https://doi.org/10.3311/PPee.37772

Periodica Polytechnica Electrical Engineering and Computer Science, 69(2), pp. 188–197, 2025

Mathematical Remodeling: Hierarchical Sensitivity Analysis Approach Based on Analysis of Finite Fluctuations

Anton Sysoev[1*], Pavel Saraev[2]

[1] Department of Applied Mathematics, Institute of Computer Sciences, Lipetsk State Technical University, Moskovskaya str. 30, RU-398070 Lipetsk, Russia

[2] Department of Automated Control Systems, Institute of Computer Sciences, Lipetsk State Technical University, Moskovskaya str. 30, RU-398070 Lipetsk, Russia

[*] Corresponding author, e-mail: sysoev_as@stu.lipetsk.ru

Received: 25 June 2024, Accepted: 03 April 2025, Published online: 05 May 2025

Abstract

To make the process of evaluating complicated hierarchical systems more tractable, it is beneficial to represent them through models derived from a unified class of frameworks that are conducive to subsequent analytical endeavors. This method of interchanging the original model with ones from predetermined categories possessing desirable characteristics is referred to as Mathematical Remodeling. Among remodeling classes there can be mentioned Neural Networks as structures being universal approximators and which are acceptable to be analyzed. The question related to the importance of model inputs (Sensitivity Analysis) has a great practical meaning, for example, this information can be used for model reduction, control of the system, etc. There are a wide variety of methods of Sensitivity Analysis, which can be classified both by the mathematical approaches used and by the types of models to which they are relevant. However, there are almost no unified approaches to assessing sensitivity in the case of hierarchical systems. The paper introduces the approach to estimate sensitivity measures for hierarchical system obtained by applying remodeling concept. The proposed method is based on Analysis of Finite Fluctuations built on the Lagrange mean value theorem. The proposed approach provides both end-to-end analysis (investigation of the influence of inputs of sub-systems on the output of the main system) and analysis of the components of the hierarchical system. The paper also contains a numerical example which demonstrates the ability of the proposed approach to deliver sensitivity measures of hierarchical system.

Keywords

remodeling, sensitivity analysis, hierarchical sensitivity measures, neural networks, analysis of finite fluctuations, Lagrange mean value theorem

1 Introduction

Complicated systems can be described by various types of subsystems having sometimes different representations. For example, the system on its highest level can be presented as a kind of indicator with predefined well-known structure, but for elements forming factors of such indicator may be used distinct approaches and models to construct them. To unify the analysis of these models it is possible based on the original model to create a more suitable for further usage model from another class. The approach to constructing a model based on an existing one is called Mathematical Remodeling [1]. One of the ways to carry out the remodeling procedure is the following: based on the original model, an array of data on the input and output values is formed a model on the resulting array of approximation algorithms. Technically remodeling is the process of formation a new model from the predefined class based on an existing model [2]. The method provides the option of using common approaches to analysis, optimization and control of the object. With the obvious advantages of the proposed approach, it also has some disadvantages, among them are the following: ignoring the physical basis of the studied process or system; additional loss of accuracy during the transition from one model to another; difficulties caused by developing and implementing additional algorithms for model transformation.

The idea of remodeling in accordance with defined reasons and criteria was stated in connection with the applied

Cite this article as: Sysoev, A., Saraev, P. "Mathematical Remodeling: Hierarchical Sensitivity Analysis Approach Based on Analysis of Finite Fluctuations", Periodica Polytechnica Electrical Engineering and Computer Science, 69(2), pp. 188–197, 2025. https://doi.org/10.3311/PPee.37772

problems of the metallurgical production (cf. [3]) and is actively developing, including the same applications and new applied problems.

Close to remodeling scheme (remodeling, cf. [4]) are surrogate modeling (cf. [5]), metamodeling (cf. [6]), co-simulation (cf. [7]) and some others similar approaches (re-run, repeat, reproduce, reuse, replicate, cf. [8]). And it should be noted that many classical and modern approaches and methods of fundamental and applied mathematics can also be interpreted as Mathematical Remodeling.

Traditionally remodeling is used to approximate static models using classical numerical methods of function approximation; in this case remodeling classes are Taylor polynomials, Fourier polynomials, etc. The other traditional approach is an equivalent remodeling, in which, in contrast to the approximation, the model is not approximated, but substituted by the equivalent one. The equivalent remodeling is also associated with the transformation of dynamic systems into equivalent "input-state-output" models, i.e. the solution of the direct problem for these systems.

As it was mentioned above one of the advantages of remodeling is the simplification of the further analysis of the model. In this regard neural network models can be indicated as one of the most unified approximators. Following the concept of multiple models describing different submodels of the system it is possible to use classical artificial neural networks to unify the representation of different components of the system, which can present by itself the hierarchical system. And the prominent problem in this field is to estimate the sensitivity measures of submodels inputs, and sensitivity measures of system units (outputs of submodels and inputs of the submodel of the higher level at the same time). This point leads to the Hierarchical Sensitivity Analysis (HSA).

The authors have previously proposed a Sensitivity Analysis (SA) approach described in a series of papers and verified on different types of models. The presented study extends the developed method to the case of hierarchical systems. The proposed original method enables both end-to-end SA (the influence of inputs of different subsystems on the output of the upper-level system) and SA of individual subsystems and groups of subsystems. Also, it is used Neural Network model as a unified remodeling class of models.

The paper is organized as follows. Section 2 presents a review on existing methods of Sensitivity Analysis (SA), special attention is given to hierarchical SA, Section 3 describes SA based on applying Analysis of Finite Fluctuations (AFF), Section 4 introduces the new approach to hierarchical SA, in Section 5 are given numerical examples and Section 6 concludes the paper.

2 State-of-art in sensitivity analysis

The study of how input uncertainty affects output uncertainty is known as Sensitivity Analysis [9]. A task such as that could be useful in solving the following issues, depending on the goal:

1. determining the robustness of a system's or model's results;
2. comprehending the relationships between inputs and outputs;
3. minimizing uncertainty by selecting the most important inputs;
4. identifying errors in a system or model by defining unexpected relationships between inputs and outputs;
5. streamlining a model by eliminating inputs that have little bearing on outputs;
6. making a model more interpretative by determining an understandable explanation of inputs; and so on.

Numerous approaches to test the sensitivity of mathematical models might be suggested, depending on the instruments being utilized. Certain features are applicable to all models, while others are restricted to those with a predetermined structure.

The SA taxonomy proposed in the study [10] splits all methods into two categories: local and global SA. The scalar function (indicator, output) y is influenced by the vector variable (factors, inputs) $X = (x_1, \ldots, x_n)$ through a black-box relation:

$$y = f\left(x_1, \ldots, x_n\right) = f\left(X\right). \tag{1}$$

The first category of methods (local SA) relies on gathering data on the degree of uncertainty in the resultant value concerning variations in one of its inputs (that is, i.e., the function's partial derivative). Such an understanding of sensitivity has two obvious weaknesses: first, the obtained result of the sensitivity assessment depends on the range of selected xi values in the case of a non-linear nature of Eq. (1); second, in the event that interactions between factors exist, the change to the partial derivative depends on factors other than the selected x_i values. This indicates that methods for the local SA produce sufficient and precise outcomes in situations where the structure of Eq. (1) is somewhat constrained, which restricts the use of this group of methods.

Methods of the second group (global SA) propose producing findings accounting modifications to all parameters, not just one, in order to take potential interactions between them into consideration. The first-order sensitivity index proposed by Sobol is the most well-known example of a global sensitivity measure (cf. [11]). Other techniques for global SA include the elementary effects approach (cf. [12]), global derivative-based measures (cf. [13]), moment-independent methods (cf. [14]), variogram-based approaches (cf. [15]), etc.

2.1 Local SA approaches

In local SA we have to estimate

$$A_i = \frac{\partial y}{\partial x}\left(x_1^{(0)},\dots,x_n^{(0)}\right) \tag{2}$$

characterizing how the change of x_i affects the output y near the value of $X^{(0)}$ (cf. [16]).

The "one at a time" (OAT) method is frequently used to evaluate the estimation of Eq. (2) based on the first partial derivatives. This method keeps all factors fixed except the one that is under perturbation [17]. The contrast approach of global SA taking into account all factors and is called "all in one" (AIO) strategy.

Another popular local SA approach is the Morris method [18]. According to this approach, measures that can be used to assess sensitivity are means and standard deviations of the absolute value of elementary effects. The following interpretations of the obtained results can be used: mean is a measure of the influence of an input on the output (a larger value indicates greater influence); standard deviation is a measure of non-linear and/or interaction effects of an input (a variable with a large measure is considered to have non-linear effects or is implied to have interaction with at least one other variable).

2.2 Global SA approaches

Methods based on the study of linear models make up the first category of techniques. Assume that inputs and outputs of the studied model are readily available and that a linear fit to the current relationship, Eq. (1) is achievable. A fitted linear model is intended to be used in certain methods for evaluating the sensitivity measures. The often used indices in this instance are: Pearson correlation coefficient (if it is equal to −1 or 1, the system output is dependent on the tested input value and the output is connected in a linear manner; if every set of an input and output have no connections, the correlation coefficient is equal to 0), standard regression coefficients (the square of this coefficient describes the portion of the output variance that is explained by an input), partial correlation coefficient (is the degree to which the output is affected by an input when the effects of the other inputs have been nullified).

When it is not possible to use a linear model to identify Eq. (1) or this structure is non-monotonic, the decomposition of the output variance can be used to assess sensitivity. In the case when Eq. (1) is a square-integrable function, defined on the unit hypercube $[0,1]^n$, it can be represented as a sum of elementary functions [19]:

$$f(X) = f_0 + \sum_{i=1}^{n} f_i(x_i) + \sum_{i<j}^{n} f_{ij}(x_i, x_j) + \ldots + f_{12\ldots n}(X). \tag{3}$$

According to the study [20]

$$\int_0^1 f_{i_1\ldots i_s}\left(x_{i_1},\dots,x_{i_s}\right)dx_{i_k} = 0, \quad 1 \le k \le s,$$
$$\left\{x_{i_1},\dots,x_{i_s}\right\} \subseteq \{1,\dots,n\}$$

the expansion, Eq. (3) is unique.

When the random vector $X \in R^n$ with mutually independent components is connected with the output y, according to [20], a functional decomposition of the variance is available:

$$Var(y) = \sum_{i=1}^{n} D_i(y) + \sum_{i<j}^{n} D_{ij}(y) + \ldots + D_{12\ldots n}(y), \tag{4}$$

where

$$D_i(y) = Var(y)\left[E\left(y|x_i\right)\right],$$
$$D_{ij}(y) = Var(y)\left[E\left(y|x_i,x_j\right)\right] - D_i(y) - D_j(y),$$

and so on for higher-order interactions. The sensitivity measures (also known as Sobol's indices [21]) are obtained from Eq. (4) as:

$$S_i = \frac{D_i(y)}{Var(y)}, \quad S_{ij} = \frac{D_{ij}(y)}{Var(y)}, \quad \ldots .$$

According to [22], the total indices can be defined as

$$S_{T_i} = S_i + \sum_{i<j} S_{ij} + \sum_{i\neq j,\ k\neq i,\ i<k} S_{ijk} + \ldots = \sum_{l \in \#i} S_l,$$

where $\#i$ are all the subsets of $\{1, \ldots, n\}$ including i.

The described above techniques are most-frequently used, but not the only existing. The other classifications can be found in reviews, for example in the case study [23] there is presented the classification taking into account the structure of studied model, number of inputs, available computational resources (cf. Fig. 1).

2.3 Hierarchical SA approaches

Hierarchical models are becoming a common strategy for studying complex problems. The growing need for hierarchical design approaches is also evident in the development of multilevel optimization methodology, to support decision making.

Among the studies on HSA there are cases proposing the top-down strategy of conducting SA [24]. This strategy contains three features: instead of performing SA to a complex AIO model, SA is applied separately, step by step, to submodels at each level of hierarchy, which allows independent or parallel executions of SA on submodels; SA is applied first to the top-level model with the final output of interest following the top-down sequence; to save cost, SA is applied only to critical lower-level submodels whose performances have a significant impact on the upper-level models. These mentioned above studies also propose an aggregation approach to evaluating the global statistical sensitivity index.

The hierarchical model is shown in Fig. 2. Following the existing strategy, SA is applied first to the top-level model (Model A), whose input variables are the performance responses passed from Submodels B and C at the second level. If the performance of Submodel C is found to be more significant than the one from Submodel B, then SA is applied only to Submodel C. Similarly, the local SA results from Submodel C will indicate whether further SA are needed for Submodels E or F.

3 Sensitivity analysis based on analysis of finite fluctuations

3.1 Background of the approach

Analysis of Finite Fluctuations was developed based on Economic Factor Analysis [25] and consists of constructing

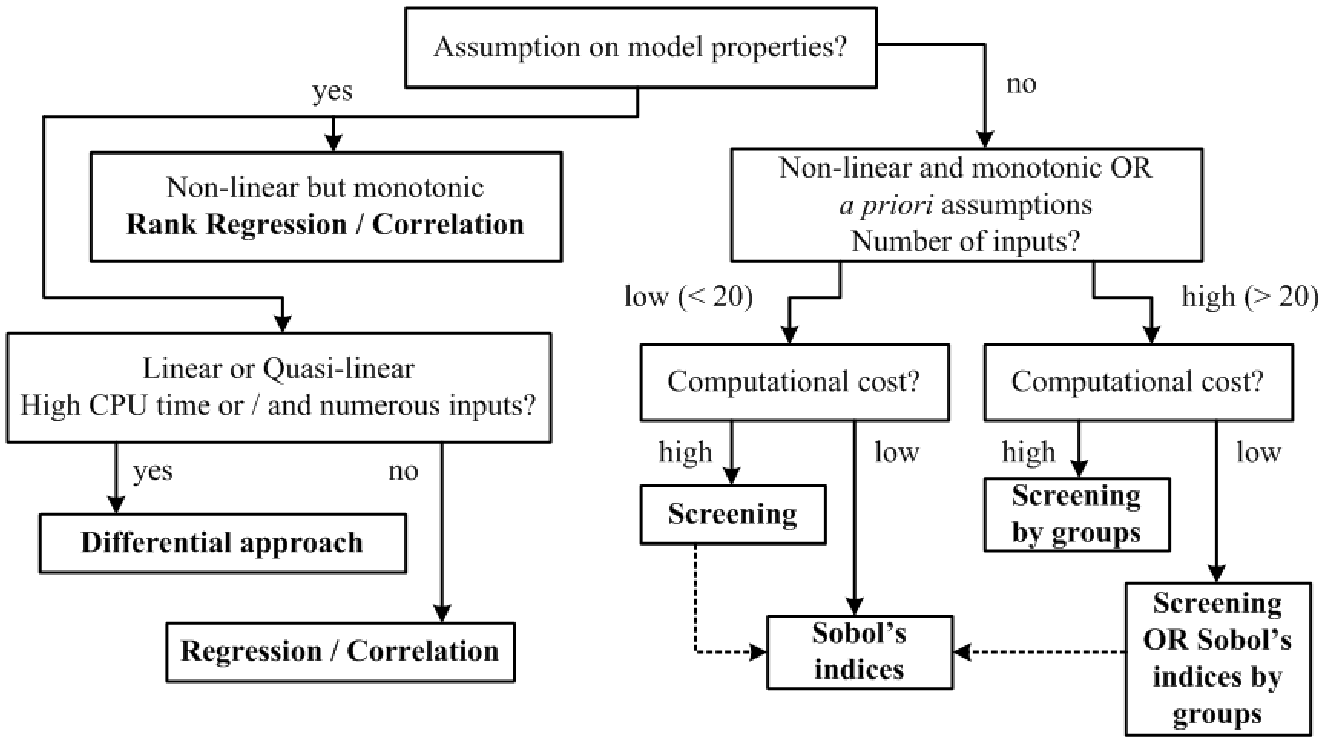

Fig. 1 Brief classification of sensitivity analysis techniques

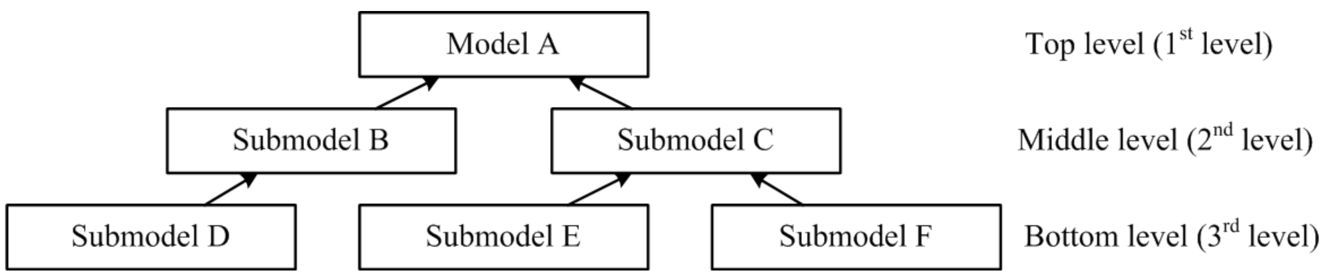

Fig. 2 Example of hierarchical model

models linking finite fluctuations of inputs (factors) and finite fluctuation of output. There are many types to measure the finite fluctuation of the variable (index, relative index, etc.), but the most frequently used is the finite increment defined as the simple difference between the current value of variable and the value at the previous moment of time ($\Delta x = x - x^{(0)}$). In case when fluctuations are small in classical Mathematical Analysis there is the model setting such connection. If the function $y = f(x)$ describing the model is defined to be continuous and differentiable in a closed domain, the approximate relationship between the response and the small fluctuations of its arguments is

$$\begin{aligned}\Delta y &= f\left(X^{(0)} + \Delta X\right) - f\left(X^{(0)}\right) \\ &= f\left(x_1^{(0)} + \Delta x_1, \dots, x_n^{(0)} + \Delta x_n\right) \\ &- f\left(x_1^{(0)}, \dots, x_n^{(0)}\right) \approx \sum_{i=1}^{n} \frac{\partial f\left(X^{(0)}\right)}{\partial x_i} \cdot \Delta x_i.\end{aligned} \tag{5}$$

But for some practical issues, fluctuations might be regarded as finite values rather than tiny ones; and Eq. (5) cannot be applied.

When increments are used as finite fluctuations, there exists a model that is structurally exactly in the form connecting the finite fluctuation of the output and the inputs' (factors) finite fluctuations. This is the Lagrange mean value theorem for functions, which is defined and continuous in a closed domain and has continuous partial derivatives inside this domain. It is formulated in the following manner:

$$\Delta y = \sum_{i=1}^{n} \frac{\partial f\left(X^{(m)}\right)}{\partial x_i} \cdot \Delta x_i, \tag{6}$$

$$X^{(m)} = \left(x_1^{(m)}, \dots, x_n^{(m)}\right),$$
$$x_i^{(m)} = x_i^{(0)} + \alpha \cdot \Delta x_i, \quad 0 < \alpha < 1.$$

Here, the mean (or intermediate) values of arguments (factors, inputs) $x_i^{(m)}$ are defined by the value of α.

3.2 Sensitivity indices based on AFF

Let's have the current moment of time. The initial state of the inputs vector is presented in the form $X^{(0)} = (x_1^{(0)}, \dots, x_n^{(0)})$, and, according to the connection to Eq. (1) the output can be presented as $y^{(0)} = f(X^{(0)})$. After a while, at the next moment of fixation, the inputs change and now they are $X^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)})$, and the output of the system is consequently $y^{(1)} = f(X^{(1)})$. Thus, the increment of the output can be defined, on the one hand, as the difference in the new and previous values of the outputs and, on the other hand, by the Lagrange theorem, Eq. (6), i.e., the following equation can be composed and solved with respect to the parameter α:

$$y^{(1)} - y^{(0)} = \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\left(\dots, x_i^{(0)} + \alpha \cdot \Delta x_i, \dots\right) \cdot \Delta x_i,$$

which allows estimating the so-called factor loadings to obtain a model of the form

$$\begin{aligned}\Delta y &= \sum_{i=1}^{n} \frac{\partial y}{\partial x_i}\left(\dots, x_i^{(0)} + \alpha \cdot \Delta x_i, \dots\right) \cdot \Delta x_i \\ &= A_{x_1} \Delta x_1 + \dots + A_{x_n} \Delta x_n.\end{aligned}$$

The procedure above is repeated m times (where m is the number of available observations); the numerical results of the analysis have to be averaged to construct the sensitivity measure [26]. It should be noted, that in the study [26] it is proposed the algorithm to build the point and the interval estimate for obtained factor loadings to construct sensitivity measures. The proposed approach is based on applying Tukey's weighted average.

4 Hierarchical sensitivity analysis based on analysis of finite fluctuations

Complex systems are mainly described by the system of indicators. Such system can be constructed with a set of functions, each of which is dependent on the certain collection of factors (inputs). The proposed approach assumes that each of the bottom level functions has its own individual set of inputs.

Let there exist $m \times n$ factors, define them by x_{ij}, $i = 1, \dots, m$, $j = 1, \dots, n$; functions f_j, $j = 1, \dots, n$ are dependent on these factors, i.e. there is defined vector-function dependent of vector-argument, but each element of the function has its own unique set of inputs: $f = f(X)$, $X \in R^{m \times n}$, $f \in R^n$, where each function f_j is a scalar function dependent on vector argument.

Let the inputs change, and their changes are described by finite increments Δx_{ji}, the vector of outputs also has the corresponding finite increment, which is defined by $\Delta f = f\left(x + \Delta x\right) - f\left(x\right)$, where $f\left(x + \Delta x\right) = f\left(x\right) + \Delta f$. Each output has its own finite increment

$$\Delta f_j = f_j\left(x_j + \Delta x_j\right) - f_j\left(x_j\right), \quad j = 1, \dots, n.$$

We suppose that functions are differentiable, and the Lagrange mean value theorem can be applied:

$$\Delta f_j = \left(\left(\Delta f_j\right)_L\right)_{\alpha_j} = \sum_{i=1}^{m} \frac{\partial f_j}{\partial x_{ji}}\left(X + \alpha_j \cdot \Delta X\right) \cdot \Delta x_{ji}, \tag{7}$$

where $\alpha \in (0,1)$.

Let p is the variable aggregating outputs f_j as a scalar function dependent on vector argument in the form $p = p(f) = p(\ldots, f_j, \ldots)$. Its finite increment is $\Delta p = p(f + \Delta f) - p(f)$ can also be presents using the Lagrange mean value theorem as

$$\Delta p = \left((\Delta p)_L\right)_\beta = \sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(f + \beta \cdot \Delta f\right) \cdot \Delta f_j, \tag{8}$$

where $\beta \in (0,1)$.

Each output has its own finite increment, and the expression, Eq. (7) can be re-written as

$$\Delta p = \left((\Delta p)_L\right)_\beta = \sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(\ldots, f_j + \beta \cdot \Delta f_j, \ldots\right) \cdot \Delta f_j. \tag{9}$$

Let's use Eq. (7) into Eq. (9)

$$\Delta p = \left(\left(\left((\Delta p)_L\right)_\beta\right)_L\right)_{(\ldots,\alpha_j,\ldots)} = \sum_{j=1}^{n} \left[\frac{\partial p}{\partial f_j}\left(\ldots, f_j + \beta \cdot \left((\Delta f_j)_L\right)_{\alpha_j}, \ldots\right) \times \left(\sum_{i=1}^{m} \frac{\partial f_j}{\partial x_{ji}}\left(x + \alpha_j \Delta x\right) \cdot \Delta x_{ji} \right) \right]$$

and after changing the summing order we receive

$$\Delta p = \left(\left(\left((\Delta p)_L\right)_\beta\right)_L\right)_{(\ldots,\alpha_j,\ldots)} = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(\ldots, f_j + \beta \cdot \left((\Delta f_j)_L\right)_{\alpha_j}, \ldots\right) \times \frac{\partial f_j}{\partial x_{ji}}\left(x + \alpha_j \Delta x\right) \cdot \Delta x_{ji} \right]. \tag{10}$$

Let $\hat{p}(x) = p(f(x))$, then

$$\Delta \hat{p} = \hat{p}(x + \Delta x) - \hat{p}(x) = \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{\partial \hat{p}}{\partial x_{ji}}\left(x + \gamma \Delta x\right) \cdot \Delta x_{ji} = \left((\Delta \hat{p})_L\right)_\gamma$$

where $\gamma \in (0,1)$ can be calculated for each function.

Applying the form invariance property of the differential of a complex function to the function $\hat{p}(x) = p(f(x))$ leads to the following representation of its derivatives:

$$\frac{\partial \hat{p}}{\partial x_{ji}}(x) = \frac{\partial p}{\partial x_{ji}}\left(f(x)\right) = \frac{\partial p}{\partial f_j}\left(f(x)\right) \cdot \frac{\partial f_j}{\partial x_{ji}}(x).$$

Then finite increment of the final output presented by the Lagrange mean value theorem is

$$\Delta p = \Delta \hat{p} = \left((\Delta \hat{p})_L\right)_\gamma = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(f(x + \gamma \Delta x)\right) \times \frac{\partial f_j}{\partial x_{ji}}\left(x + \gamma \Delta x\right) \cdot \Delta x_{ji} \right]. \tag{11}$$

By equating Eqs. (10) and (11), we find α_j, $j = 1, \ldots, n$, $\beta, \gamma \in (0,1)$:

$$\sum_{i=1}^{m} \left[\sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(\ldots, f_j + \beta \cdot \left((\Delta f_j)_L\right)_{\alpha_j}, \ldots\right) \times \frac{\partial f_j}{\partial x_{ji}}\left(x + \alpha_j \Delta x\right) \cdot \Delta x_{ji} \right] = \sum_{i=1}^{m} \left[\sum_{j=1}^{n} \frac{\partial p}{\partial f_j}\left(f(x + \gamma \Delta x)\right) \times \frac{\partial f_j}{\partial x_{ji}}\left(x + \gamma \Delta x\right) \cdot \Delta x_{ji} \right].$$

The received model links parameters α_j, β, $\gamma \in (0,1)$. If suppose that $p(x)$ describes the whole complex system with its inputs x_{ij} and at the same time with its nodes (subsystems) f_j, and f_j describes the dependence of each subsystem on its inputs, then the received expression can link the finite fluctuations of the whole system with the finite fluctuations of inputs of its subsystems.

5 Numerical example

5.1 Structure of hierarchical model

The structure of the hierarchical system under consideration is presented in Fig. 3.

On the top level as well as on bottom level are applied neural networks models from the unified remodeling class. All three models are fully connected neural networks with the following meta parameters:

- Submodel Y_1: 1 hidden layer consisting of 2 neurons,
- Submodel Y_2: 1 hidden layer consisting of 1 neurons,
- Model Z: 1 hidden layer consisting of 2 neurons,

activation functions are the same for all neurons and described as $\sigma(\text{net}) = \dfrac{1}{1 + \exp(-\text{net})}$.

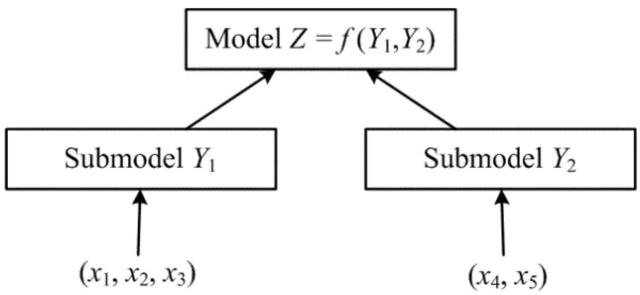

Fig. 3 Structure of numerical example system

The general structure of remodeling class is

$$y = \sigma\left(\beta_0 + \sum_{i=1}^{t} w_i \sigma\left(\beta_i + \sum_{j=1}^{k} w_{ij} x_j\right)\right), \quad (12)$$

where x_j are inputs ($j = 1, \ldots, k$, j is the number of inputs), w_{ij}, w_i are weights on hidden and output layers respectively ($i = 1, \ldots, t$, t is the number of neurons on the hidden layer), β_i and β_0 are biases on hidden and output layers respectively.

5.2 Scope of experiment

To assess the adequacy of the results obtained using the proposed approach, a series of computational experiments were conducted. The neuraldat data set from the NeuralNetTools package [27] of the R data processing environment was used as data. The data set consists of 2 non-correlated outputs and 3 non-correlated inputs; 2000 realizations, which allows us to obtain 1999 finite response increments and their corresponding arguments. The number of inputs was extended to 5, there was used non-linear connection between the outputs to simulate the structure of the system which is under consideration (there was supposed that Model Z has the following form $Z = \exp(Y_1) \cdot Y_2$). The remodeling approximation error is 3.78%.

5.3 Results of HSA

Following the mentioned above top-down strategy, firstly it was estimated the sensitivity measures of outputs of Submodels Y_1 and Y_2 as inputs of Model Z. The results are presented on Fig. 4. Here and further all obtained results are compared with sensitivity indices calculated by Garson's approach. This method is based on the study of weights constructed neural network model. It is believed that the variation of the studied coefficients can explain the characteristics of the "black box" neural network. According to the study [28], for three-layer neural network with a classical structure, factor sensitivity coefficients can be found as

$$S_k^p(i) = \frac{\sum_{j=1}^{n}\left(w_{ij} \cdot v_{jk} \Big/ \sum_{i=1}^{n} w_{ij}\right)}{\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(w_{ij} \cdot v_{jk} \Big/ \sum_{i=1}^{n} w_{ij}\right)\right)},$$

where i, j and k are indices for weights of input, hidden, and output layer weights respectively.

Based on comparison of Garson's algorithm and proposed approach it is possible to conclude that applying analysis of final fluctuations gives consistent result.

Similar situations can be seen for Submodels Y_1 and Y_2 (cf. Figs. 5 and 6 respectively). It should be noted that for Submodel Y_2 both approaches give the same outputs.

The proposed approach has an undeniable advantage. In contrast to Sobol indices described above it does not use an approximation procedure to model statistical parameters of the studied structure and in contrast to Garson's strategy, our approach operates with both parameters and factors of the model. Thus, the proposed approach does not include the source of uncertainty. And this fact allows us to construct global sensitivity measures showing the sensitivity of model output (in the presented example Model Z output) depending on the inputs of lower-level models (Submodels Y_1 (x_1, x_2, x_3) and Y_2 (x_4, x_5)).

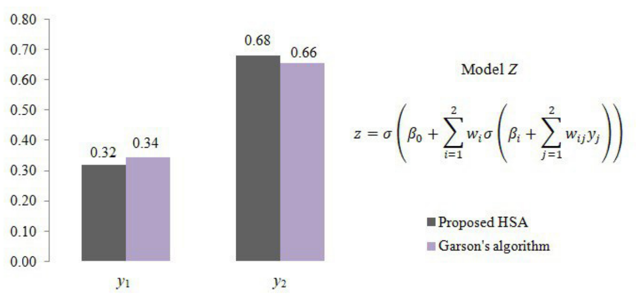

Fig. 4 Results of HSA for Model Z

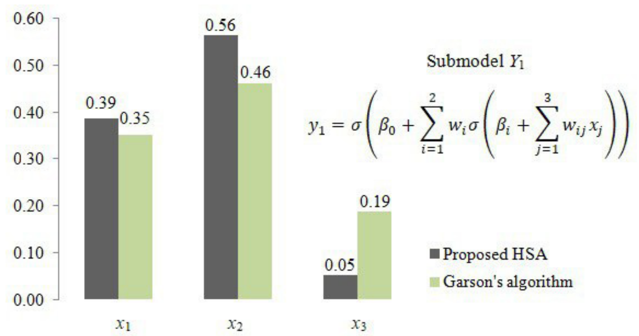

Fig. 5 Results of HSA for Submodel Y_1

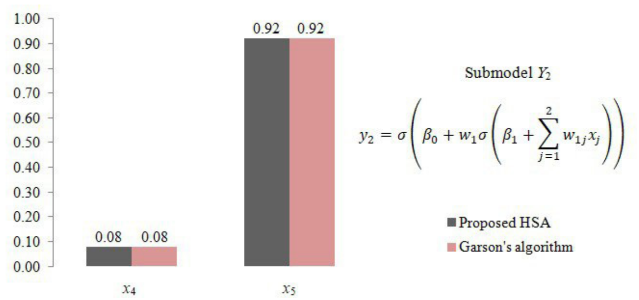

Fig. 6 Results of HSA for Submodel Y_2

5.4 Constructing global sensitivity indices

We propose the following model for estimating the global indices for sensitivity measures in case on hierarchical sensitivity analysis:

$$A_{f(x_i)} = \left(\prod_{j=1}^{n-1} A_{f_j}(x) \right) \cdot A_{x_i}, \tag{13}$$

where $A_{f_j}(x)$ are sensitivity measures of nods elements (submodels), n is the number of hierarchy levels of the system.

In our example global sensitivity measure for the input x_1 is equal to (according to Eq. (12)):

$$A_{Z(x_1)} = A_{y_1(x)} \cdot A_{x_1}.$$

Indices $A_{Z(x_2)}$, $A_{Z(x_3)}$, $A_{Z(x_4)}$ and $A_{Z(x_5)}$ can be estimated similarly. The histogram of sensitivity measures for the top-level model based on the inputs of the bottom-level models is presented in Fig. 7.

Based on the results presented in Fig. 7 we can conclude that inputs on the bottom-level are arranged in the following order (affecting the top-level output): x_5, x_2, x_1, x_4, x_3.

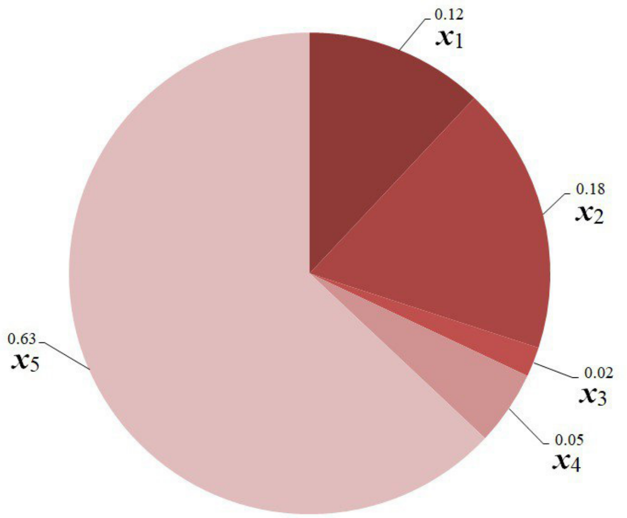

Fig. 7 Global HSA indices

5.5 Discussing results

Calculations presented in the study give an exact sensitivity measures estimate. But it should be mentioned that these measures are easy to be found for submodels (in case when each element is under consideration separately from the whole hierarchical structure). The presented approach allows estimating sensitivity measures for the whole system taking into account its hierarchy, starting from sensitivity of the highest level output (indicator) based on its inputs (outputs of models which are one level lower), to sensitivity of highest level output based on inputs of models on the lowest level (as it is presented in Section 3.2). But such procedures are more complicated and take more resources. To avoid these lacks, we have proposed an approach to the aggregation of sensitivity measures for constructing global indices (cf. Eq. (13)). Obtained results for the presented numerical example were compared with indices found by direct computations using the Lagrange mean value theorem. Both sets of measures are similar, which proves that Eq. (13) can be used with this regard.

6 Conclusion and outlook

The presented paper introduces an approach to hierarchical sensitivity analysis for remodeling concept. The strategy is based on applying Analysis of Finite Fluctuations as a tool to measure importance of studied factors. Starting with an applied problem, how to choose the most important inputs for the neural network model, it was synthesized the strategy how to measure sensitivity for factors affecting studied model taking into account its hierarchical structure. The following research will be devoted to the extension of studied models and using another type of models of remodeling classes and numerical estimation of partial derivatives of the models.

Acknowledgement

The research is supported by the Russian Science Foundation grant No. 24-21-00474 [29].

References

[1] Saraev, P. V., Blyumin, S. L., Galkin, A. V. "Nekotoryye sovremennyye podhody k modelirovaniyu i avtomatizatsii slozhnykh sistem" (Some Modern Approaches to Modeling and Automation of Complex Systems), Conduct the Higher Educational Institutions of the Black Earth, 47(1), pp. 53–66, 2017. (in Russian)

[2] Blyumin, S. L., Galkin, A. V. Saraev, P. V. "Formalizacija kriteriev kachestva matematicheskogo remodelirovanija dinamicheskih sistem" (Formalizing quality criteria for mathematical remodeling of dynamical systems), In: Proceedings of the 10th Russian multiconference "Modeli, metody i tehnologii intellektual'nogo upravlenija (IU-2017)", Gelendzhik, Russia, 2017, pp. 33–35. ISBN 978-5-9275-2462-4 (in Russian)

[3] Kotsar, S. L., Blyumin, S. L., Baryshev, V. G. Polyakov, B. A. "Primenenie nesimmetrichnogo ortogonalnogo planirovaniya pri issledovanii protsessov prokatki" (Supplementing of asymmetric orthogonal planning while exploration of the operation of rolling), In: Proceedings of the first Soviet Technical Conference "Primenenie EVM v metallurgii", Moscow, USSR, 1973, pp. 118–119. (in Russian)

[4] Li, T., Su, Y., Zhong, B. "Remodeling for fuzzy PID controller based on neural networks", In: Proceedings of the Second International Conference of Fuzzy Information and Engineering (ICFIE), Gaungzhu, China, 2007, pp. 714–725. ISBN 978-3-540-71440-8 https://doi.org/10.1007/978-3-540-71441-5_77

[5] Grihon, S., Alestra, S., Burnaev, E. Prikhodko, P. "Surrogate Modeling of Buckling Analysis in Support of Composite Structure Optimization", presented at DYNACOMP 2012 1st International Conference on Composite Dynamics, Arcachon, France, May, 22-24, 2012.

[6] Zhao, D., Xue, D. "A multi-surrogate approximation method for metamodeling", Engineering with Computers, 27(2), pp. 139–153, 2011. https://doi.org/10.1007/s00366-009-0173-y

[7] Gomes, C., Thule, C., Broman, D., Larsen, P. G., Vangheluwe, H. "Co-simulation: State of the Art", [preprint] arXiv, arXiv:1702.00686v1, 01 February 2017. https://doi.org/10.48550/arXiv.1702.00686

[8] Benureau, F., Rougier, N. "Re-run, Repeat, Reproduce, Reuse, Replicate: Transforming Code into Scientific Contributions", [preprint] arXiv, arXiv:1708.08205v1, 28 August 2017. https://doi.org/10.48550/arXiv.1708.08205

[9] Saltelli, A., Ratto, M., Andres, T., Campolongo, F., Cariboni, J., Gatelli, D., Saisana, M., Tarantola, S. "Global Sensitivity Analysis. The Primer", John Wiley & Sons, 2008. ISBN 978-0-470-05997-5 https://doi.org/10.1002/9780470725184

[10] Campolongo, F., Saltelli, A., Tarantola, S. "Sensitivity analysis as an ingredient of modeling", Statistical Science, 15(4), pp. 377–395, 2000. https://doi.org/10.1214/ss/1009213004

[11] Sobol, I. M. "Global sensitivity indices for nonlinear mathematical models and their Monte Carlo estimates", Mathematics and Computers in Simulation, 55(1–3), pp. 271–280, 2001. https://doi.org/10.1016/S0378-4754(00)00270-6

[12] Garcia Sanchez, D., Lacarrière, B., Musy, M., Bourges, B. "Application of sensitivity analysis in building energy simulations: Combining first- and second-order elementary effects methods", Energy and Buildings, 68, pp. 741–750, 2014. https://doi.org/10.1016/j.enbuild.2012.08.048

[13] Lamboni, M., Kucherenko, S. "Multivariate sensitivity analysis and derivative-based global sensitivity measures with dependent variables", Reliability Engineering & System Safety, 212, 107519, 2021. https://doi.org/10.1016/j.ress.2021.107519

[14] Borgonovo, E., Castaings, W., Tarantola, S. "Model emulation and moment-independent sensitivity analysis: An application to environmental modelling", Environmental Modelling & Software, 34, pp. 105–115, 2012. https://doi.org/10.1016/j.envsoft.2011.06.006

[15] Rana, S., Ertekin, T., King, G. R. "An efficient assisted history matching and uncertainty quantification workflow using Gaussian processes proxy models and variogram based sensitivity analysis: GP-VARS", Computers & Geosciences, 114, pp. 73–83, 2018. https://doi.org/10.1016/j.cageo.2018.01.019

[16] Pujol, G. "Simplex-based screening designs for estimating metamodels", Reliability Engineering & System Safety, 94(7), pp. 1156–1160, 2009. https://doi.org/10.1016/j.ress.2008.08.002

[17] Hamby, D. M. "A comparison of sensitivity analysis techniques", Health Physics, 68(2), pp. 195–204, 1995. https://doi.org/10.1097/00004032-199502000-00005

[18] Dean, A., Lewis, S. "Screening: Methods for experimentation in industry, drug discovery, and genetics", Springer, 2006. ISBN 978-0-387-28013-4 https://doi.org/10.1007/0-387-28014-6

[19] Hoeffding, W. "A Class of Statistics with Asymptotically Normal Distribution", The Annals of Mathematical Statistics, 19(3), pp. 293–325, 1948. https://doi.org/10.1214/aoms/1177730196

[20] Efron, B., Stein, C. "The jacknife estimate of variance", The Annals of Statistics, 9(3), pp. 586–596, 1981. https://doi.org/10.1214/aos/1176345462

[21] Sobol', I. M. "Sensitivity estimates for nonlinear mathematical models", Mathematical Modelling and Computational Experiments, 1(4), pp. 407–414, 1993.

[22] Homma, T., Saltelli, A. "Importance measures in global sensitivity analysis of nonlinear models", Reliability Engineering & System Safety, 52(1), pp. 1–17, 1996. https://doi.org/10.1016/0951-8320(96)00002-6

[23] Iooss, B., Lemaître, P. "A review on global sensitivity analysis methods", In: Dellino, G., Meloni, C. (eds.) Uncertainty Management in Simulation-Optimization of Complex: Algorithms and Applications, Springer, 2015, pp. 101–122. ISBN 978-1-4899-7546-1 https://doi.org/10.1007/978-1-4899-7547-8_5

[24] Xu, C., Zhu, P., Liu, Z., Tao, W. "Mapping-Based Hierarchical Sensitivity Analysis for Multilevel Systems with Multidimensional Correlations", Journal of Mechanical Design, 143(1), 011707, 2021. https://doi.org/10.1115/1.4047689

[25] Sysoev, A., Ciurlia, A., Sheglevatych, R., Blyumin, S. "Sensitivity Analysis of Neural Network Models: Applying Methods of Analysis of Finite Fluctuations", Periodica Polytechnica Electrical Engineering and Computer Science, 63(4), pp. 306–311, 2019. https://doi.org/10.3311/PPee.14654

[26] Sysoev, A. "Sensitivity Analysis of Mathematical Models", Computation, 11(8), 159, 2023. https://doi.org/10.3390/computation11080159

[27] Rdocumentation "neuraldat: Simulated dataset for function examples", [online] Available at: https://www.rdocumentation.org/packages/NeuralNetTools/versions/1.5.3/topics/neuraldat [Accessed: 02 April 2025]

[28] Maozhun, S., Ji, L. "Improved Garson algorithm based on neural network model", In: 2017 29th Chinese Control And Decision Conference (CCDC), Chongqing, China, 2017, pp. 4307–4312. ISBN 978-1-5090-4657-7 https://doi.org/10.1109/CCDC.2017.7979255

[29] Russian Science Foundation "Kartochka proekta fundamental'nyh i poiskovyh nauchnyh issledovanij, podderzhannogo Rossijskim nauchnym fondom" (Card of the project of fundamental and prospecting scientific research supported by the Russian Science Foundation), [online] Available at: https://rscf.ru/project/24-21-00474/ [Accessed: 02 April 2025] (in Russian)