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RESEARCH ARTICLE

# Symmetric distance formula in kantor spaces and the radius of the circumscribed sphere of affinely independent set of points

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***Abstract***

*Mass points are very useful objects not only in physics but also in geometry. There are several ways to approach the mathematics of mass points. In this paper we give an independent interpretation. We define kantor space and kantors as the elements of it. We prove that this is a vector space and give a short overview of the types of bases and the connections between them. One of our important tools is the symmetric distance formula for kantors, which expresses the distance of two points in terms of their kantric coordinates. We introduce the kantric scalar product, which allows us to prove easily the existence of an orthogonal point and give a formula of the radius of the circumscribed sphere of affinely independent set of points, which is our main result.*

***Keywords***

*kantor · mass point · circumscribed sphere · indefinit inner product*

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## 1 The kantor space

In this paper we give an independent interpretation of the mass-point theory. Of course, it has numerous common points with another existing theories. For more details see [1, 3, 5, 6]. Possible applications are not detailed here but mentioned in Section 6.

First, we give the basic definitions and the explanation will be detailed after them.

**Definition 1 (Kantor space)** *Let $K_r = \mathbb{R}^n \times (\mathbb{R}\backslash\{0\})$ and $K_s = \mathbb{R}^n \times \{0\}$. Define the addition and the multiplication by scalar on the set $K_r \cup K_s$ in the following way:*

- $(P, p) + (Q, q) = \left(\frac{pP+qQ}{p+q}, p + q\right)$ *if* $(P, p), (Q, q) \in K_r$ *and* $p + q \neq 0$,
- $(P, p) + (Q, -p) = (p(P - Q), 0)$ *if* $(P, p), (Q, -p) \in K_r$,
- $(P, p) + (\underline{v}, 0) = (P + \frac{\underline{v}}{p}, p)$ *if* $(P, p) \in K_r$ *and* $(\underline{v}, 0) \in K_s$,
- $(\underline{v}, 0) + (P, p) = (P + \frac{\underline{v}}{p}, p)$ *if* $(P, p) \in K_r$ *and* $(\underline{v}, 0) \in K_s$,
- $(\underline{v}, 0) + (\underline{w}, 0) = (\underline{v} + \underline{w}, 0)$ *if* $(\underline{v}, 0), (\underline{w}, 0) \in K_s$,
- $\lambda(P, p) = (P, \lambda p)$ *if* $(P, p) \in K_r$ *and* $\lambda \in \mathbb{R}\backslash\{0\}$,
- $0 \cdot (P, p) = (\underline{0}, 0)$ *if* $(P, p) \in K_r$,
- $\lambda(\underline{v}, 0) = (\lambda\underline{v}, 0)$ *if* $(\underline{v}, 0) \in K_s$ *and* $\lambda \in \mathbb{R}$.

*The algebraic structure obtained in this way is called a kantor space of dimension $n + 1$ and denoted by $\mathbb{K}^n$.*

The word kantor is a compound consists of "quan(tity)" and "(vec)tor", since the concept of kantors was based on common properties of particular physical quantities and was introduced as the counterpart of the vector concept (“quan” has been changed to “kan” because the word quantor was already reserved).

The elements of $\mathbb{K}^n$ will be denoted by dotted capital letters. The elements of $K_r$ and $K_s$ are called regular and singular kantors respectively. In the above representation, the coordinates are called the vector-mass coordinates of the kantor.

For a regular kantor $\dot{P} = (P, p)$, $P \in \mathbb{R}^n$ is called the center of $\dot{P}$. We say that $\dot{P}$ is unite if $|\dot{P}| = 1$.

For a singular kantor $\dot{V} = (\underline{v}, 0)$, $\underline{v} \in \mathbb{R}^n$ is called the translation vector of $\dot{V}$. The translation value of a singular kantor is the norm of its translation vector. The mass of a kantor is the last coordinate of it and sometimes denoted by $|\cdot|$. For example, $|\dot{P}| = p$, $|\dot{V}| = 0$.

It is left to the reader to verify that $\mathbb{K}^n$ is a vector space over $\mathbb{R}$ with the above operations and $K_s$ is a subspace of it. Moreover, the mass is a linear functional on $\mathbb{K}^n$ and its kernel is $K_s$. The zero vector of $\mathbb{K}^n$ is the kantor $(\underline{0}, 0)$. The additive inverse of $(P, p)$, $p \neq 0$ is $(P, -p)$ and of $(\underline{v}, 0)$ is $(-\underline{v}, 0)$.

The idea behind the kantor space is to form a closed algebraic structure from the mass points of $\mathbb{R}^n$. The naive concept is that a mass point has a center and a positive mass. These kind of mass points can be added together and multiplied by positive scalars based on physical analogies. We can easily define points with negative mass and multiplication by negative scalars as well. The major problem is the case of zero masses. How can one add mass points with opposite masses or multiply by 0?

Let $\dot{P} = (P, p)$ and $\dot{Q} = (Q, -p)$ be two mass points centered at $P$ and $Q$ with opposite masses, $p \neq 0$, and let $\dot{V} = \dot{P} + \dot{Q}$ be a hypothetic mass point. If $\dot{R} = (R, r)$, $r \neq 0$, $r \neq -p$ and we want to preserve the associativity of addition, then $\dot{R}+\dot{V} = \dot{R}+(\dot{P}+\dot{Q})$ must be equal to $(\dot{R} + \dot{P}) + \dot{Q}$, which we can compute without introducing the unfamiliar zero mass.

$$(\dot{R} + \dot{P}) + \dot{Q} = \left(\frac{rR + pP}{r + p}, r + p\right) + (Q, -p)$$
$$= \left(\frac{rR + pP - pQ}{r}, r\right) = \left(R + \frac{p}{r}(P - Q), r\right).$$

So, $\dot{V}$ acts on $\dot{R}$ like a mass-preserving translation with vector $\frac{p}{r}(P - Q)$. It depends on the mass of $\dot{R}$, which is some kind of inertia property. To define $\dot{V}$, it is enough to know the impact of $\dot{V}$, which is entirely described by the translation vector $p(P - Q) \in \mathbb{R}$. These kantors are called singular refering to their zero mass.

It is easy to see that the translation-vector of the sum of singular kantors is the sum of the translation-vectors, and the translation vector of $\lambda\dot{V} = \lambda\dot{P} + \lambda\dot{Q}$ is $\lambda$ times the translation vector of $\dot{V}$. Hence, the set of singular kantors is isomorphic to $\mathbb{R}^n$ as a vector space.

The zero element of $\mathbb{K}^n$ is the singular kantor with translation vector $\underline{0}$. We have to clarify the case of a mass-point with zero mass. Let $\dot{Q} = (Q, 0)$, $\dot{P} = (P, p)$, where $p \neq 0$. By the naive definition, $\dot{P} + \dot{Q} = \left(\frac{pP+0Q}{p}, p + 0\right) = (P, p) = \dot{P}$. It means that kantors of the form $(Q, 0)$ exactly acts like the singular kantor $\dot{0}$, therefore we can identify these kantors with $\dot{0}$.

We can represent kantors on $n + 1$ coordinates: the last coordinate is the mass and the $n$-tuple of the first $n$ coordinates is the translation vector or the center of the kantor depending on its mass is zero or not. These considerations lead us to the above definition.

First, we have to determine the dimension of $\mathbb{K}^n$.

**Theorem 2** *The dimension of $\mathbb{K}^n$ over $\mathbb{R}$ is $n + 1$.*

**Proof:** Let $\underline{e}_i$ denote the $i$th standard basis vector of $\mathbb{R}^n$. Then $\mathcal{B} = \{(\underline{e}_1, 0), (\underline{e}_2, 0), \ldots, (\underline{e}_n, 0), (\underline{0}, 1)\}$ is a basis of $\mathbb{K}^n$.

First we prove the linear independence of $\mathcal{B}$. Put a linear combination: $\sum_{i=1}^n \lambda_i(\underline{e}_i, 0) + \lambda_{n+1}(\underline{0}, 1)$.

$\sum_{i=1}^n \lambda_i(\underline{e}_i, 0) + \lambda_{n+1}(\underline{0}, 1) = (\sum_{i=1}^n \lambda_i\underline{e}_i, 0) + (\underline{0}, \lambda_{n+1})$. If the linear combination is $\dot{0}$, then $\lambda_{n+1} = 0$, otherwise the mass of the linear combination would not be 0. In this case, however, $(\sum_{i=1}^n \lambda_i\underline{e}_i, 0) = \dot{0} = (\underline{0}, 0)$, which means $\sum_{i=1}^n \lambda_i\underline{e}_i = \underline{0}$. The linear independence of the standard basis in $\mathbb{R}^n$ implies $\lambda_i = 0$ for $i = 1, 2, \ldots, n$.

The other step is to prove that $\mathcal{B}$ spans $\mathbb{K}^n$. It is easy to see that $\mathcal{B}\backslash\{(\underline{0}, 1)\}$ spans the subspace of singular kantors because $\{\underline{e}_1, \underline{e}_2, \ldots, \underline{e}_n\}$ spans $\mathbb{R}^n$. If $\dot{P} = (P, p)$ is a regular kantor, then $\dot{S} = \dot{P} - p(\underline{0}, 1)$ is a singular kantor. Now, $\dot{S}$ can be written in the form $\sum_{i=1}^n \lambda_i(\underline{e}_i, 0)$ with some $\lambda_i \in \mathbb{R}$. Hence, $\dot{P} = \sum_{i=1}^n \lambda_i(\underline{e}_i, 0)+ p(\underline{0}, 1)$. □

$\mathcal{B}$ contains $n$ singular and 1 regular kantor. So, we can make a distinction between bases based on the number of singular kantors contained in them.

**Definition 3**

- *A basis of $\mathbb{K}^n$ is called $r$-singular if it contains exactly $r$ singular kantors.*
- *A basis is regular if it is 0-singular.*
- *A basis is kernel-singular if it is $n$-singular. The single regular kantor of this basis is called a kernel.*

**Remark 4** *For an $r$-singular basis $\mathcal{B}$, $r \leq n$. Namely, if $r = n + 1$, then the span of $\mathcal{B}$ would consists of singular kantors, hence it would not be $\mathbb{K}^n$.*

**Remark 5** *There are regular bases in $\mathbb{K}^n$: if $\{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{M}\}$ is a kernel-singular basis, then $\mathcal{B} = \{\dot{M} + \dot{S}_1, \dot{M} + \dot{S}_2, \ldots, \dot{M} + \dot{S}_n, \dot{M}\}$ is a regular basis because $\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{M}$ can be expressed as a linear combination of the elements of $\mathcal{B}$.*

**Definition 6** *A regular basis $\mathcal{B} = \{\dot{B}_1, \dot{B}_2, \ldots, \dot{B}_{n+1}\}$ is called standard if for all $i$, $j$, $|\dot{B}_i| = 1$ and $|B_i - B_j| = 1$ (i.e., $B_1, \ldots, B_{n+1}$ are the vertices of an $n$-dimensional regular simplex).*

One can assign a kernel-singular basis to every regular basis in the following way.

Let $\mathcal{B} = \{\dot{B}_1, \dot{B}_2, \ldots, \dot{B}_{n+1}\}$ be a regular basis, $b_i = |\dot{B}_i|$ and $\dot{S}_i = \frac{1}{b_i}\dot{B}_i - \frac{1}{b_{n+1}}\dot{B}_{n+1}$ for $i = 1, 2, \ldots, n$. Then $\mathcal{B}' = \{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{B}_{n+1}\}$ is a kernel-singular basis because $\dot{B}_1, \dot{B}_2, \ldots, \dot{B}_{n+1}$ can be expressed as a linear combination of the elements of $\mathcal{B}'$.

## 2 Distance formula

In this section, we would like to determine the distance of the centers of two regular kantors based on their regular coordinates related to $\mathcal{B}$.

As in a general vector space, we can coordinate the elements of $\mathbb{K}^n$ relative to a fixed basis. A natural question is that what is the correspondence between the the vector-mass and the regular coordinates of a kantor.

Let $\mathcal{B} = \{\dot{B}_1, \dot{B}_2, \dots, \dot{B}_{n+1}\}$ be a fixed regular basis of $\mathbb{K}^n$. Denote the corresponding kernel-singular basis by $\mathcal{B}'$ ($\dot{S}_i = \frac{1}{b_i}\dot{B}_i - \frac{1}{b_{n+1}}\dot{B}_{n+1}$ for $i = 1, 2, \dots, n$). If $\underline{s}_i = (B_i - B_{n+1})$ denotes the translation vector of $\dot{S}_i$, then $\{\underline{s}_1, \underline{s}_2, \dots, \underline{s}_n\}$ forms a basis of $\mathbb{R}^n$. We express the first $n$ vector-mass coordinates relative to this basis. For a regular kantor $\dot{A} = (a_1, a_2, \dots, a_{n+1})_{\mathcal{B}}$,

$$\begin{aligned}\dot{A} = \sum_{i=1}^{n+1} a_i\dot{B}_i &= \sum_{i=1}^{n} a_i\left(b_i\dot{S}_i + \frac{b_i}{b_{n+1}}\dot{B}_{n+1}\right) + a_{n+1}\dot{B}_{n+1}\\ &= \sum_{i=1}^{n} a_ib_i\dot{S}_i + \frac{1}{b_{n+1}}\left(\sum_{i=1}^{n+1} a_ib_i\right)\dot{B}_{n+1},\end{aligned}$$

hence $\dot{A} = \left(a_1b_1, a_2b_2, \dots, a_nb_n, \frac{1}{b_{n+1}}\sum_{i=1}^{n+1} a_ib_i\right)_{\mathcal{B}'}$ coordinated with respect to the associated kernel-singular basis.

The singular kantor $\sum_{i=1}^{n} a_ib_i\dot{S}_i$ translates $\frac{1}{b_{n+1}}\left(\sum_{i=1}^{n+1} a_ib_i\right)\dot{B}_{n+1}$ by the vector $\frac{\sum_{i=1}^{n} a_ib_i\underline{s}_i}{\sum_{i=1}^{n+1} a_ib_i}$. Thus, the vector-mass coordinates of $\dot{A}$ is

$$\left(B_{n+1} + \frac{\sum_{i=1}^{n} a_ib_i(B_i - B_{n+1})}{\sum_{i=1}^{n+1} a_ib_i}, \sum_{i=1}^{n+1} a_ib_i\right) = \left(\frac{\sum_{i=1}^{n+1} a_ib_iB_i}{\sum_{i=1}^{n+1} a_ib_i}, \sum_{i=1}^{n+1} a_ib_i\right),$$

which is a weighted average of the points $B_i$ for $i = 1, \dots, n+1$.

Let $\dot{A} = (a_1, a_2, \dots, a_{n+1})_{\mathcal{B}}$ and $\dot{C} = (c_1, c_2, \dots, c_{n+1})_{\mathcal{B}}$ be two regular kantors. We suppose that $\dot{A}$ and $\dot{B}$ are unite kantors i.e., $|\dot{A}| = |\dot{B}| = 1$ (if not, we can divide by the masses without changing the centers).

The above argument shows that $A = \sum_{i=1}^{n+1} a_ib_iB_i$ and $C = \sum_{i=1}^{n+1} c_ib_iB_i$. Hence, the square distance of $A$ and $C$ is

$$\begin{aligned}d_{AC}^2 &= \langle A - C|A - C\rangle\\ &= \left\langle\sum_{i=1}^{n+1}(a_i - c_i)b_iB_i \,\middle|\, (a_i - c_i)b_iB_i\right\rangle\\ &= \sum_{i,j=1}^{n+1} b_ib_j(a_i - c_i)(a_j - c_j)\left\langle B_i|B_j\right\rangle.\end{aligned}$$

We know that $d_{B_iB_j}^2 = \langle B_i|B_i\rangle - 2\left\langle B_i|B_j\right\rangle + \left\langle B_j|B_j\right\rangle$, so $\left\langle B_i|B_j\right\rangle = \frac{1}{2}\left(\langle B_i|B_i\rangle + \left\langle B_j|B_j\right\rangle - d_{B_iB_j}^2\right)$. Thus,

$$d_{AC}^2 = \sum_{i,j=1}^{n+1} -\frac{1}{2}d_{B_iB_j}^2 b_ib_j(a_i-c_i)(a_j-c_j) + \sum_{i,j=1}^{n+1} b_ib_j(a_i-c_i)(a_j-c_j)\langle B_i|B_i\rangle.$$

$$\sum_{i,j=1}^{n+1} b_ib_j(a_i - c_i)(a_j - c_j)\langle B_i|B_i\rangle = \left\langle\sum_{i=1}^{n+1} b_i(a_i - c_i)B_i \,\middle|\, \sum_{j=1}^{n+1} b_j(a_j - c_j)B_i\right\rangle$$

and

$$\sum_{j=1}^{n+1} b_j(a_j - c_j)B_i = \left(\sum_{j=1}^{n+1} b_j(a_j - c_j)\right)B_i = \left(\sum_{j=1}^{n+1} b_ja_j - \sum_{j=1}^{n+1} b_jc_j\right)B_i = (|\dot{A}| - |\dot{C}|)B_i = 0.$$

We have arrived to the point to give the distance formula for regular kantors.

**Theorem 7** *Let $\dot{A} = (a_1, a_2, \dots, a_{n+1})_{\mathcal{B}}$ and $\dot{C} = (c_1, c_2, \dots, c_{n+1})_{\mathcal{B}}$ be two unite kantors. Then the square-distance of A and C is*

$$d_{AC}^2 = \sum_{i,j=1}^{n+1} -\frac{1}{2}d_{B_iB_j}^2 b_ib_j(a_i - c_i)(a_j - c_j).$$

□

This formula is symmetric in the regular coordinates.

We introduce the notation $\left\langle\dot{B}_i|\dot{B}_j\right\rangle = -\frac{1}{2}d_{B_iB_j}^2 b_ib_j$ for the "kantric scalar product" of the basis kantors. If we bilinearly extend this definition, the distance formula can be written in the form

$$d_{AC}^2 = \left\langle\dot{A} - \dot{C}|\dot{A} - \dot{C}\right\rangle = \sum_{i,j=1}^{n+1}\left\langle\dot{B}_i|\dot{B}_j\right\rangle(a_i - c_i)(a_j - c_j).$$

## 3 The kantric scalar product

**Definition 8** *Let $\dot{A}$ and $\dot{C}$ be arbitrary kantors in $\mathbb{K}^n$ coordinated with respect to the regular basis $\{\dot{B}_1, \dots, \dot{B}_{n+1}\}$. Then $\langle\dot{A}|\dot{C}\rangle_{\mathcal{B}} = \sum_{i,j=1}^{n+1}\langle\dot{B}_i|\dot{B}_j\rangle a_ic_j$ defines the kantric scalar product of the two kantors, where $\langle\dot{B}_i|\dot{B}_j\rangle = -\frac{1}{2}d_{B_iB_j}^2 b_ib_j$.*

The kantric scalar product is a bilinear form, but it is not positive. For example, $\langle\dot{B}_i|\dot{B}_i\rangle = 0$ and $\langle\dot{B}_i + \dot{B}_j|\dot{B}_i + \dot{B}_j\rangle = 2\langle\dot{B}_i|\dot{B}_j\rangle < 0$ if $i \neq j$ and $b_i, b_j > 0$. The following theorem describes the kantric scalar product of singular kantors.

**Theorem 9** *For any two singular kantors $\dot{S} = (\underline{s}, 0)$, $\dot{T} = (\underline{t}, 0)$, $\left\langle\dot{S}|\dot{T}\right\rangle = \left\langle\underline{s}|\underline{t}\right\rangle$.*

**Proof:** If $\dot{S} = (s_1, s_2, \dots, s_{n+1})_{\mathcal{B}}$ is a singular kantor, $\dot{A}$ is a unite kantor and $\dot{C} = \dot{A} + \dot{S}$, then $\dot{C}$ is also unite and the square of the translation value of $\dot{S}$ is

$$\left|\underline{s}\right|^2 = d_{AC}^2 = \sum_{i,j=1}^{n+1}\left\langle\dot{B}_i|\dot{B}_j\right\rangle(a_i - c_i)(a_j - c_j) =$$

$$\sum_{i,j=1}^{n+1}\left\langle\dot{B}_i|\dot{B}_j\right\rangle s_is_j = \left\langle\dot{S}|\dot{S}\right\rangle.$$

Thus, $\left\langle\dot{S}|\dot{T}\right\rangle = \frac{1}{2}(\left\langle\dot{S} + \dot{T}|\dot{S} + \dot{T}\right\rangle - \left\langle\dot{S}|\dot{S}\right\rangle - \left\langle\dot{T}|\dot{T}\right\rangle) = \frac{1}{2}(\left\langle\underline{s} + \underline{t}|\underline{s} + \underline{t}\right\rangle - \left\langle\underline{s}|\underline{s}\right\rangle - \left\langle\underline{t}|\underline{t}\right\rangle) = \left\langle\underline{s}|\underline{t}\right\rangle$. □

This means that the kantric scalar product on $K_s$ is the same as the euclidean scalar product of the translation vectors.

**Definition 10**

(1) *The regular kantor $\dot{O} \in \mathbb{K}^n$ is an orthogonal kantor of the regular basis $\mathcal{B}$ if for all singular kantor $\dot{S} \in \mathbb{K}^n$, $\langle\dot{O}|\dot{S}\rangle_{\mathcal{B}} = 0$.*

(2) *The regular kantor $\dot{O} \in \mathbb{K}^n$ is a circumkantor of the regular basis $\mathcal{B}$ if for all $i = 1, 2, \dots, n+1$, $d_{OB_i} = R$ with some real $R$ (here, $R$ must be the circumradius of the set $\{B_1, B_2, \dots, B_{n+1}\}$).*

(3) *The kantor $\dot{O} \in \mathbb{K}^n$ is a kernel of the mass-functional with respect to $\mathcal{B}$ if there exists a real number $\alpha \neq 0$ such that for all $\dot{A} \in \mathbb{K}^n$, $\langle\dot{O}|\dot{A}\rangle_{\mathcal{B}} = \alpha|\dot{A}|$, i.e., the scalar multiplication with $\dot{O}$ is a linear function of the mass.*

(4) *The regular kantor $\dot{O} \in \mathbb{K}^n$ is a minimum kantor of the scalar product related to $\mathcal{B}$ if for all regular kantor $\dot{A} \in \mathbb{K}^n$,*

$$\frac{\langle \dot{O}|\dot{O}\rangle_{\mathcal{B}}}{o^2} \leq \frac{\langle \dot{A}|\dot{A}\rangle_{\mathcal{B}}}{a^2},$$

*where $o = |\dot{O}|$ and $a = |\dot{A}|$.*

**Theorem 11** *Definitions (1)-(4) are equivalent provided that $n \geq 2$.*

**Proof:** First, we prove the equivalence of definitions (*1*)-(*3*).

*(1) ⇒ (2)*
Let $\dot{O}$ be an orthogonal kantor, $o = |\dot{O}| \neq 0$ and $\frac{\langle \dot{O}|\dot{O}\rangle}{o^2} = \theta$. Furthermore, let $\dot{A} \in \mathbb{K}^n$ be an arbitrary regular kantor and $a = |\dot{A}|$. Then

$$\frac{\langle \dot{A}|\dot{A}\rangle}{a^2} = \left\langle \frac{\dot{A}}{a} \,\middle|\, \frac{\dot{A}}{a} \right\rangle = \left\langle \left(\frac{\dot{A}}{a} - \frac{\dot{O}}{o}\right) + \frac{\dot{O}}{o} \,\middle|\, \left(\frac{\dot{A}}{a} - \frac{\dot{O}}{o}\right) + \frac{\dot{O}}{o} \right\rangle =$$

$$= \left\langle \frac{\dot{A}}{a} - \frac{\dot{O}}{o} \,\middle|\, \frac{\dot{A}}{a} - \frac{\dot{O}}{o} \right\rangle + 2\left\langle \frac{\dot{A}}{a} - \frac{\dot{O}}{o} \,\middle|\, \frac{\dot{O}}{o} \right\rangle + \left\langle \frac{\dot{O}}{o} \,\middle|\, \frac{\dot{O}}{o} \right\rangle =$$

$$= d_{AO}^2 + \frac{2}{o}\left\langle \frac{\dot{A}}{a} - \frac{\dot{O}}{o} \,\middle|\, \dot{O} \right\rangle + \frac{\langle \dot{O}|\dot{O}\rangle}{o^2} = d_{AO}^2 + \theta$$

because $\dot{O}$ is orthogonal to $\frac{\dot{A}}{a} - \frac{\dot{O}}{o}$. This implies that for all $1 \leq i \leq n+1$, $\frac{\langle \dot{B}_i|\dot{B}_i\rangle}{b_i^2} = d_{B_iO}^2 + \theta$. By definition, $\frac{\langle \dot{B}_i|\dot{B}_i\rangle}{b_i^2} = 0$, hence $d_{B_iO} = \sqrt{-\theta}$ for all $i$, where $R = \sqrt{-\theta}$ is a real number. This means that $\dot{O}$ is a circumkantor of the basis $\mathcal{B}$ and $\theta < 0$.

*(2) ⇒ (3)*
If $\dot{O}$ is a circumkantor of $\mathcal{B}$, then for all $1 \leq i \leq n+1$, $d_{OB_i} = R$ with some positive real $R$. Let $\theta = \frac{\langle \dot{O}|\dot{O}\rangle}{o^2}$.
For all index $i$,

$$R^2 = \left\langle \frac{\dot{B}_i}{b_i} - \frac{\dot{O}}{o} \,\middle|\, \frac{\dot{B}_i}{b_i} - \frac{\dot{O}}{o} \right\rangle = \frac{1}{b_i^2}\langle \dot{B}_i|\dot{B}_i\rangle - \frac{2}{b_i o}\langle \dot{B}_i|\dot{O}\rangle + \frac{1}{o^2}\langle \dot{O}|\dot{O}\rangle \;\Rightarrow$$

$$\langle \dot{B}_i|\dot{O}\rangle = \frac{b_i o}{2}(\theta - R^2).$$

Let $\alpha = \frac{o}{2}(\theta - R^2)$, which is nonzero due to the regularity of $\dot{O}$ and because $n \geq 2$ (if $\theta$ would be equal to $R^2$, then $\langle \dot{B}_i|\dot{O}\rangle = 0$ for all $i$, which implies $\langle \dot{O}|\dot{O}\rangle = 0$ and $R^2 = \theta = 0$ in contradiction with the linear independence of $\mathcal{B}$). Then for arbitrary regular kantor $\dot{K} = \sum_{i=1}^{n+1} k_i \dot{B}_i$,

$$\langle \dot{K}|\dot{O}\rangle = \left\langle \sum_i k_i \dot{B}_i \,\middle|\, \dot{O} \right\rangle = \sum_i k_i \langle \dot{B}_i|\dot{O}\rangle = \alpha \sum_i k_i b_i = \alpha|\dot{K}|$$

holds. Thus, $\dot{O}$ is a kernel of the mass-functional.

*(3) ⇒ (1)*
If $\dot{O}$ is a kernel of the mass-functional, then for each singular kantor $\dot{S} \in \mathbb{K}^n$, $\langle \dot{O}|\dot{S}\rangle = \alpha|\dot{S}| = 0$. Obviously, $\dot{O} \neq 0$. If $|\dot{O}| = 0$, then $\dot{O}$ would be a nonzero singular kantor with $\langle \dot{O}|\dot{O}\rangle > 0$ and on the other hand $\langle \dot{O}|\dot{O}\rangle = \alpha|\dot{O}| = 0$, which is impossible. Hence, $\dot{O}$ is a regular kantor and an orthogonal kantor of the basis $\mathcal{B}$.
Finally, we show that definitions *(1)* and *(4)* are equivalent.

*(1) ⇒ (4)*
Suppose that $\dot{O}$ is an orthogonal kantor. We know that $\dot{O}$ is regular and for any regular kantor $\dot{A}$

$$\frac{\langle \dot{A}|\dot{A}\rangle}{a^2} = \frac{\langle \dot{O}|\dot{O}\rangle}{o^2} + d_{AO}^2 \geq \frac{\langle \dot{O}|\dot{O}\rangle}{o^2}$$

holds by the *(1) ⇒ (2)* part of the proof. So, $\dot{O}$ is the minimum point of the scalar product.

*(4) ⇒ (1)*
Let $\dot{O}$ be the minimum point of the scalar product. It is clear that $|\dot{O}| \neq 0$. Assume to the contrary that there exists a singular kantor $\dot{S} \in \mathbb{K}^n$ for which $\langle \dot{O}|\dot{S}\rangle \neq 0$. Now, $\dot{O} - \dot{S}$ is a regular kantor, so property *(4)* implies

$$\frac{\langle \dot{O} - \dot{S}|\dot{O} - \dot{S}\rangle}{o^2} \geq \frac{\langle \dot{O}|\dot{O}\rangle}{o^2} \;\Rightarrow\; \langle \dot{O}|\dot{O}\rangle - 2\langle \dot{O}|\dot{S}\rangle + \langle \dot{S}|\dot{S}\rangle \geq \langle \dot{O}|\dot{O}\rangle \;\Rightarrow$$

$$\langle \dot{S}|\dot{S}\rangle \geq 2\langle \dot{O}|\dot{S}\rangle \neq 0.$$

This statement also holds for $-\dot{S}$, so $\langle \dot{S}|\dot{S}\rangle = \langle -\dot{S}|-\dot{S}\rangle \geq 2\langle \dot{O}|-\dot{S}\rangle = -2\langle \dot{O}|\dot{S}\rangle$. Therefore, $\langle \dot{S}|\dot{S}\rangle \geq 2|\langle \dot{O}|\dot{S}\rangle| > 0$.
Let $t = \frac{\langle \dot{S}|\dot{S}\rangle}{|\langle \dot{O}|\dot{S}\rangle|} > 1$. Then, applying the previous argument one more times, $\left\langle \frac{\dot{S}}{t} \,\middle|\, \frac{\dot{S}}{t} \right\rangle \geq 2\left|\left\langle \dot{O} \,\middle|\, \frac{\dot{S}}{t} \right\rangle\right|$. Hence,

$$\frac{\langle \dot{S}|\dot{S}\rangle}{t^2} \geq \frac{2|\langle \dot{O}|\dot{S}\rangle|}{t} \;\Rightarrow\; \langle \dot{S}|\dot{S}\rangle \geq 2t|\langle \dot{O}|\dot{S}\rangle| \;\Rightarrow$$

$$\langle \dot{S}|\dot{S}\rangle \geq 2\langle \dot{S}|\dot{S}\rangle$$

in contradiction with $\dot{S} \neq 0$. Thus, $\dot{O}$ is an orthogonal kantor of the basis $\mathcal{B}$. □

**Theorem 12** *The set $\{B_1, B_2, \ldots, B_{n+1}\}$ of $n+1$ points is affinely independent if and only if $\mathcal{B} = \{\dot{B}_1, \dot{B}_2, \ldots, \dot{B}_{n+1}\}$ is a basis of $\mathbb{K}^n$, where $\dot{B}_i = (B_i, 1)$ for $i = 1, 2, \ldots, n+1$.*

**Proof:** If the points are affinely independent and $\underline{s}_i = B_i - B_{n+1}$, then $\{\underline{s}_1, \underline{s}_2, \ldots, \underline{s}_n\}$ is a basis of $\mathbb{R}^n$, hence $\{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{B}_{n+1}\}$ forms a kernel-singular basis of $\mathbb{K}^n$, where $\dot{S}_i = (\underline{s}_i, 0)$ for $i = 1, 2, \ldots, n$. This means exactly that $\mathcal{B}$ is a basis too (see Remark 5).

If $\mathcal{B}$ is a regular basis and $\dot{S}_i = \frac{1}{b_i}\dot{B}_i - \frac{1}{b_{n+1}}\dot{B}_{n+1}$, then $\mathcal{B}' = \{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{B}_{n+1}\}$ is a kernel-singular basis. This implies that the points $B_1, B_2, \ldots, B_{n+1}$ are affinely independent. □

An important consequence of Theorems 11 and 12 is that for every basis, there exists an orthogonal kantor and it is unique up to scalar multiplication: the points of the basis are affinely independent, so the circumsribed sphere exists and there is a unique circumkantor up to scalar multiplication.

## 4 Diagonalisation of the kantric scalar product

The kantric scalar product related to the regular basis $\mathcal{B}$ can be written in a simplier form using coordinate-transformation. Let $\dot{O}$ be the orthogonal kantor of $\mathcal{B}$ such that $\langle \dot{O}|\dot{O}\rangle = -1$. From the proof of Theorem 11, $\frac{\langle \dot{O}|\dot{O}\rangle}{o^2} = -R^2$, thus $o$ must be $\frac{1}{R}$. Let $\{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n\}$ be an orthonormal system of singular kantors in $\mathbb{K}^n$ (i.e., $\left\langle \dot{S}_i|\dot{S}_j \right\rangle = \delta_{ij}$). The translation vectors of these kantors form an orthonormal basis of $\mathbb{R}^n$, therefore $Q = \{\dot{S}_1, \dot{S}_2, \ldots, \dot{S}_n, \dot{O}\}$ is a kernel-singular basis of $\mathbb{K}^n$.

Since $\langle \dot{O}|\dot{O}\rangle = -1$ and for all $1 \leq i, j \leq n$, $\langle \dot{S}_i|\dot{S}_j\rangle = \delta_{ij}$ and $\langle \dot{O}|\dot{S}_j\rangle = 0$, thus for arbitrary kantors $\dot{A} = (a'_1, a'_2, \ldots, a'_{n+1})_Q$, $\dot{C} = (c'_1, c'_2, \ldots, c'_{n+1})_Q \in \mathbb{K}^n$ coordinated with respect to the basis $Q$,

$$\langle \dot{A}|\dot{C}\rangle_{\mathcal{B}} = \left\langle \sum_{i=1}^{n} a'_i \dot{S}_i + a'_{n+1}\dot{O} \,\middle|\, \sum_{j=1}^{n} c'_j \dot{S}_j + c'_{n+1}\dot{O} \right\rangle =$$

$$= \sum_{i,j=1}^{n} a'_i c'_j \langle \dot{S}_i|\dot{S}_j\rangle + \sum_{i=1}^{n} a'_i c'_{n+1}\langle \dot{S}_i|\dot{O}\rangle + \sum_{j=1}^{n} a'_{n+1} c'_i \langle \dot{O}|\dot{S}_j\rangle +$$

$$+ a'_{n+1}c'_{n+1}\langle \dot{O}|\dot{O}\rangle = \sum_{i=1}^{n} a'_i c'_i - a'_{n+1}c'_{n+1},$$

which is a sum of $n + 1$ terms oppositely to the $\frac{n(n+1)}{2}$ terms of the original formula of Definition 8. If the coordinates of $\dot{A}$ and $\dot{C}$ are known with respect to the basis $Q$ or the matrix of the coordinate transformation is of simple form, then it is worth to switch to this scalar product formula.

**Remark 13** *The fact $\langle \dot{O}|\dot{O}\rangle < 0$ does not depend on the mass of $\dot{O}$ and $\langle \dot{S}_i|\dot{S}_i\rangle > 0$ does not depend on the translation value of $\dot{S}_i$. This causes n positive and 1 negative sign in the above formula and means that the signature of the kantric scalar product is* $(+, +, \ldots, +, -)$.

## 5 The radius of the circumscribed sphere of affinely independent set of points

Let $B = \{B_1, B_2, \ldots, B_{n+1}\}$ be an affinely independent set of points in $\mathbb{R}^n$, $n \leq 2$ and denote the distance of $B_i$ and $B_j$ by $d_{ij}$. Let $\mathbf{1} \in \mathbb{R}^{n+1}$ denote the column vector consists of all ones and $\mathbf{D}$ be the $(n + 1) \times (n + 1)$ matrix for which $[\mathbf{D}]_{ij} = d_{ij}^2$.

**Theorem 14** *The radius of the circumscribed sphere of the set B is*

$$R = \frac{1}{\sqrt{2\mathbf{1}^T\mathbf{D}^{-1}\mathbf{1}}}.$$

**Proof:** Let $\dot{B}_i = (B_i, 1)$ be regular kantors for $i = 1, \ldots, n + 1$. Then $\mathcal{B} = \{\dot{B}_1, \dot{B}_2, \ldots, \dot{B}_{n+1}\}$ is a regular basis of $\mathbb{K}^n$. Denote the center of the circumscribed sphere by $O$ and let $\dot{O} = (O, 1)$ be a circumkantor.

Since $\dot{O}$ is an orthogonal kantor by Theorem 11, thus $\left\langle \dot{O}|\dot{B}_j - \dot{O}\right\rangle = 0$ for all $j = 1, \ldots, n + 1$. This means that

$$\left\langle \dot{O}|\dot{B}_j\right\rangle = \left\langle \dot{O}|\dot{O}\right\rangle = -R^2.$$

If $\dot{O} = (o_1, \ldots, o_{n+1})_{\mathcal{B}}$, then by the definition of the kantric scalar product,

$$\left\langle \dot{O}|\dot{B}_j\right\rangle = \sum_{i=1}^{n+1} -\frac{1}{2} d_{ij}^2 o_i.$$

Let $\underline{o}^T = [o_1, o_2, \ldots, o_{n+1}]$ be the row vector related to $\dot{O}$. The above equations can be collected to the system of equations:

$$\mathbf{D}\underline{o} = 2R^2\mathbf{1}.$$

$\mathbf{D}$ is invertible because this system has a unique solution: $\underline{o}$ must be the unit kantor placed in the center of the circumscribed sphere. This shows that $\underline{o} = 2R^2\mathbf{D}^{-1}\mathbf{1}$.

Since $\mathbf{1}^T\underline{o} = 1$, thus

$$1 = 2R^2\mathbf{1}^T\mathbf{D}^{-1}\mathbf{1} \;\Rightarrow\; R^2 = \frac{1}{2\mathbf{1}^T\mathbf{D}^{-1}\mathbf{1}}.$$

We have just obtained the radius of the circumsribed sphere as a function of the distances of the points. □

**Remark:** On the plain, $R$ is the radius of the circumscribed sphere of the triangle $\Delta_{B_1B_2B_3}$. If we denote the length of the sides by $a$, $b$ and $c$ and the area of the triangle by $T$, then Theorem 14 and the well known formula $T = \frac{abc}{4R}$ together gives the Heron's formula, which expresses the area of the triangle as a symmetric function of the side-lengths. It means that our formula is equivalent to Heron's formula in 2-dimension, and it is a generalisation of it to higher dimensions. An equivalent form of Theorem 14 was known in 3-dimension (see [2]), though our proof is more elegant and more general.

Finally, a related open problem is to determine the radius of the incircle of an $n$ dimensional simplex with a kantric method. It is $\frac{nV}{A}$ by a simple elementary calculation, where $V$ is the volume and $A$ is the area of the surface of the simplex.

## 6 Applications

Kantors are effective tools to solve planar geometric problems. The method is the same as in case of mass-points and is well detailed in [3, 5, 6]. Let there be given a triangle with vertices $A$, $B$ and $C$. Then the unite kantors $\dot{A} = (A, 1)$, $\dot{B} = (B, 1)$ and $\dot{C} = (C, 1)$ form a basis of $\mathbb{K}^2$. One can determine the kantric coordinates of the centroid, the orthocenter, the circumcenter and the center of the incircle with respect to this basis up to the mass. Then one can prove classical geometric properties of these points with a simple calculation, for example the existence of the Euler's line or Feuerbach's circle. On can also calculate distances of given points by the distance-formula of Theorem 7. It can be practical to introduce the concept of kantric line and kantric circle.

*Example:* Determine the kantric coordinates of the center of the incircle. We are looking for a regular kantor $\dot{Q} = (Q, 1)$ in the form $\dot{Q} = \alpha\dot{A} + \beta\dot{B} + \gamma\dot{C}$, where $Q = (q_x, q_y, q_z)$ is the center of the incircle and $\alpha, \beta, \gamma \in \mathcal{R}$. Classical theorems say that $Q$ is the intersection point of the angle bisectors, and an angle bisector divide the opposite side into segments of relative length equal to the relative length of the nearby sides of the triangle. Let $f_\alpha$ denote the bisector of the angle $\alpha$, and $P_\alpha$ denote the intersection of $f_\alpha$ and the side $a$. The center of the kantor $\dot{P}_\alpha = (0, b, c)$ (the coordinates are relative to the basis $\{\dot{A}, \dot{B}, \dot{C}\}$) is $P_\alpha$ because $P_\alpha$ is on the segment $\overline{BC}$, and $|\overline{BP_\alpha}|/|\overline{P_\alpha C}| = c/b$. This means that $q_y/q_z = c/b$ because $\dot{Q}$ is a linear combination of $\dot{A}$ and $\dot{P}_\alpha$. Similarly, $q_x/q_y = c/a$ and $q_x/q_z = b/a$. So, $\dot{Q} = (a, b, c)$ is a good choice.

Our main result, Theorem 14, can be applied in engineering and architecture to determine the radius, the surface and the volume of the circumscribed sphere of four spatial points by measuring the distances between them. An other form of Theorem 14 is known in 2-dimension (Heron's formula) as well as in 3-dimension (see [2]). However, it can be a useful tool in higher dimensional geometry.

Another area of applications is projective geometry. In $\mathbb{K}^n$, there is a natural one-to-one correspondence between the $k$ dimensional and the $n-k$ dimensional kantric subspaces given by the orthogonality via kantric scalar product. Actually, in case of $k = 0$, the assignment can be expressed by a sphere inversion with center $O$ and ratio $R^2$.

A kantric subspace of dimension $k$ is a $k+1$ dimensional subspace of $\mathbb{K}^n$. The 0-dimensional kantric subspaces are the ones generated by one kantor.

Let $\mathcal{H} = \left\langle \dot{P} \right\rangle$ be the 0-dimensional kantric subspace generated by $\dot{P}$. It is clear that $\mathcal{H}^\perp = \{\dot{Q} \in \mathbb{K}^n : \left\langle \dot{P}|\dot{Q} \right\rangle = 0\}$, the orthocomplement of $\mathcal{H}$, is an $n-1$ dimensional kantric subspace of $\mathbb{K}^n$.

If $\dot{P} = \dot{O}$, then $\mathcal{H}^\perp$ is the subspace of singular kantors. If $\dot{P}$ is a singular kantor, then $\mathcal{H}^\perp$ is spanned by $\dot{O}$ and $n-1$ independent singular kantor orthogonal to $\dot{P}$. Finally, if $\dot{P}$ is a regular kantor distinct from $\dot{O}$, then $\mathcal{H}^\perp$ is spanned by $n-1$ independent singular kantor orthogonal to $\dot{P} - \dot{O}$ and a regular kantor $\dot{T}$ centered on the $OP$ halfline satisfying $|\overline{OP}||\overline{OT}| = R^2$. The proof of the latter statement does not require new ideas, therefore left to the reader.

Let us extend $\mathbb{R}^n$ with a point at infinity in each direction to form the projective space $\mathbb{P}^n$. Let $P$ be the center of $\dot{P}$ in $\mathbb{P}^n$ (even if $\dot{P}$ is singular), $H^\perp$ be the set of the centers of the kantors in $\mathcal{H}^\perp$ and $S$ be the sphere with diameter $\overline{OP}$. Then $H^\perp$ is the unique hyperplain containing the image of $S \setminus \{O\}$ under the sphere inversion with center $O$ and ratio $R^2$. Finally, $\mathcal{H}^\perp$ is uniquely determined by $H^\perp$.

There are numerous quantities with kantric nature in physics, such as mass, electric charge and even force in special cases, to which our theory could be well applied. It is an interesting question which the singular forms of these "regular" quantities are.

In $\mathbb{K}^3$, the kantric scalar product is a Minkowski inner product with signature $(+,+,+,-)$ (see Section 4), hence $\mathbb{K}^3$ can also be considered as a representation of the Minkowski spacetime where mass plays the role of time. However, the space dimensions are not in direct accordance with the dimensions of the underlying $\mathbb{R}^3$ space of $\mathbb{K}^3$ (vector coordinates), but the mixtures of vector and mass coordinates. Hence, a simple euclidean distance formula should be applied instead of the kantric distance formula. For more details on Minkowski spacetime see [4].

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