(2\arccos\frac{1}{\sqrt{\frac{w_A^2-(w_B-w_C)^2}{(w_B+w_C)^2-w_A^2}+1}}\right)$$

$$=2\frac{1}{\frac{w_A^2-(w_B-w_C)^2}{(w_B+w_C)^2-w_A^2}+1}-1=\dots$$

$$=\frac{w_B^2+w_C^2-w_A^2}{2w_Bw_C}$$

Eqs. (2b) and (2c) can be proven in the same way. □

THEOREM 2.4 *If $a=b$ and $w_A=w_B=1$ in the triangle, then the optimal P point divides the area of the $\triangle ABC$ into the following areas.*

$$\text{area}(ABP)=\frac{c^2\cdot w_C}{4\sqrt{4-w_C^2}} \tag{3a}$$

$$\text{area}(CAP)=\text{area}(BCP)=\frac{c}{4}\left[\sqrt{a^2-\left(\frac{c}{2}\right)^2}-\frac{c\cdot w_C}{2\sqrt{4-w_C^2}}\right] \tag{3b}$$

Note that the areas of the small triangles are equal to the barycentric coordinates of point P.

Proof. To calculate the area of $\triangle ABP$, compute first its height based on the tangent of $\frac{\hat{\gamma}}{2}$: $h_{ABP}=\frac{c}{2}/\tan\frac{\hat{\gamma}}{2}$. After substituting Equation (2c) for $\hat{\gamma}$ we get: $\text{area}(ABP)=\frac{c\cdot h_{ABP}}{2}=\frac{c^2\cdot w_C}{4\sqrt{4-w_C^2}}$.

Then we can calculate the areas of triangles CAP and BCP. The areas of them are equal, so we express them as:

$$\text{area}(CAP)=\text{area}(BCP)=\frac{\text{area}(ABC)-\text{area}(ABP)}{2}$$

$$=\frac{c}{4}\left[\sqrt{a^2-\left(\frac{c}{2}\right)^2}-\frac{c\cdot w_C}{2\sqrt{4-w_C^2}}\right].$$

□

3. Connection between the Fermat-Problem and Local Topology Optimization

In the field of network planning, the Fermat-problem can be applied to merge two links of a network at an extra point if it results in cost saving. Namely, the cost of the original links (AC and BC) is greater than the cost of the links to the extra point (AP and BP) and the cost of the merged link (PC).

In practice, the cost of the links can be calculated according to the following formula:

$$\mathcal{C}_{link} = l \cdot f(t), \tag{4}$$

where the first term l denotes the length of the link and the second term $f(t)$ indicates the capacity-related cost of the link. This latter component can be linear, piece-wise constant, etc.

Let us consider the weight of the links to be equivalent to their traffic-related cost: $w_A \equiv f(t_A)$ and $w_B \equiv f(t_B)$ (where t_A and t_B represent the traffic of node A and B, respectively). The cost of the multiplexed traffic w_C is supposed to be

1. $w_C \leq w_A + w_B$ and
2. $w_C \geq \min(w_A, w_B)$.

In practice, the multiplexing gain gives how much capacity can be spared by merging links, and the multiplexing gain indirectly determines the required capacity on link PC and w_C.

DEFINITION 3.1 *According to the above demands, the formula for calculating w_C by the multiplexing gain M is*

$$w_C = (1 - M)\min(w_A, w_B) + \max(w_A, w_B), \tag{5}$$

where $0 \leq M \leq 1$.

Note that *Angle-technique* presented in Section 2 is just a way to find the solution of the Fermat-problem. So if *Angle-technique* cannot be used in the local topology optimization, then the weights trivially determine P (see Claims 2.1 and 2.2). However, in most cases, P is equal to C, so improvement cannot be achieved. In some other cases P is equal to A or B. The rest of the section focuses on the cases connected to Claim 2.3.

In the following, let w_{min} denote $\min(w_A, w_B)$ and w_{max} denote $\max(w_A, w_B)$.

If we know the multiplexing gain, then it may be important to know how great the angle between link AC and BC can be in order to efficiently apply *Angle-technique*.

THEOREM 3.1 *The* Angle-technique *is applicable if angle γ between link AC and BC is at most*

$$\gamma_{\max} = 2\operatorname{arccot}\sqrt{\frac{2 - M}{M} \cdot \frac{2w_{\max} - Mw_{\min}}{2w_{\max} + (2 - M)w_{\min}}}. \tag{6}$$

Proof. By substituting *Eq.* (5) into *Eq.* (2c), one gets that

$$\hat{\gamma} = 2\operatorname{arccot}\sqrt{\frac{(w_{\min}(1-M)+w_{\max})^2-(w_B-w_A)^2}{(w_B+w_A)^2-(w_{\min}(1-M)+w_{\max})^2}}.$$

It can be easily seen that one can substitute $(w_B+w_A)^2$ with $(w_{\min}+w_{\max})^2$, and $(w_B-w_A)^2$ with $(w_{\min}-w_{\max})^2$. Simplifying the above equation, one gets that

$$\hat{\gamma} = 2\operatorname{arccot}\sqrt{\frac{2-M}{M}\cdot\frac{2w_{\max}-Mw_{\min}}{2w_{\max}+(2-M)w_{\min}}}. \tag{7}$$

Since Theorem 2.1 says $\gamma \leq \hat{\gamma}$ must be satisfied, the above equation is an upper bound for γ. □

Before the further analysis of the connection between γ and M, let us discuss two interesting connections between $\hat{\gamma}$ and M.

LEMMA 3.1 *$\hat{\gamma}$ is inversely proportional to w_C and directly proportional to M.*

Proof. Consider *Eq.* (2c), the arc cotangent expression is monotonically decreasing function of w_C, thus $\hat{\gamma}$ is inversely proportional to w_C. Consider Definition 3.1, w_C inversely proportional to M, thus $\hat{\gamma}$ directly proportional to M. □

LEMMA 3.2 *If $M = 1$, then $\frac{1}{2}\pi \leq \hat{\gamma} \leq \frac{2}{3}\pi$.*

Proof. Substitute $M = 1$ into *Eq.* (7). Then we get:

$$\hat{\gamma} = 2\operatorname{arccot}\sqrt{\frac{2w_{max}-w_{min}}{2w_{max}+w_{min}}} = 2\operatorname{arccot}\sqrt{Q} \tag{8}$$

By proving $\frac{1}{3} \leq Q \leq 1$, we prove the statement of the lemma.

Let us change $w_{\min}$ between 0 and w_{max}, where $w_{\max}$ is any fixed positive value. If $w_{\min} = 0$, then $Q = 1$. If $w_{\min}$ increases, then Q decreases until $w_{\min} = w_{\max}$. If $w_{\min} = w_{\max}$, then $Q = \frac{1}{3}$.

Let us change $w_{\max}$ from $w_{\min}$ to ∞, where $w_{\min}$ is any fixed positive value. If $w_{\max} = w_{\min}$, then $Q = \frac{1}{3}$. If $w_{\max}$ increases, then Q increases. If $w_{\max} \to \infty$, then $Q \to 1$.

Altogether, Q is between $\frac{1}{3}$ and 1. Thus $\frac{1}{2}\pi \leq \hat{\gamma} \leq \frac{2}{3}\pi$. □

From a practical point of view, it is important to analyse the cases when the cost of the network always or never can be decreased by merging two links at a legal multiplexing gain $0 \leq M \leq 1$. The following two theorems present these cases.

THEOREM 3.2 *If $\gamma > \frac{2}{3}\pi$, then there is no legal multiplexing gain $0 \leq M \leq 1$, for which* Angle-technique *is applicable.*

Proof. Theorem 2.1 says that *Angle-technique* is applicable only if $\gamma \leq \hat{\gamma}$. Lemma 3.2 gives that $\hat{\gamma} \leq \frac{2}{3}\pi$ if $M = 1$. So if $\gamma > \frac{2}{3}\pi$, then $\hat{\gamma} > \frac{2}{3}\pi$ has also to be true. According to Lemma 3.1, $\hat{\gamma}$ is directly proportional to M, so $M > 1$ has also to be true. But a legal M must be less or equal to 1 (see Definition 3.1). Thus if $\gamma > \frac{2}{3}\pi$, then there is no legal multiplexing gain $0 \leq M \leq 1$. □

THEOREM 3.3 *If $\gamma < \frac{1}{2}\pi$, then there always exists a legal multiplexing gain $0 \leq M \leq 1$ for which* Angle-technique *is applicable.*

Proof. Theorem 2.1 says that *Angle-technique* is applicable only if $\gamma \leq \hat{\gamma}$. Lemma 3.2 says if $M = 1$, then $\hat{\gamma} \geq \frac{1}{2}\pi$. So if $\gamma < \frac{1}{2}\pi$ and $M = 1$, then $\gamma < \hat{\gamma}$. Thus if $\gamma < \frac{1}{2}\pi$, then there always exists a legal multiplexing gain $0 \leq M \leq 1$. □

In the case of network planning, the most important question is how much cost saving (gain) can be achieved by applying the *Angle-technique*. The gain can be defined as $1 - \frac{\text{new cost}}{\text{original cost}}$. First let us determine how the three nodes of $\triangle ABC$ have to be placed in the plane to provide the maximal achievable gain. Then let us calculate the value of this maximal gain.

LEMMA 3.3 *At any given w_A, w_B and legal M, the maximal gain is achieved when $\gamma = 0$ and $|AC| = |BC|$.*

Proof. *Eq.* (5) says that $w_C \leq w_A + w_B$ for $0 \leq M \leq 1$. Let us apply the following notations. S denotes the node closer to C and its weight is denoted by w_S. L denotes the node farther from C and its weight is denoted by w_L. We want to find the lowest possible new cost for the network, since that case would result the maximal gain.

$$\begin{aligned} N_{\text{new}} &= w_L|LP| + w_S|SP| + w_C|PC| \\ &= w_L|LP| + w_L|PC| + w_S|SP| + w_S|PC| \\ &\quad + (w_C - w_L - w_S)|PC| \\ &\geq w_L|LC| + w_S|SC| + (w_C - w_L - w_S)|PC| \end{aligned}$$

If node P is both in sections LC and BC, then N_{new} is equal to the last expression and $\gamma = 0$. In that case, nodes A, B and C are collinear, so the optimal node P is equal to S (see Property 2.1 and 2.2). Then N_{new} is

$$N_{\text{new}} = w_L|LC| + (w_C - w_L)|SC|.$$

Independently from γ, the original cost of the network is

$$N_{\text{orig}} = w_A|AC| + w_B|BC| = w_L|LC| + w_S|SC|.$$

It is easy to see if γ tends to 0, then the gain of *Angle-technique* converges to the gain of the collinear case

$$
\begin{aligned}
G = 1 - \frac{N_{\text{new}}}{N_{\text{orig}}} &= \frac{N_{\text{orig}} - N\text{new}}{N_{\text{orig}}} \\
&= \frac{w_L|LC| + w_S|SC| - w_L|LC| - (w_C - w_L)|SC|}{w_L|LC| + w_S|SC|} \\
&= \frac{(w_S + w_L - w_C)|SC|}{w_L|LC| + w_S|SC|}.
\end{aligned}
$$

In order to prove that the gain is maximal if $|AC| = |BC|$, namely $|LC| = |SC|$, it is enough to show the following inequality.

$$
G = \frac{(w_S + w_L - w_C)|SC|}{w_L|LC| + w_S|SC|} \leq \frac{(w_S + w_L - w_C)}{w_L + w_S} \tag{9}
$$

If $w_C = w_S + w_L$, then both sides are 0. Otherwise we can simplify the inequality as follows

$$
\begin{aligned}
\frac{|SC|}{w_L|LC| + w_S|SC|} &\leq \frac{1}{w_L + w_S} \\
|SC|(w_L + w_S) &\leq w_L|LC| + w_S|SC| \\
|SC| &\leq |LC|.
\end{aligned}
$$

Of course, $|SC| \leq |LC|$, so at any given w_A, w_B and legal M, the gain is maximal if $\gamma = 0$ and $|AC| = |BC|$. □

THEOREM 3.4 *The maximal gain that can be achieved by* Angle-technique *is*

$$
G_{\max} = \frac{w_{\min} M}{w_A + w_B}. \tag{10}
$$

Proof. Lemma 3.3 says that at any given w_A, w_B and legal M, the gain tends to the maximum if γ tends to 0 and $|AC| = |BC|$. According to *Eq.* (9) the maximal gain can be formulated as

$$
G_{\max} = \frac{(w_A + w_B - w_C)}{w_A + w_B}. \tag{11}
$$

According to *Eq.* (5), the gain can be formulated as

$$
\begin{aligned}
G_{\max} &= \frac{w_A + w_B - [(1 - M)w_{\min} + w_{\max}]}{w_A + w_B} \\
&= \frac{w_A + w_B - w_{\min} - w_{\max} + M w_{\min}}{w_A + w_B} \\
&= \frac{w_{\min} M}{w_A + w_B}.
\end{aligned}
$$

□

The above maximal gain is a general upper bound of course, however, in particular cases, a tighter upper bound is needed. Before the further analysis of the gain, a connection between sides of $\triangle ABC$ and $|PC|$ is given.

LEMMA 3.4 *If M is legal* $(0 \leq M \leq 1)$*, then* $|PC| \leq \min(|AC|, |BC|)$*.*

Proof. If M is legal, then $w_C \geq w_{\max}$. If w_C is the greatest weight, then $\hat{\gamma}$ is the smallest Fermat-angle. Since $\hat{\alpha} + \hat{\beta} + \hat{\gamma} = 2\pi$ and $\hat{\alpha}, \hat{\beta}, \hat{\gamma} \leq \pi$, so $\hat{\alpha}, \hat{\beta} \geq \frac{\pi}{2}$. In *Fig.* 3, two viewing circles are shown.

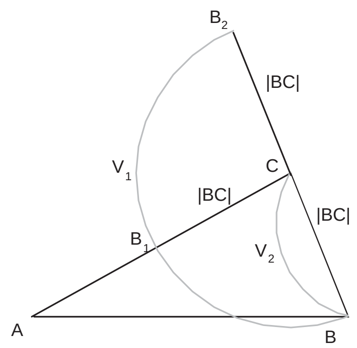

Fig. 3. Connection between the sides of $\triangle ABC$ and $|PC|$.

V_1 is the $\frac{\pi}{2}$ viewing circle of section B_2B, where B_2 is constructed by reflecting B to C. So $|B_2C| = |B_1C| = |BC|$. The other viewing circle is V_2, which is the $\hat{\alpha}$ viewing circle of section BC. Since $\hat{\alpha} \geq \frac{\pi}{2}$, thus V_2 is not outside of V_1. P is on V_2, so $|PC|$ is not greater than $|BC|$. Consequently, $|PC| \leq |BC|$. In a similar way, we get that $|PC| \leq |AC|$, so $|PC| \leq \min(|AC|, |BC|)$. □

From a technological point of view, it is important to know the value of the gain by taking the actual $\triangle ABC$ into consideration more accurately. An upper bound for the gain can be given by a linear function of γ as follows.

THEOREM 3.5

$$G(\gamma) \leq G_{\max}\left(1 - \frac{\gamma}{\gamma_{\max}}\right). \tag{12}$$

Proof. By definition, $G(\gamma) = 1 - \frac{N_{\text{new}}}{N_{\text{orig}}} = \frac{N_{\text{orig}} - N_{\text{new}}}{N_{\text{orig}}}$, where $N_{\text{orig}} = w_A|AC| + w_B|BC|$ and $N_{\text{new}} = w_A|AP| + w_B|BP| + w_C|PC| \geq \ldots = N_{\text{orig}} + |PC|(w_C - w_A - w_B)$ (see Lemma 3.3 for details). According to *Eq.* (5)

$$N_{\text{new}} \geq N_{\text{orig}} - w_{\min}M|PC| = N'_{\text{new}}.$$

Then the following inequality holds for $G(\gamma)$:

$$G(\gamma) = \frac{N_{\text{orig}} - N_{\text{new}}}{N_{\text{orig}}} \leq \frac{N_{\text{orig}} - N'_{\text{new}}}{N_{\text{orig}}}$$
$$G(\gamma) \leq \frac{w_{\min} M|PC|}{N_{\text{orig}}}$$

So we have to prove that

$$\frac{w_{\min} M|PC|}{N_{\text{orig}}} \leq G_{\max}\left(1 - \frac{\gamma}{\gamma_{\max}}\right) = \frac{w_{\min} M}{w_{\min} + w_{\max}}\left(1 - \frac{\gamma}{\gamma_{\max}}\right),$$

which is true if $M = 0$ or $w_{\min} = 0$. By transposition we get that

$$\frac{w_{\min}|PC| + w_{\max}|PC|}{w_{\min} Z_1 + w_{\max} Z_2} \leq 1 - \frac{\gamma}{\gamma_{\max}}, \qquad (13)$$

where $Z_1 = |AC|$ if $w_A = w_{\min}$ else $Z_1 = |BC|$ and $Z_2 = |BC|$ if $w_B = w_{\max}$ else $Z_2 = |AC|$.

According to Lemma 3.4, $|PC| \leq \min(Z_1, Z_2)$, so Inequality (13) is true for both $\gamma = 0$ and $\gamma = \gamma_{\max}$. Since for $0 \leq \gamma \leq \gamma_{\max}$ the left side is a monotonously decreasing convex function and the right side is a monotonously decreasing linear function, so Inequality (13) holds on the whole region. □

If $w_{\max}$ is much greater than w_{min}, then the upper bound given by *Eq.* (12) delivers a practically satisfying estimation. If the weights are relatively close to each other, then an estimation formula of the gain can be given.

CLAIM 3.1 If $\frac{w_{\max}}{w_{\min}} < 10$, then the gain can be estimated by the following formula:

$$\max_{\alpha,\beta} G(\gamma) \approx G_{\max}\left[\vartheta_1\left(\frac{\gamma}{\gamma_{\max}}\right)^2 - \vartheta_2\frac{\gamma}{\gamma_{\max}} + 1\right], \qquad (14)$$

where

$$\vartheta_1 = 1 - M\left(0.0846 + 0.0679\frac{w_{\max}}{w_{\min}}\right)$$
$$\vartheta_2 = 2 - M\left(0.0503 + 0.0691\frac{w_{\max}}{w_{\min}}\right).$$

(Note that the above formula underestimates the gain if γ is close to $\gamma_{\max}$, however, then G tends to 0.)

Table 1. Precision of the evaluation of G.

M	1	1.5	$w_{max} \diagdown w_{min}$ 2	4	8
0.1	0.002%	0.003%	0.002%	0.002%	0.006%
0.2	0.011%	0.010%	0.012%	0.005%	0.020%
0.3	0.024%	0.023%	0.024%	0.007%	0.044%
0.4	0.049%	0.035%	0.043%	0.011%	0.072%
0.5	0.070%	0.056%	0.070%	0.021%	0.102%
0.6	0.100%	0.086%	0.106%	0.040%	0.133%
0.7	0.137%	0.129%	0.152%	0.061%	0.164%
0.8	0.185%	0.185%	0.193%	0.110%	0.184%
0.9	0.244%	0.223%	0.269%	0.167%	0.192%
1.0	0.270%	0.307%	0.339%	0.261%	0.183%

The precision of the formula is shown in *Table 1*.

In the rows, different values for M, in the columns, different w_{max} and w_{min} ratios are evaluated. The results show how the formula estimates the maximal gain. The results are the average of the absolute difference between the 'exact values' and the estimation for $\gamma = 0 \ldots \gamma_{max}$. (Note that the 'exact values' numerically can be computed by the combination of the Weiszfeld algorithm and a successive approximation process.)

For example, the achievable gain at $w_{max} = w_{min}$ is shown in *Fig. 4*. The lowest curve is for $M = 0.1$, the uppermost is for $M = 1$ and between them M increases with 0.1 steps.

4. Conclusions

The *Fermat-angles* are proposed as a new representation of P, which is the solution to the weighted Fermat-problem. The *Angle-technique* is proposed as a new construction method of P. Several geometrical properties of the solution are analysed including the connection of the Fermat-angles with the angles of the triangle and with the weights of the vertices.

For the case of local topology optimization applications (when the merging of links may result in cost savings), some formulae are given, which tell us whether it is worth constructing P at all and if it is worth, then the formulae approximate the achievable cost saving in advance (without constructing P). Based on the presented results, some applications using the solution of the Fermat-problem for local topology optimization may be accelerated or may be extended to the weighted case (e. g. [7]).

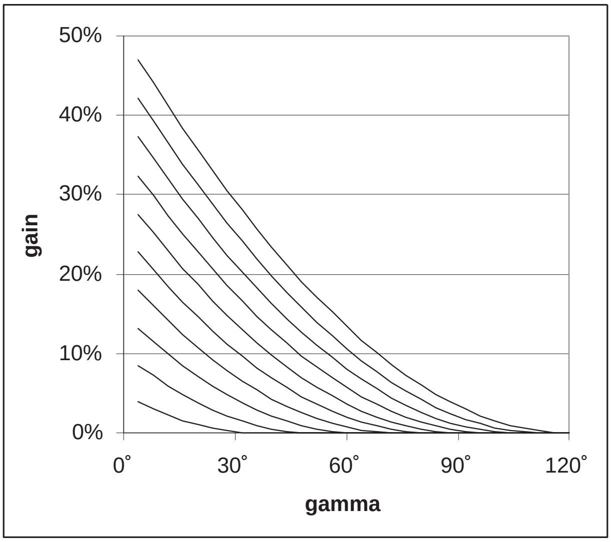

Fig. 4. Connection between γ and G with different M values.

Acknowledgements

The author thanks Antal Bányai, Alpár Jüttner and the anonymous referee for their valuable comments and advice.

References

[1] BRIMBERG, J. – CHEN, R. – CHEN, D., Accelerating Convergence in the Fermat-Weber Location Problem, OPER. RES. LETTERS, **22** (1998), pp. 151–157.

[2] JALAL, G. – KRARUP, J., Geometrical Solution to the Fermat Problem with Arbitrary Weights, *Annals of Operations Research*, **123** (2003), pp. 67–104.

[3] MARTELLI, M., *Geometrical Solution of Weighted Fermat Problem about Triangles*, New Trends in Mathematical Programming, Kluwer Academic Publishers, 1998, pp. 171–180.

[4] MOLNÁR, E., (ed): *Elemi matematika II.*, Tankönyvkiadó, Budapest, 1967.

[5] WEISZFELD, E., Sur le point pour lequel la somme des distances de *n* points donnés est minimum, *Thoku Math. J.* **43** (1937), pp. 355–386.

[6] YAGLOM, I. M., *Geometric Transformations I.*, Random House, 1962.

[7] ZACHARIASEN, M. – WINTER, P., Concatenation-Based Greedy Heuristics for the Euclidean Steiner Tree Problem, *Algorithmica* **25** (1999), pp. 418–437.