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Parameter Estimation of a Noisy Real Sinusoid Using Quartic Polynomial Approach

Gayathri Narayanan[1*], Dhanesh G. Kurup[2]

[1] Department of Electronics and Communication Engineering, Amrita Vishwa Vidyapeetham, Amritapuri Campus, Clappana P. O., Kollam 690525, Kerala, India
[2] Department of Electronics and Communication Engineering, Amrita Vishwa Vidyapeetham, Bengaluru Campus, Kasavanahalli, Carmelaram P. O., Bengaluru 560035, Karnataka, India
[*] Corresponding author, e-mail: gayathrin@am.amrita.edu

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Abstract

An analytical polynomial expression, for accurate and computationally efficient frequency estimation of a single real sinusoid under Additive White Gaussian Noise (AWGN), is derived and proposed in this paper. The method, which can be easily adapted for real time frequency estimation, is based on transforming the frequency estimation problem as the solution of a fourth order (quartic) expressed as powers of a trigonometric function containing the unknown frequency. The coefficients of the quartic polynomial can be found using the complex magnitudes of three Discrete Fourier Transform (DFT) bins, centered at the maximum magnitude value of the DFT coefficients. Simulated results illustrate that, the performance of the proposed estimator has a mean squared-error (MSE) performance which is very close to the Cramer Rao Lower Bound (CRLB) for high signal-to-noise ratio (SNR) region, as well as close to previously published estimators in the low SNR region.

Keywords

frequency estimation, quartic polynomials, Cramer Rao Lower Bound (CRLB), Discrete Fourier Transform (DFT), real sinusoid

1 Introduction

In several engineering disciplines, the problem of estimation of unknown frequencies is often encountered, particularly of sinusoidal signals. Notwithstanding the application, poorly estimated frequency values can lead to detrimental results. It is, therefore, imperative to devise an accurate and efficient method for the estimation of signal parameters. In this work, a novel, accurate and computationally efficient frequency estimation technique for a real sinusoid in Additive White Gaussian Noise (AWGN) is proposed. The sinusoidal signal is often captured by analog to digital converters, delivering a finite number of samples of the original sinusoid, which is also corrupted by noise [1]. Further signal processing operations undertaken on the sampled data presents an exigent task and the frequency estimation of a sinusoid becomes a challenging statistical problem [2, 3]. From comprehensive literature studies, it is inferred that the approaches adopted in prior works in real sinusoidal detection under noise can be broadly classified under time domain as well as frequency domain methods. The most widely accepted time-domain methods are those methods which include the computation of half-length autocorrelation [4], Prony's and Pisarenko's methods [5–8], the multiple signal classification algorithm (MUSIC) [9], estimation of signal parameters via rotational invariant techniques (ESPRIT) [10] and matrix-pencil method [11]. More recently proposed high resolution methods, such as [12], uses iterative minimization schemes. Such high-resolution estimators, although accurate, are computationally intensive due to the adoption of methods involving complex matrix processing operations. Therefore, the implementation of these methods in real-time applications is difficult and may not always be desirable. The approach based on FFT (Fast Fourier Transform) is a widely explored and accepted method since it can also relate to the ML (Maximum Likelihood) estimation of frequency. Although the FFT/DFT based methods [13, 14] are computationally simpler in comparison with many other methods, it is challenging to achieve an acceptable estimation accuracy due to the inherent estimation bias in these methods.

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A sizeable majority of literature for frequency estimation is proposed, implemented, and analyzed for complex sinusoidal signals. For example [15–20] deals with estimation of complex sinusoids in noise to achieve varying levels of estimation accuracy and computational speed in estimation. However, such methods cannot be directly applied for the frequency estimation of real sinusoids as they suffer from the drawback posed by the superposition of positive frequency and negative frequency spectrum of the real sinusoidal signals. The use of windowing techniques [21] can reduce the spectral leakage, but it is challenging to achieve higher estimation accuracy using such methods. A direct approach to minimize the effects of interference is presented in [22]. Alternately, a method with smaller estimation bias is proposed in [23] at the cost of increased computational overload. Frequency estimation approach for real sinusoids based on an expression for Maximum Likelihood (ML) is derived in [24]. An estimation technique based on the implementation of weighted least squares, which involves extensive computations, is discussed in [25]. In [26], the estimation is performed for a real sinusoid, by modulation and Discrete Fourier Transform (DFT) bin excision approach applied to filter out the negative frequency component. The modulating functions method is proposed in [27] which uses the modulation approach to estimate the frequency of noisy sinusoidal signals, by using a recursive algorithm. In [28], the interference attributed to the negative frequency spectrum is computed, resulting in increased estimation accuracy at low signal-to-noise ratio (SNR). The frequency estimation techniques mentioned above fail when there is a requirement for real time frequency estimation and when the signal's center frequency varies over a wide range. This is because, such methods suffer from severe computational overload. Therefore, there is a requirement for more efficient real time sinusoidal frequency estimators which can be adapted easily and ported to embedded computing platforms such as micro-controllers and FPGAs.

In this article, the frequency estimation of a real sinusoid based on DFT bins centered at the maximum magnitude value is formulated as the solution of a quartic polynomial. The quartic polynomial is derived as a solution of three equations corresponding to three DFT bins. Since the exact solution for a polynomial up to degree four exists, the unknown frequency can be determined analytically and with increased speed of computation. Unlike several existing methods, the quartic polynomial approach adopted is non-iterative, which makes it simple, powerful, and faster than other methods for real time applications. Hence, the proposed method is desirable, for faster frequency estimation, especially in hardware implementations for real time applications. In continuation with the introduction, background study and motivation of the work as presented in Section 1, the rest of the article is organized as stated. In Section 2, the fundamental theory and the frequency estimation problem statement is presented. The proposed polynomial solution supplemented with the derivations of the polynomial coefficients are also presented. In Section 3, the simulation results are presented in the subsections for different scenarios considered by varying parameters like frequency, phase, SNR and signal length. The inferences observed from the simulation studies of the above scenarios are presented. An attempt was also made to compare the obtained results with many previously published estimators to establish the validity of the work. The conclusion and contribution of this work is presented in Section 4.

2 Problem statement and the proposed polynomial solution

Consider a real sinusoid of amplitude a, frequency ω_c and phase ϕ_c embedded in samples of white noise y_n and sampled at frequency f_s:

$$s[n] = a\cos(\omega_c t_n + \phi_c) + y_n. \tag{1}$$

The value of n ranges between 0 to $N-1$ and $t_n = n\Delta = n/f_s$, where the parameter Δ represents the sampling period given by $1/f_s$. Let the estimate of ω_c be given by

$$\hat{\omega}_c = \omega_k + \delta, \tag{2}$$

where $\delta \in \left[-\frac{\omega_s}{N}, \frac{\omega_s}{N}\right]$ and ω_k corresponds to the frequency of bin k corresponding to the maximum value of the magnitude of the DFT coefficients. The frequency estimate can be obtained from the solution of the quartic polynomial:

$$\mathbb{P}(\chi) = P_0 + P_1\chi + P_2\chi^2 + P_3\chi^3 + P_4\chi^4. \tag{3}$$

The estimate of ω_c is given by the solution of Eq. (3), where:

$$\chi = \tan\left(\frac{\delta\Delta}{2}\right), \tag{4}$$

and the coefficients P_n are real. Coincidental to the formulation dealt in this article, the exact analytical solutions of polynomials are known up to fourth degree [29, 30], which results in an efficient approach to tackle the frequency estimation problem. The mathematical derivation of the polynomial coefficients is presented in Section 2.1.

2.1 Derivation of polynomial coefficients

In Section 2.1, the derivation of the polynomial coefficients is presented starting from the sampled signal given by Eq. (1). The complex N point DFT coefficients $F(\omega)$ corresponding to an arbitrary frequency ω can be written as

$$F(\omega)=\sum_{n=0}^{N-1}s[n]e^{-j\omega t_n}=\frac{a}{2}\sum_{m=1}^{2}T_m(\omega)e^{-j\phi_c(2m-3)}, \tag{5}$$

where:

$$T_m(\omega)=\sum_{n=0}^{N-1}e^{-j(\omega_c-\omega)(2m-3)t_n},\ \ m=1,2. \tag{6}$$

The terms $T_1(\omega)$ and $T_2(\omega)$ have been formulated to represent the positive and negative frequency components of the spectrum respectively. Using the terms represented by T_1 and T_2, the expressions for the spectral components of the three adjacent DFT bins for positive and negative frequencies can be represented as given in Eqs. (7) to (12):

$$T_1\left(\omega_k-\frac{\omega_s}{N}\right)=\frac{\sin\left(\frac{\delta\Delta N}{2}\right)}{\sin\left(\frac{\delta\Delta}{2}+\frac{\pi}{N}\right)}\frac{e^{j\left(\frac{\delta\Delta N}{2}\right)}}{e^{j\left(\frac{\delta\Delta}{2}+\frac{\pi}{N}\right)}} \tag{7}$$

$$T_1(\omega_k)=\frac{\sin\left(\frac{\delta\Delta N}{2}\right)}{\sin\left(\frac{\delta\Delta}{2}\right)}\frac{e^{j\left(\frac{\delta\Delta N}{2}\right)}}{e^{j\left(\frac{\delta\Delta}{2}\right)}} \tag{8}$$

$$T_1\left(\omega_k+\frac{\omega_s}{N}\right)=\frac{\sin\left(\frac{\delta\Delta N}{2}\right)}{\sin\left(\frac{\delta\Delta}{2}-\frac{\pi}{N}\right)}\frac{e^{j\left(\frac{\delta\Delta N}{2}\right)}}{e^{j\left(\frac{\delta\Delta}{2}-\frac{\pi}{N}\right)}} \tag{9}$$

$$T_2\left(\omega_k-\frac{\omega_s}{N}\right)=\frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2}-\frac{\pi}{N}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta}{2}-\frac{\pi}{N}\right]}} \tag{10}$$

$$T_2(\omega_k)=\frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta}{2}\right]}} \tag{11}$$

$$T_2\left(\omega_k+\frac{\omega_s}{N}\right)=\frac{\sin\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta N}{2}\right]}}{\sin\left[(2\omega_k+\delta)\frac{\Delta}{2}+\frac{\pi}{N}\right]e^{-j\left[(2\omega_k+\delta)\frac{\Delta}{2}+\frac{\pi}{N}\right]}}. \tag{12}$$

Algebraic manipulations are performed on Eqs. (7) to (12) to yield the following three equations in terms of both T_1 and T_2 as represented in Eqs. (13) to (15):

$$\frac{2F\left(\omega_k-\frac{\omega_s}{N}\right)}{ae^{j\phi_c}}=T_1\left(\omega_k-\frac{\omega_s}{N}\right)+G^-(\delta)T_2(\omega_k)e^{-j2\phi_c}, \tag{13}$$

$$\frac{2F(\omega_k)}{ae^{j\phi_c}}=T_1(\omega_k)+T_2(\omega_k)e^{-j2\phi_c}, \tag{14}$$

$$\frac{2F\left(\omega_k+\frac{\omega_s}{N}\right)}{ae^{j\phi_c}}=T_1\left(\omega_k+\frac{\omega_s}{N}\right)+G^+(\delta)T_2(\omega_k)e^{-j2\phi_c}, \tag{15}$$

where:

$$G^-(\delta)=\left[\frac{C_2\tan\frac{\delta\Delta}{2}+S_2}{C_1\tan\frac{\delta\Delta}{2}+S_1}\right]e^{-j\frac{\pi}{N}}, \tag{16}$$

$$G^+(\delta)=\left[\frac{C_2\tan\frac{\delta\Delta}{2}+S_2}{C_3\tan\frac{\delta\Delta}{2}+S_3}\right]e^{j\frac{\pi}{N}}, \tag{17}$$

$$S_i=\sin\left[\left(2(k-1)+i\right)\frac{\pi}{N}\right], \tag{18}$$

$$C_i=\cos\left[\left(2(k-1)+i\right)\frac{\pi}{N}\right], \tag{19}$$

where $i \in 1, 2, 3$.

Equation (13) − Eq. (16) × Eq. (14):

$$\frac{2F\left(\omega_k-\frac{\omega_s}{N}\right)}{ae^{j\phi_c}}-\frac{2G^-(\delta)F(\omega_k)}{ae^{j\phi_c}}=T_1\left(\omega_k-\frac{\omega_s}{N}\right)-G^-(\delta)T_1(\omega_k). \tag{20}$$

Equation (15) − Eq. (17) × Eq. (14):

$$\frac{2F\left(\omega_k+\frac{\omega_s}{N}\right)}{ae^{j\phi_c}}-\frac{2G^+(\delta)F(\omega_k)}{ae^{j\phi_c}}=T_1\left(\omega_k+\frac{\omega_s}{N}\right)-G^+(\delta)T_1(\omega_k). \tag{21}$$

From Eqs. (20) and (21), it is observed that the negative frequency component is thus implicitly eliminated through algebraic manipulations. Further, dividing Eq. (20) with Eq. (21) and rearranging the terms, the expression for the quartic polynomial is obtained:

$$F^-\left[T_1^+(\delta)-G^+(\delta)\right]-F^+\left[T_1^-(\delta)-G^-(\delta)\right] \\ +G^+(\delta)T_1^-(\delta)-G^-(\delta)T_1^+(\delta)=0, \tag{22}$$

where, $F^\pm$ and $T_1^\pm$ are respectively given by

$$F^\pm=\frac{F\left(\omega_k\pm\frac{\omega_s}{N}\right)}{F(\omega_k)}, \tag{23}$$

$$T_1^\pm(\delta)=\frac{T_1\left(\omega_k\pm\frac{\omega_s}{N}\right)}{T_1(\omega_k)}. \tag{24}$$

On equating the real part of Eq. (22) to zero and grouping together similar powers of $\tan\frac{\delta\Delta}{2}$, the resulting expression takes the form of Eq. (3) and can be expressed as a fourth order polynomial in χ, where $\chi=\tan\frac{\delta\Delta}{2}$. The coefficients of the quartic polynomial can be given by Eqs. (25) to (29):

$$P_0=S^2S_2\psi_s, \tag{25}$$

$$P_1=S^2\left(S_2\psi_c+C_2\psi_s\right)+S\left(S_1S_3\gamma^+-S_2\alpha^+\right), \tag{26}$$

$$P_2=S^2C_2\psi_c-C^2S_2\psi_s+S\left(C_3S_1+S_3C_1\right)\gamma^+ \\ +C\left(S_3S_1\gamma^-+S_2\alpha^-\right)-S\left(C_2\alpha^++S_2\beta^+\right), \tag{27}$$

$$P_3=-C^2\left(S_2\psi_c+C_2\psi_s\right)+C\left(S_1C_3+C_1S_3\right)\gamma^- \\ +S\left(C_1C_3\gamma^+-C_2\beta^+\right)+C\left(S_2\beta^-+C_2\alpha^-\right), \tag{28}$$

$$P_4=C\left(C_3C_1\gamma^--CC_2\psi_c+C_2\beta^-\right), \tag{29}$$

where:

$$C=\cos\left(\frac{\pi}{N}\right),\ S=\sin\left(\frac{\pi}{N}\right),$$

$$\alpha^\pm=S_1\pm S_3,\ \beta^\pm=C_1\pm C_3,$$

$$\psi_c=C_1R^--C_3R^+,\ \psi_s=S_1R^--S_3R^+,$$

$$\gamma^\pm=R^-\pm R^+,\ R^\pm=Re\left[F^\pm e^{\mp j\frac{\pi}{N}}\right].$$

Hence, the solution of the unknown frequency from the DFT coefficients can be formulated as the solution of a quartic polynomial for its real valued roots. The coefficients of the polynomial, as presented in Eq. (25) to Eq. (29) can be easily programmed in a computing platform. Let $\chi=\chi_s$ be the solution of the quartic polynomial thus obtained. Therefore, according to Eq. (2), the unknown frequency is obtained as:

$$\omega_c=\omega_k+\frac{2}{\Delta}\tan^{-1}\chi_s, \tag{30}$$

where $\chi_s\in\left[-\tan(\pi/N),\tan(\pi/N)\right]$.

The root of the polynomial which falls in the DFT bin corresponding to maximum magnitude is chosen as the best estimate out of the four roots of the polynomial. The analytical method for solving the polynomial is adopted from [29, 30]. Although, the analytical solution of the quartic polynomial described is quite accurate achieving frequency estimates remarkably close to Cramer Rao Lower Bound (CRLB), a refinement of the frequency estimation using the analytical solution of the quartic polynomial is proposed. For obtaining the fine frequency estimate from the solution of the quartic polynomial Eq. (30), two-dimensional Taylor Series is used to approximate the time domain samples in Eq. (1) as specified in Eq. (31):

$$s_d[m]=s[mU],\ m\in[0,M-1], \\ M=\frac{N}{U}\approx\hat{a}_c\cos\left(\hat{\omega}_ct_m+\hat{\phi}_c\right) \\ -\hat{a}_c\sin\left(\hat{\omega}_ct_m+\hat{\phi}_c\right)t_m\Delta_\omega\hat{a}_c\sin\left(\hat{\omega}_ct_m+\hat{\phi}_c\right)\Delta_\phi, \tag{31}$$

where U is the under-sampling factor, $\Delta_\omega=\omega-\hat{\omega}_c$ and $\Delta_\phi=\phi_c-\hat{\phi}_c$, and $\hat{\omega}_c$ is the solution of the quartic polynomial in Eq. (30). The pair $\{\hat{a}_c,\hat{\phi}_c\}$, are the estimated amplitude and phase respectively from the DFT bins with maximum magnitude. Hence, Eq. (31) yields a system of M equations. This enables the computation of fine estimates of frequency and phase from $\{\Delta_\omega,\Delta_\phi\}$.

Another method for solving Eq. (3), which works equally well is the two term Taylor expansion, based on first and second derivatives of Eq. (3). To ascertain the validity of the analytical solution, the solution of the quartic polynomial is iteratively computed as shown in Eq. (32):

$$\chi_{i+1}=\frac{\chi_iP'(\chi_i)-P(\chi_i)}{P'(\chi_i)}, \tag{32}$$

where i is the iteration number. It is to be noted that Eq. (32) stems from the Taylor Series expansion of $\mathbb{P}(\chi)$ in Eq. (3) retaining terms containing the first derivative. Therefore, the quartic polynomial can be approximated near an approximate value χ_0 within the range $[-\tan(\pi/N),\tan(\pi/N)]$ using a quadratic equation. The initial solution can then be iteratively refined to converge to the exact solution of the quartic polynomial.

To determine an approximate RMSE expression of the proposed approach, the quartic polynomial is represented in terms of a cubic approximation by retaining the coefficients P_0 to P_3:

$$P(\chi) \approx P_0 + P_1\chi + P_2\chi^2 + P_3\chi^3. \quad (33)$$

Therefore, the approximate expression for RMSE can be written as

$$\text{RMSE} \approx \sqrt{\frac{1}{L}\sum_{l=0}^{L-1}\left[\omega_c^{(l)} - \left(\omega_k^{(l)} + \frac{2}{\Delta}\tan^{-1}\chi_{sa}^{(l)}\right)\right]^2}, \quad (34)$$

where L represents the number of iterations, $\omega_c^{(l)}$, $\omega_k^{(l)}$ and $\chi_{sa}^{(l)}$ are the frequency of the sinusoid, frequency corresponding to the DFT peak and the solution of the cubic polynomial in the l^{th} experiment, for a given noise variance. The authors have adopted simulation-based validation using the approximate RMSE expression for the verification of the proposed method. A detailed mathematical validation is out of the scope of this article and is intended as an extension of the present work.

2.2 Comparison of computational efficiency

The computation time for determining the unknown frequency and memory resources for the proposed method is small since the proposed estimator is based on the analytic solution of a fourth order polynomial. To the best of author's knowledge, the number of arithmetic operations required in all the previously reported works is a function of N; for instance, the iterative frequency estimation method [15], the windowing approach [16], the autocorrelation-based approach [17], the analytical signal-based approach [23], the maximum likelihood method [24] and the method based on negative frequency filtering approach [26]. An exhaustive comparison of various methods as well as the computational efficiency is also discussed in [26]. The method presented in [26] requires > $16N\log_2(N) + 50N$ arithmetic operations and the method presented in [28] required $O(N\log_2(N))$ operations. Although, the method in [15] requires > $8N\log_2(N) + 20N$ operations, the method has a large deviation from CRLB at moderate SNR.

The proposed method offers the notable advantage that the number of arithmetic operations is ≈ 70 [29] and independent of N for solving the quartic polynomial from the DFT coefficients. This includes all the arithmetic operations involving solutions of four quadratic polynomials and one cubic polynomial. Further, for refinement of the solution in Eq. (31), an overdetermined system of equations is solved using standard least squares method which has computational complexity of $O(4N/U)$ [31], where U is the under-sampling factor. As a result, the process of frequency estimation is fast and can be deployed in real time hardware with minimal programming effort. Section 3 highlights the findings and validation.

3 Results and discussion

In Section 3, the accuracy of the proposed method is analyzed by means of simulation studies for various parameter settings. In all the simulations, a sampled real sinusoid with random phase is considered. The performance of the proposed method is studied by computing the RMSE for various SNR conditions, for fixed signal lengths. The effect of the signal length on the performance of the proposed method is also studied. To scrutinize the effect of frequency on the proposed method, the value of frequency assigned in the simulation is either fixed as mentioned or varied in a specified range to include both very low and very high frequencies. Finally, the effect of harmonic distortion on frequency estimation of real sinusoids is also studied.

3.1 Variable SNR

In Section 3.1, the frequency estimation accuracy of the proposed method is studied by computing the root mean square error (RMSE) for varying values of SNR. A sampled sinusoid embedded in noise with fixed frequency $f_c = 0.1917$, and number of samples $N = 128$ is considered corresponding to Eq. (1). The phase of the sinusoid is uniformly distributed in the interval $[0, 2\pi]$.

The performance of the proposed method is compared with the iterative frequency estimation method (AM) [15], Kaiser window approach with $\beta = 5$ integrated with Candan's estimator [16], the autocorrelation based (AC) approach [17], the analytical signal-based approach (AS) [23] combined with the AM method, maximum likelihood method (ML) method [24] and the method based on negative frequency filtering approach (NFF) method [26]. For the ML method, the cost function is computed over a grid of width two FFT bins centered at FFT maximum and point spacing of $\Delta f_{\text{ML}} = \sqrt{\text{CRLB}}$. The analytic signal is calculated using the Hilbert transform in the AS-based approach. Fig. 1 shows the performance comparison of the proposed method in terms of the computation of the RMSE calculated over 10,000 Monte Carlo simulations.

The RMSE saturates and remains far from CRLB for AM method, AC and AS-based methods for SNR > 20 dB, whereas Candan's estimator has a significant bias for the considered SNR range. From Fig. 1, it is evident that the

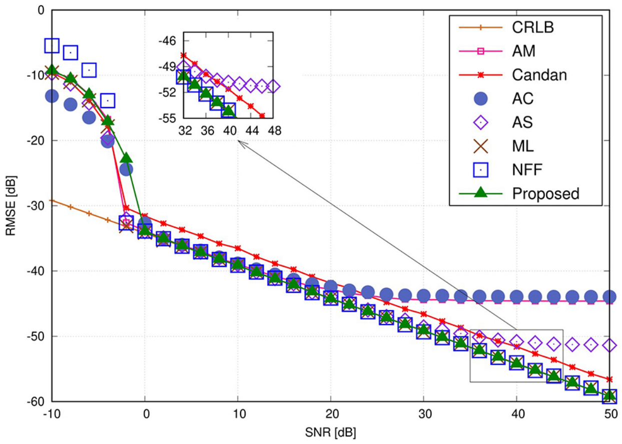

Fig. 1 Comparison of the RMSE performance of the proposed estimator for varying SNR and fixed frequency $f_c = 0.1917$

ML method, the NFF method and the proposed method outperform the other methods for moderate and high SNRs, i.e SNR > −2 dB. In the low SNR region, the performance of the ML method and the proposed method are slightly better than the NFF method in [26]. Since the SNR values are extremely low and the RMSEs of the methods are much higher than the CRLB, this marginal improvement is of comparatively less significance. However, both the ML method and NFF method are iterative in nature, which increases the computational complexity of these methods with respect to the proposed approach. The inset graph in Fig. 1 shows that there is no estimation bias at high SNR.

In Fig. 2, the proposed method is validated by comparing the RMSE performance of the estimator with the iterative Taylor Series approximation of the quartic polynomial formulated in Eq. (31) for $N = [128, 256]$. It is inferred

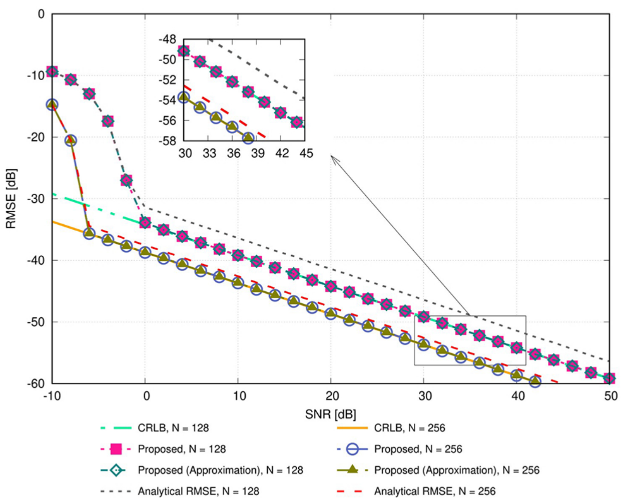

Fig. 2 Comparison of proposed quartic polynomial method with analytical and iterative solution for $f_c = 0.1917$

that, the solution corresponding to the first iteration of the Taylor Series approximation has converged to the solution yielded by the quartic polynomial approach. The performance of the proposed method for variation in signal length is also presented. From Fig. 2, it is evident that the proposed method exhibits better conformance to RMSE for larger values of signal length. Fig. 2 also shows that the proposed estimation technique meets the CRLB corresponding to SNR values of approximately 0 dB for $N = 128$ and −6 dB onwards for $N = 256$ and the analytical RMSE is closer to the CRLB for $N = 256$ than for $N = 128$. It is also observed that, for moderate and high SNR values, a noticeable estimation bias is present. However, it is evident that the derived analytical RMSE exhibits reliable performance at low SNR values for both the chosen values of N. For instance, at $N = 128$, the analytical RMSE coincides exactly with the proposed approach up to SNR = −2 dB. For larger values of signal length, $N = 256$, the analytical RMSE follows the proposed approach up to SNR = −7 dB. For $N = [128, 256]$, the RMSE performance of the proposed method is observed to be superior to the performance of the analytical RMSE. An improvement by a factor of 3 dB and 1 dB is observed for $N = [128, 256]$ respectively. Interestingly, a remarkably similar trend is observed in the performance of the proposed method when fine frequency estimation is employed, producing a strikingly similar performance improvement by the same factor.

3.2 Variable frequency

In Subsection 3.2, Fig. 3 and Fig. 4 show the effect of variable frequency on the proposed estimation method for various SNR and $N = 128$ and $N = 256$ respectively. For signal length $N = 128$, a few deviations are observed at low SNR values. As the SNR increases, the RMSE performance meets the CRLB. When N is increased to 256, the deviations from RMSE are observed to be smaller for $N = 128$ and SNR = −3 dB. In general, the RMSE performance is better for larger values of signal length. For instance, for SNR = 6 dB, the achieved RMSE is approximately −37.2 dB for $N = 128$. For the same SNR, the proposed method achieves an RMSE of −41.7 dB for $N = 256$.

A significant inference here is the dependence of the proposed algorithm on the choice of frequency, irrespective of the choice of the signal length N. At exceptionally low frequencies close to zero and very high frequencies close to 0.5, the estimated value does not converge to the actual value of the frequency resulting in sub-optimal performance of the algorithm at the extreme limits of the

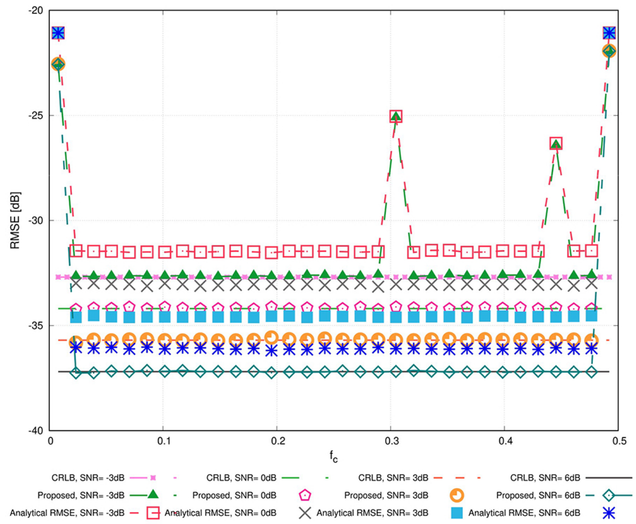

Fig. 3 Comparison of RMSE performance of the proposed estimator with CRLB for variable frequency in the range [$1/N$, $0.5 - (1/N)$] and variable SNR for $N = 128$

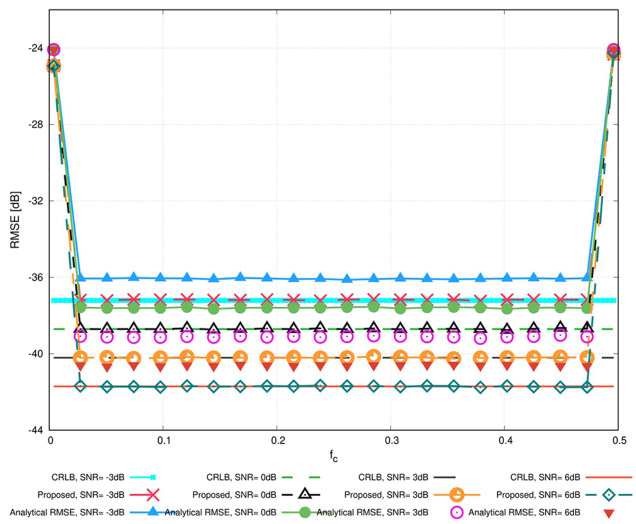

Fig. 4 Comparison of RMSE performance of the proposed estimator with CRLB for variable frequency in the range [$1/N$, $0.5 - (1/N)$] and variable SNR for $N = 256$

frequency. For L random experiments carried out for the same number of points and sampling interval, since the frequency of the sinusoid varies in the range $\left[\frac{1}{N}, \frac{1}{2} - \frac{1}{N}\right]$, the acquired number of sine wave cycles varies in the range $\left[\Delta, \frac{2\Delta N^2}{(N-2)}\right]$. Therefore, the minimum number of acquired sine wave cycles for the simulation environment in Fig. 3 and Fig. 4 is unity. From Fig. 3 and Fig. 4, the performance of the proposed method is validated against the analytical RMSE. It is observed from both the figures that, for varying values of N and sinusoid frequencies, the approximate analytical RMSE does not satisfactorily meet the CRLB. This bias in estimation is observed due to the cubic approximation adopted for the computation of the polynomial root in Eq. (33).

3.3 Variable signal length

A sinusoidal signal of variable length, $N = 2^n$, in the interval $n \in [5, 10]$ is considered for studying the SNR performance. The RMSE curves are calculated for SNR (dB) = [−3, 0, 3, 6] for $f_c = 0.05$ and $f_s = 1$ using 10000 independent experiments as indicated in Fig. 5. From the results, it can be inferred that, except for SNR = −3 dB, the estimation performance is remarkably close to the CRLB for signal lengths greater than $N = 64$.

For $N = 32$ and all values of SNR in Fig. 5, the performance deviates from the CRLB. A loosely similar comparison is presented in Fig. 4 of [28], but the authors have considered the analysis for only a single value of SNR = 40 dB. From this, it is observed that the proposed method has an estimation performance closer to CRLB for higher SNR regime. Fig. 5 also presents the performance of the analytical RMSE. It is observed that, for varying values of N and SNR, the approximate analytical RMSE does not approach the CRLB. This suboptimal performance is due to the approximation adopted in Eq. (33).

3.4 Effect of harmonic distortion

In Section 3.4, the effect of harmonic distortion on the proposed estimation method is studied. The interference due to the presence of the second harmonic is considered for varying values of signal-to-interference ratio (SIR) and SNR, for the same simulation settings as in Fig. 1. The RMSE performance of the proposed estimator is

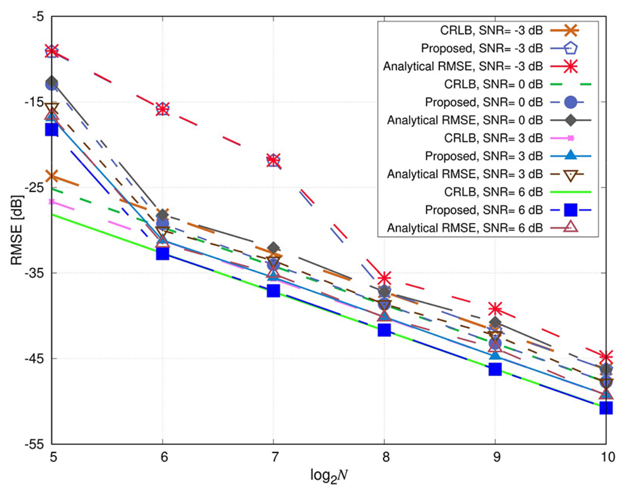

Fig. 5 Comparison of RMSE performance of the proposed estimator for variable signal length N and different SNR for $f_c = 0.05$

compared with CRLB calculated for SNR ranging from −10 dB to 50 dB for $N = 256$. From Fig. 6, it is observed that, for SNR values in the range 2 dB to 20 dB, the RMSE converges to CRLB for all values of SIR, except SIR = 0 dB. However, as the SNR increases, the signal power becomes more dominant compared to noise. Therefore, at high SNR, the effect of harmonics becomes dominant and RMSE deviates from the CRLB, see Fig. 6. Fig. 6 also presents the analytical RMSE computed for $N = 256$. It is observed that the analytical RMSE exhibits reliable performance at low SNR values like −10 dB and −8 dB. However, for SNR values greater than −8 dB, the analytical RMSE deviates slightly from the CRLB, introducing an estimation bias at these SNR values.

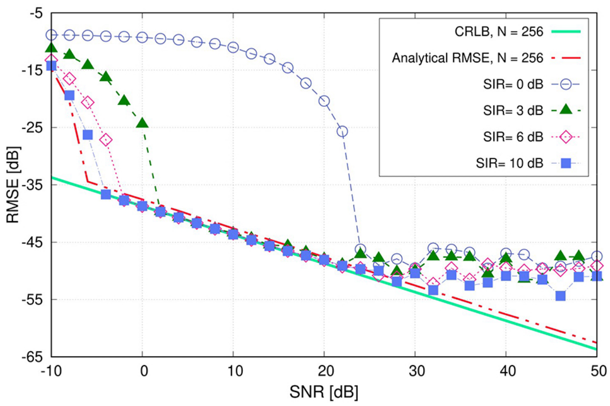

Fig. 6 Effect of harmonic distortion on the RMSE performance of the proposed estimator for variable SIR and SNR for $f_c = 0.1917$ and $N = 256$

4 Conclusion

An accurate and computationally efficient frequency estimator for a real sinusoid in AWGN based on three DFT samples is proposed. The frequency estimate is the solution of the quartic polynomial which falls in the DFT bin under consideration. On comparing the performance of the algorithm with other existing methods, the proposed estimator is accurate and performs very well for high SNR region, as well as in the low SNR region for various test cases. The performance of the proposed method is also robust against frequency estimation with harmonic distortion. Since the proposed method is computationally efficient and based on DFT, optimized FFT implementations available in digital hardware can be harnessed for applications which requires real time sinusoidal parameter estimation.

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