Generating spherical multiquadrangulations by restricted vertex splittings and the reducibility of equilibrium classes

Abstract

We study the combinatorial properties associated with an earlier published, geometric algorithm capable of generating convex bodies in any primary equilibrium class (i.e. bodies with arbitrary numbers of equilibrium points) from a single ancestor.

Primary equilibrium classes contain several topological secondary classes based on the arrangement of the equilibrium points. Here we show that the associated graph expansion algorithm is incomplete in the sense that using the same ancestor, not all secondary classes can be generated and we point out the nontrivial set of ancestors necessary to generate all secondary classes.

Keywords

plane multiquadrangulation · unrooted unsensed colored map · vertex splitting · census · convex body · equilibrium class

Richárd Kápolnai

Dept. of Control Engineering and Information Technology, BME, H-1117 Budapest, Magyar tudósok körútja 2., Hungary
e-mail: kapolnai@iit.bme.hu

Gábor Domokos

Dept. of Mechanics, Materials & Structures, BME, H-1521 Budapest, Műegyetem rakpart 1-3. K.II.42., Hungary
e-mail: domokos@iit.bme.hu

Tímea Szabó

Dept. of Mechanics, Materials & Structures, BME, H-1521 Budapest, Műegyetem rakpart 1-3. K.II.42., Hungary
e-mail: tszabo@szt.bme.hu

1 Introduction

Our work is motivated by the classification system of convex, homogeneous 3D bodies introduced by [20]. They map each body into its *primary equilibrium class* defined by the *numbers* of the stable and unstable equilibrium points of the body surface. Moreover, the isomorphism classes of the *topologies* of the equilibria provides a refined, secondary classification system, where such a topology can be genuinely represented by a 2-coloured quadrangulation [7].

1.1 Generating multiquadrangulations

A *quadrangulation* of the sphere is a loopless graph embedded in the sphere having every face bounded by a closed walk of length 4. We allow parallel edges, and the boundary walk may repeat edges or vertices. This definition was also used by [14], however, [1] applied the word "pseudoquadrangulation" instead for multigraphs. If we want to emphasize that the quadrangulation may have parallel edges, it is called a *multiquadrangulation*, if a quadrangulation has no parallel edges, it is called a *simple quadrangulation*. Note that the 2-path P_2 (the path of length 2 with two edges and three vertices) is the smallest quadrangulation, and the 4-cycle C_4 (the cycle of length 4) is the smallest simple quadrangulation, illustrated on Fig. 5.

Two quadrangulations are considered isomorphic, if there is a homeomorphism from one to another that either preserves or reverses the orientation of the embedding. Equivalently, the *cyclic ordering* of the incident edges at each vertex is either preserved or reversed, so e.g. a graph is isomorphic to its reflection. Such an isomorphism class is also called an *unsensed, unrooted map* in the literature [22]. Let Q denote the family of all multiquadrangulations, and Q_1 the family of all simple quadrangulations.

We say a graph family $\mathcal{F}$ is *generated* from the starting set $K \subset \mathcal{F}$ by some given graph operations, if each graph in $\mathcal{F}$ can be constructed from some graph of K by applying a finite series of the given graph operations. All our graph operations are based on the vertex splitting, depicted on Fig. 1, explained as follows.

The embedding (the cyclic ordering) of the graphs showed is also important so the small open triangles denote that other
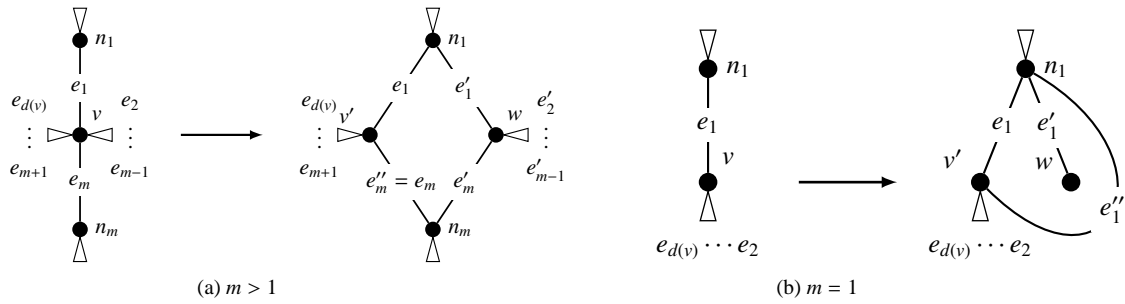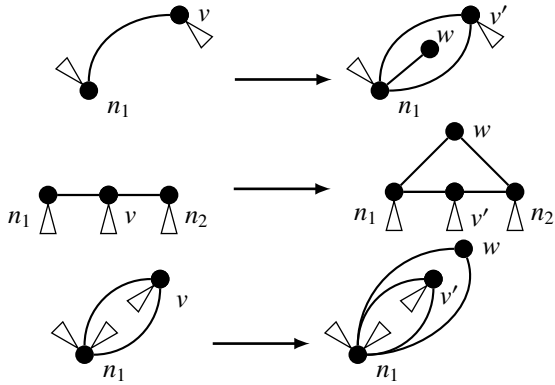
Fig. 1. Vertex splitting of degree m.

Fig. 2. The monotone vertex splittings $S_{1,2}$. The 2-splitting has two variants: simple case and parallel case.

edges *may* occur only at that position so they clarify the cyclic ordering of the edges in the original (left to the arrow) and in the resulting (right to the arrow) graph. Vertex splitting is also known in the literature as the inverse of face contraction.

[3, 18] showed that the splittings generate Q_1 from C_4. There are a number of related results regarding inductive generation of certain simple quadrangulation families, e.g. [3] gave inductive definition of 3-connected quadrangulations, [16] generated quadrangulations with minimum degree 3, [5] improved efficiency for all these families above. While [3, 17, 18] mainly focus on simple quadrangulations, the following observation is a straightforward extension of their results: the splitting generalized for parallel case as well ($m = 1$ on Fig. 1b.) generates Q from P_2.

As shown on Fig. 1, the splittings replace a vertex v with vertices w and v' dividing the edges of v. The *degree of a splitting* is $D := \min\{d(v'), d(w)\}$ where $d(v)$ denotes the degree of the vertex v. Note that $1 \le D \le \lfloor d(v)/2 \rfloor$ always holds. We consider *restricted splittings* $S_{i,j}$ with a limited range of D such that $1 \le i \le D \le j \le \lfloor d/2 \rfloor$. [5] showed that the restricted splittings $S_{2,3}$ (with $2 \le D \le 3$) are enough to generate Q_1 from C_4. Their result can be easily extended for parallel graphs using the generalized vertex splitting, so our previous observation can be improved as follows:

Theorem 1. *The restricted splittings $S_{1,3}$ generate Q from P_2.*

In this paper we investigate the hierarchy generated by restricting the splittings $S_{1,2}$ (see Fig. 2), and we also show some application of restrictions $S_{1,1}$, $S_{2,2}$ and $S_{3,3}$.
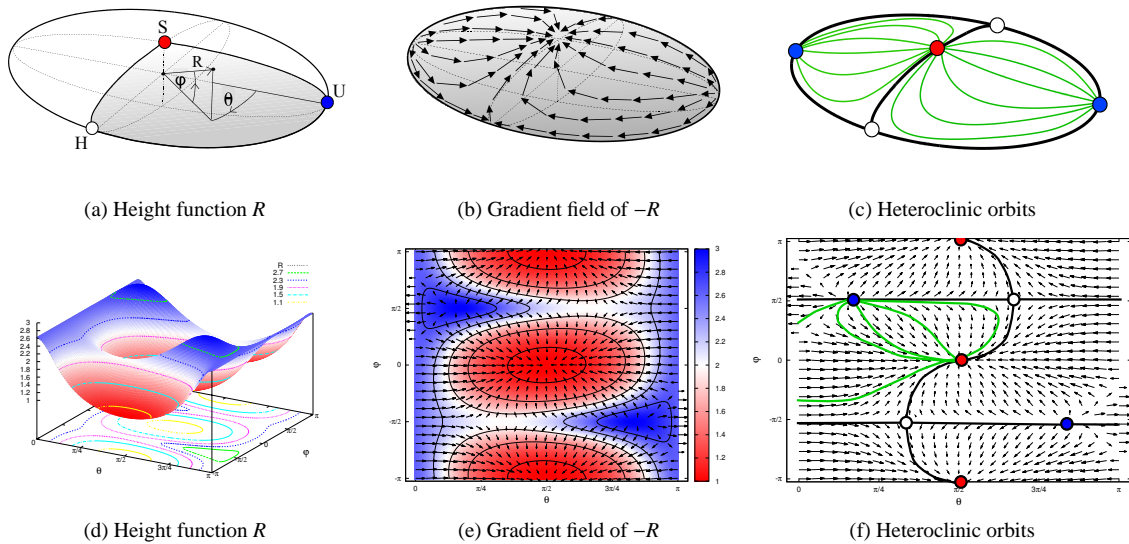
The main reason we focus on splittings $S_{1,2}$ is that these are not only local modifications but they purely extend the graph without removing any edge. Formally, we say a graph operation is *monotone* if the original graph is the embedded subgraph of the resulting one. Actually $S_{1,2}$ are the only monotone operations on quadrangulations introducing only one new vertex. The local monotone operations are also called *face subdivisions* in the literature meaning one face is divided into smaller regions (e.g. [14, 19]). [19] called the *order* of a face subdivision the number of the introduced vertices in the operation, although he applied this concept on triangulations. So on quadrangulations the monotone vertex splittings are exactly the face subdivisions of order 1.

For the splitting $S_{D,D}$ with a given D, we use the shorthand *D-splitting*, and we call its inverse *D-contraction*. If a D-contraction is applicable to a graph, we say the graph is *D-contractible*, otherwise we say the graph is *D-irreducible*. It is known that among the simple quadrangulations, C_4 is the only graph which is D-irreducible for any D [3, 18]. It follows from Theorem 1 that among the multiquadrangulations, only P_2 is D-irreducible for any D, hence P_2 is also referred to as the only ancestor of all quadrangulations. However, restricting the splittings to $S_{1,2}$ also admits other, nontrivial ancestors. Throughout this paper we say a graph is an *irreducible ancestor* (or shortly *irreducible*) if it is 1-irreducible and 2-irreducible. We mention that the concept of irreducible graphs with respect to some given graph operations was analogously used in e.g. [17, 18].

We characterise the irreducible graphs as the ones with minimum degree 3, and extend a result of [3] that the small graphs with less than 8 vertices are not irreducible, except P_2:

Theorem 2. *Every quadrangulation with less than 8 vertices is generated from P_2 by monotone splittings.*

We strengthen this theoretical result with observations on a data set generated by our computer program which we developed to explicitly enumerate every possible quadrangulation for a given size, based on the software `plantri` [6]. Furthermore, our data set also shows that there are only three irreducible graphs up to a size of 10 vertices. Besides P_2, the other ones are the radial graphs (a.k.a. vertex-face incidence graphs [15]) of the skeletons of the two smallest polyhedra. The *radial graph $R(G)$* of an embedded graph G is a bipartite embedded graph such that one of the independent vertex sets of $R(G)$ corresponds to

(a) Height function R (b) Gradient field of $-R$ (c) Heteroclinic orbits

(d) Height function R (e) Gradient field of $-R$ (f) Heteroclinic orbits

Fig. 3. Morse–Smale graph of the ellipsoid. The axes of the ellipsoid are rotated for presentation purposes. Colouring notation: red: stable, blue: unstable, white: saddle.

the vertex set of G, the other one to the face set of G. Two vertices are connected in $R(G)$ with the same multiplicity as the incidence multiplicity of their preimages in G (i.e. the appearance count of a vertex in the boundary walk of a face), with the obvious cyclic orderings. In addition, we prove

Theorem 3. *The radial graph of any polyhedral skeleton is irreducible, and they are generated from the radial graphs of the pyramids with polygonal base by the restricted splitting $S_{3,3}$.*

Then we prove that the irreducible ancestor of any graph is unique, i.e. a graph cannot be generated starting from two different irreducible graphs:

Theorem 4. *If a quadrangulation can be generated from both irreducible ancestors A_1 and A_2 by monotone splittings, then A_1 is isomorphic to A_2.*

1.2 Generating secondary equilibrium classes

The main goal of this paper is twofold: one is to contribute some observations on the reachability of quadrangulations i.e. which quadrangulations can be generated from others, the other is to apply them on the geometric interpretation mentioned earlier. So finally, we translate our observations into the context of topology of the equilibrium points of convex bodies. A generic convex body is given by its scalar height function $R(\theta, \varphi)$ which gives the distance between the surface and the mass center for any direction. A surface point is an *equilibrium* if the gradient of the height function R is zero at that point (illustrated on the ellipsoid on Fig. 3). An equilibrium is *stable* if R takes a local minimum value, *unstable* if R takes a local maximum value, otherwise it is a *saddle point*. According to the Poincaré–Hopf theorem [2], the cardinality of stable, unstable and saddle equilibria of a body, denoted respectively by s, u, h, are deeply related: $s + u - h = 2$, thus the pair $\{s, u\}$ defines unambiguously

the *primary equilibrium class* of the body [20]. The primary equilibrium classes can be refined into *secondary equilibrium classes* which determine also the topology of the equilibria defined by the Morse–Smale complex of the height function R [7]. This topology is usually given by a 3-coloured spherical quadrangulation, where the vertices are the equilibria, the colour of a vertex defines its equilibrium type (stable, unstable or saddle), and an edge is a particular path on the surface connecting two equilibria, whose tangent vectors agree with the gradient vectors of R (Fig. 3). We transform these 3-coloured surface graphs with a bijection to generic 2-coloured quadrangulations keeping the underlying geometric meaning, and we use this latter form to define the secondary class as a fully combinatorial object. We mention that a spherical quadrangulation is always bipartite [1, 12] hence the 2-colouring is possible.

The data set yielded by our program enables us to present some statistics on the number of possible secondary classes as well. Because of the 1-1 correspondence of 2-coloured quadrangulations and connected plane graphs [5, 11], our numbers agree with the numbers of the unrooted and unsensed maps calculated by [24], and exhaustively enumerated by [21, 23]. [22] surveys different types of map census results achieved both exhaustive search and with formulae. We are also able to give statistics on the number of multiquadrangulations using a trivial relation between the number of maps, quadrangulations and self-dual quadrangulations for a fixed size.

[20] constructed the geometry of a representative body of the primary class $\{1, 1\}$ referred as mono-monostatic body, also known as *Gömböc*. They also concluded that every primary class for all $s, u \geq 1$ is generated from the Gömböc via their specific geometric transformations called *Columbus' algorithm*. Columbus' algorithm is a sequence of sensitive modifications on a body perturbing the surface only at the vicinity of an equilib-

(a) Cutting the surface with a plane near U_0

(b) Modified gradient field of $-R$
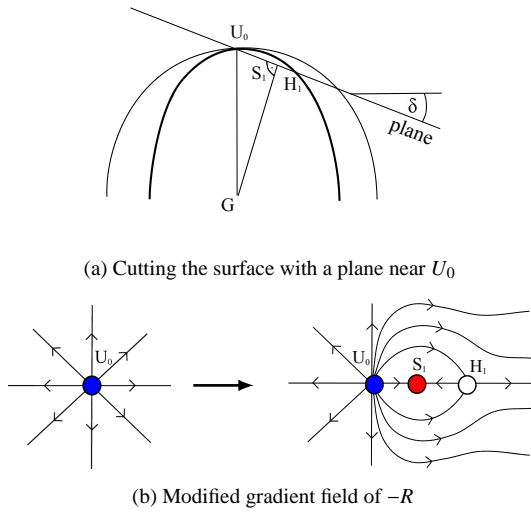
Fig. 4. Applying a step of Columbus' algorithm on an egg-shaped surface. Two equilibria appear on the surface: S_1, H_1.

rium, such that the body belonging to the primary class $\{s, u\}$ is transformed to another one belonging to $\{s + 1, u\}$ or $\{s, u + 1\}$ (illustrated on Fig. 4). Because of the geometric feasibility of Columbus' algorithm, i.e. the transformations always can be applied around any equilibrium, they referred to the Gömböc as the ancestor of every primary class.

The combinatorial equivalent of the original Columbus' algorithm is the sequence of monotone coloured splittings. The result of [20] can be reformulated as follows: for any s, u, some secondary class in the primary class $\{s, u\}$ is generated from the Gömböc using the monotone coloured splittings. (We detail the secondary class of Gömböc in Section 4.) We extend this statement with the following corollaries of our theorems:

Corollary 1. *Not every secondary class is generated from the Gömböc using the monotone coloured splittings, but the ones with $s + u < 8$ are all generated from the Gömböc.*

We say a polyhedron is a *minimal polyhedron* if its every face contains one stable and its every vertex is an unstable equilibrium. With respect to the monotone coloured splittings (and the original Columbus' algorithm), they are also ancestors:

Corollary 2. *The secondary classes of the minimal polyhedra are irreducible so not generated from any other secondary class.*

Finally, if we aim to generate a secondary class from every primary class, it is enough to use one of the two monotone coloured splittings:

Corollary 3. *It is enough to use either the coloured splitting $S_{1,1}$ or $S_{2,2}$ to generate every primary class from a finite starting set of secondary classes.*

We mention that the coloured splittings are also a restricted subset of the operations called "cancellations" in computational geometry by [10] and [4, Fig. 7.3], who used them to simplify a multi-resolution mesh structure. Domokos, Lángi and Szabó

[7] showed recently that the geometric counterparts of the unrestricted coloured splittings generate the whole family of secondary classes, however, this is beyond the scope of this paper.

1.3 Organization of this paper

The paper is organized as follows. Section 2 analyzes the vertex splitting and establishes its properties on the reachability of quadrangulations. Section 3 presents the hierarchy of irreducible quadrangulations: the reachability of small graphs, the polyhedral irreducible quadrangulations and the uniqueness of ancestors. Section 4 interprets our results on the equilibrium topologies of convex bodies. Finally we show some statistics on our data set in Section 5.

2 The properties of vertex splitting

In order to generalize the splitting for parallel case (with the notations illustrated on Fig. 1), let $\sigma(v)$ denote the cyclic ordering of the vertex v in clock-wise order, so $\sigma(v) = (e_1, \ldots, e_{d(v)})$. Let n_i denote the other endpoint of the edge e_i.

Definition 1. A *vertex splitting* on a quadrangulation G is a graph operation transforming G to G', specified by a walk $n_1 e_1 v e_m n_m$ of G. The vertex v of G is replaced by vertices w and v' in G' dividing the edges of v. The cyclic ordering of w is $\sigma(w) = (e'_1, \ldots, e'_m)$, where e'_i has the same other endpoint as e_i had; the cyclic ordering of v' is $\sigma(v') = (e_1, e''_m, e_{m+1}, \ldots, e_{d(v)})$, keeping some edges of the former vertex v, and $e''_m = e_m$ if $m > 1$, otherwise e''_m has the same other endpoint as e'_m and e_m had.

Note that if $m > 1$ and n_1 is identical to n_m then e_1 and e_m are distinct parallel edges. The new face introduced by the splitting is bounded by the walk $v' e_1 n_1 e'_1 w e'_m n_m e''_m v'$. Observe that the degree of the new vertices are $d(w) = m$ and $d(v') = d(v) - m + 2$. In addition, the splitting specified by the walk $n_1 e_1 v e_m n_m$ creates the very same graph that the one specified by the walk $n_m e_m v e_1 n_1$. Hence the *degree of a splitting*, denoted by D, is invariant to reflection, and $D := \min\{d(v'), d(w)\} = \min\{m, d(v) - m + 2\}$.

Let n denote the number of vertices of the quadrangulation. There are exactly four quadrangulations such that $n \le 4$, shown on Fig. 5, where C_4 is generated from P_2 with a 2-splitting, and Q_3, Q_4 are generated from P_2 with a 1-splitting. For larger graphs, we have

Proposition 1. *For a quadrangulation such that $n > 4$,*

 (i) *if it has a vertex of degree 1, then it is 1-contractible even for $n = 4$,*

 (ii) *if it has a vertex of degree 2, then it is 2-contractible,*

 (iii) *if its minimum degree is k, then it is k-contractible, and not l-contractible for any $l < k$,*

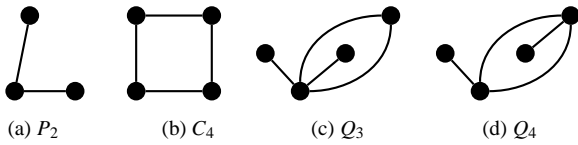 (iv) *it is irreducible if and only if its minimum degree is 3.*

(a) P_2 (b) C_4 (c) Q_3 (d) Q_4

Fig. 5. The four smallest quadrangulations with $n \le 4$.
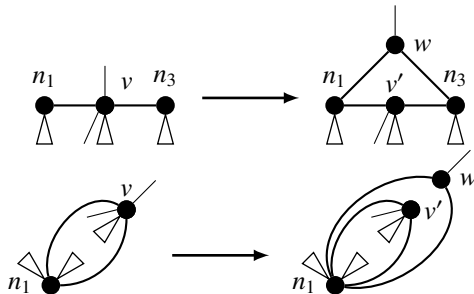

Fig. 6. 3-splitting. The half edge at v denotes that there must be an edge otherwise it would be a 2-splitting.

Proof. Part (i) is a straightforward consequence of the quadrangulated faces.

For part (ii), suppose a vertex w is incident to two edges going to n_1 and n_2. If n_1 and n_2 are distinct vertices, then the graph is 2-contractible as depicted on Fig. 2. If $n_1 = n_2$ and n_1 was not incident to any other parallel edge, the graph would be isomorphic to Q_3 (see Fig. 5). If there are other parallel edges incident to n_1, then the graph is 2-contractible, the parallel case is applicable (see Fig. 2).

Part (iii) follows from the definition of the degree of splitting and from the observation that a k-splitting introduces a vertex of degree k.

Part (iv) follows from (i)-(iii) as the monotone splittings introduce a vertex of degree 1 or 2. $\square$

Proposition 1 provides the necessary observations to prove that the splittings $S_{1,3}$ generate $\mathcal{Q}$.

Proof of Theorem 1. The statements follows from Proposition 1 and from the fact that the minimum degree of a multiquadrangulation is either 1, 2 or 3 by Euler's formula. $\square$

It is clear that the 1-splitting and the 2-splitting are monotone, and modify only one face of the graph in the vicinity of a point. Hence the perturbation of the graph is minimal. Monotonicity is also sufficient to characterize these splittings:

Proposition 2. *Every plane graph operation defined on quadrangulations which adds one vertex and is monotone, is a restricted splitting $S_{1,2}$.*

Proof. Consider any operation satisfying the conditions, putting the new vertex w into an original quadrilateral face bounded by the walk $x_1 x_2 x_3 x_4 x_1$, listing only the vertices of the walk. Note that some of the vertices x_i may coincide. By Euler's formula, we need to add two edges in addition to the new vertex, so connect, without loss of generality, w and x_1 with an edge. Then, we have an "almost quadrangulation": except for one face which is bounded by the walk $x_1 w x_1 x_2 x_3 x_4 x_1$ of length 6. This face can

be divided into two quadrilateral faces by adding one edge in three ways: connecting x_1 and x_2 (the fourth and the seventh elements of the walk sequence), or x_1 and x_4 (third and sixth elements), or w and x_3 (second and fifth) with an edge. The first and the second case is applying a 1-splitting, the third case is applying a 2-splitting. $\square$

3 Irreducible quadrangulations

As being irreducible for $n > 3$ is equivalent to having minimum degree of 3, one can easily verify that e.g. the radial graph of the skeleton of the tetrahedron is irreducible (see Fig. 7(a)). According to Theorem 2, it is the smallest irreducible quadrangulation after P_2. In preparation of proving it, we need to generalize slightly an earlier result:

Proposition 3 ([3]). *Every simple 3-connected quadrangulation has at least 8 vertices of degree 3.*

The generalization of Batagelj's proof for our need is straightforward as his statement still holds for multigraphs and instead of 3-connectivity it is enough to assume the minimum degree is 3. The latter observation has already been made in [5]. Hence we use the following statement:

Proposition 4. *Every quadrangulation with minimum degree 3 has at least 8 vertices of degree 3.*

Proof. The layout of this proof is similar to Batagelj's proof, but the conditions are relaxed. From Euler's formula we have that the sum of the degrees of a quadrangulation of size $n := s + u$ is $4n - 8$. If it has k vertices of degree 3 and $n - k$ other vertices of degree at least 4, then the sum of the degrees is also at least $3k + 4(n - k)$, implying $k \ge 8$. $\square$

Proof of Theorem 2. By Proposition 1 and Proposition 4, we have that the only irreducible ancestor such that $n < 8$ is P_2. So the theorem can be proved by induction. $\square$

The data set generated by our program has strengthened Theorem 2, as we could count easily the irreducible ones for $n \le 10$ by Proposition 1. The statistics showed that there are exactly three irreducible graphs among them: P_2 and the radial graphs of the two polyhedral skeletons shown on Fig. 7. Theorem 3 states that the radial graphs of polyhedral skeletons are all irreducible. Moreover, it gives an inductive definition for them, as they cannot be generated by monotone splittings.

Proof of Theorem 3. Any polyhedral skeleton is simple and 3-connected by Steinitz' theorem [15]. Let $\mathcal{Q}_4$ denote the family of simple and 3-connected quadrangulations having no separating 4-cycles. It is known [5] that a graph is simple and 3-connected if and only if its radial graph belongs to $\mathcal{Q}_4$, proving the first part of the theorem.

It was also shown [5] that $\mathcal{Q}_4$ is generated by the restricted splitting $S_{3,3}$ starting from the pseudo-double wheels. A *pseudo-double wheel* is a cycle of even length, with its inner and outer face subdivided by a vertex, such that the inner vertex is adjacent
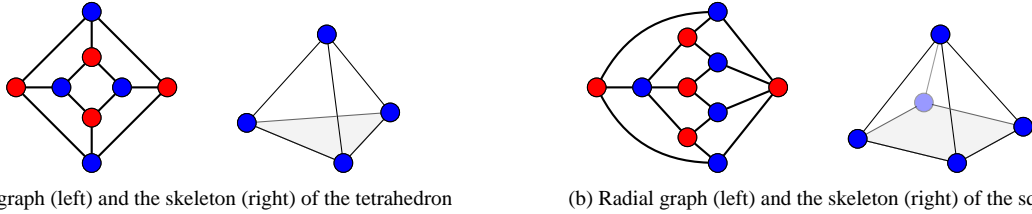
(a) Radial graph (left) and the skeleton (right) of the tetrahedron

(b) Radial graph (left) and the skeleton (right) of the square pyramid

Fig. 7. Irreducible quadrangulations.

to the odd-numbered vertices of the cycle, the outer vertex is adjacent to the even-numbered vertices of the cycle [6]. It is easy to verify that a pseudo-double wheel of size $n = 2k + 2$ is the radial graph of the skeleton of a pyramid with a k-sided base which completes the proof. $\qquad\square$

Now we show that the irreducible ancestor of a quadrangulation is unambiguously defined, so a quadrangulation cannot be generated from different irreducible ancestors at the same time. Consequently, for any two different irreducible quadrangulations $A_1 \neq A_2$ the families generated starting from $\{A_1\}$ and from $\{A_2\}$ are disjoint, thus the set of irreducible quadrangulations leads to a natural partition of Q.

Proof of Theorem 4. Without loss of generality, suppose A_1 is not isomorphic to P_2 thus the minimum degree of A_1 is 3. Let $G_0, G_1, \ldots, G_p$ be the series of graphs yielded by the generation process of G from A_1 where $G_0 = A_1$ and $G_p = G$, and, indexed in reverse order, $G_{p+q}, \ldots, G_{p+1}, G_p$ the graphs yielded by the generation process from A_2 where $G_{p+q} = A_2$, for some $p, q > 0$. We will prove that the edges of G_0 remain intact despite applying any monotone vertex splitting or monotone face contraction, i.e. G_0 is the embedded subgraph of every G_k for all $0 \leq k \leq p + q$, so A_1 is the subgraph of A_2, and vice versa.

We say an edge of a graph G_i is *ancient* if it is also in G_0. By induction, assume that G_0 is the embedded subgraph of the graphs $G_0, \ldots, G_{k-1}$ for some k, which trivially holds for $k = 1$. If G_k is created from G_{k-1} by a monotone splitting, clearly no edge is removed due to the definition of monotonicity. If G_k is created from G_{k-1} by a 2-contraction, then we use a basic property of the 2-contraction that if it removes an edge, then one endpoint of the edge is also removed and the degree of the removed endpoint is 2. However, according to the induction hypothesis, each ancient edge of G_{k-1} has endpoints with degree at least 3 thus no 2-contraction is applicable to G_{k-1} to remove an ancient edge.

The last case is where G_k is created from G_{k-1} by a 1-contraction, removing two edges: e_1' and e_1'' (see the definition of vertex splitting and Fig. 1b.). Edge e_1' cannot be ancient because of having and endpoint of degree 1 as explained in the previous case. If edge e_1'' is ancient and e_1 is not, then again no ancient edge is removed as it is only a technicality if we actually remove the edge e_1'' or e_1 because of the symmetry of the 1-contraction. If both edges e_1'' and e_1 would be ancient and e_1' would not be, then, because of the induction hypothesis, there would be a face

in G_0 bounded by the walk $v'e_1 n_1 e_1'' v'$ contradicting the assumption that G_0 is a quadrangulation. Thus a 1-contraction cannot remove an ancient edge either.

Consequently, as neither the splittings nor the contractions alter the embedding of the rest of the graph, G_0 remains the embedded subgraph of G_k for all k. $\qquad\square$

4 Generating secondary equilibrium classes

In this section we interpret our results on generating secondary equilibrium classes as coloured surface graphs of convex bodies. Some of these observations were already outlined in the conference version of this paper [13].

The idea of representing a 3D body with some surface graph appears in multiple disciplines. Most famous are the polyhedral graphs that are the skeletons of the convex polyhedra, characterized by Steinitz's theorem [15]. Another related appearance is the mesh generation of physical shapes in computational geometry, when some Morse–Smale complex of a body is drawn as a surface graph, see e.g. [8]. However, the underlying function used for meshing is not necessarily the height function R, thus the nodes of the mesh and the equilibrium points do not necessarily coincide.

4.1 Geometric interpretation of quadrangulations and vertex splittings

To determine the secondary class of a body, we need to introduce some concepts from Morse theory [2, 10], illustrated on Fig. 3. In generic case, we say a path on the surface is a *heteroclinic orbit*, if its tangent vectors agree with the gradient vectors of the height function R, and its endpoints are two equilibria of different type (see Fig. 3(b) and Fig. 3(f)). It is known the heteroclinic orbits incident to a saddle point are isolated on the surface, and there are only a finite number of them.

These isolated orbits divide the body surface into quadrilateral cells (see Fig. 8(a)), and in each cell an infinite number of non-isolated, heteroclinic orbits are going from the unstable to the stable point which we disregard for now. In this way, the body surface defines a vertex-coloured multigraph embedded on the sphere, where the vertices are the equilibria, the edges are the isolated heteroclinic orbits connecting saddle and non-saddle vertices, the faces are the quadrilateral cells, and the colour of a vertex gives its type of equilibrium.

This graph is referred as the *Morse–Smale graph*, which means possessing three properties [2, 10]. (1) The graph is a *quadrangulation* of the plane unless the body is in the primary
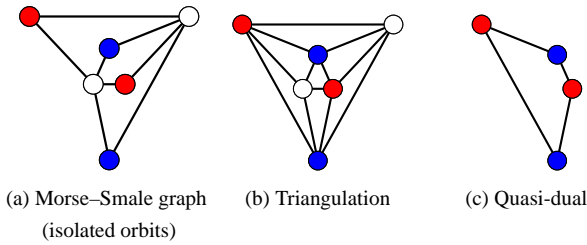
(a) Morse–Smale graph   (b) Triangulation   (c) Quasi-dual
(isolated orbits)

Fig. 8. Determining the secondary class of the ellipsoid.
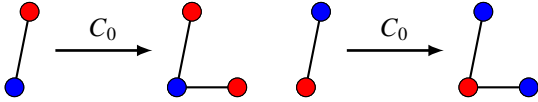

Fig. 9. Columbus' algorithm: auxiliary coloured splitting C_0 defined only on the quasi-dual P_1 of the Gömböc.

class of the Gömböc. (2) Every quadrilateral boundary walk is a sequence of a saddle, a stable, a saddle and an unstable vertex. (3) The degree of every saddle vertex is 4.

We perform two invertible transformations on the Morse–Smale graph, to yield a more compact graph which is still a genuine representation of the geometry. First, we connect the stable and unstable points in each quadrangulated face, creating a triangulation (see Fig. 8(b)). The geometric interpretation of this operation could be that from each cell, we pick one orbit arbitrarily from the infinite set of unstable-stable orbits. Then we remove the saddle points and the edges incident to them, creating another quadrangulation leaving only the edges created in the previous step (see Fig. 8(c)). This idea also appears in [8] as the *quasi-dual* of the Morse–Smale complex, however, as their goal is to simplify of the mesh structure, they do not mention that it can be shown that these transformations are invertible, so we lost no information by removing the saddles. We use the resulting quasi-dual graph to define the secondary class of the body, summarized in the following

Definition 2. A *quasi-dual* is a 2-coloured multiquadrangulation, where the independent sets are called the stable and the unstable equilibria, of size s and u, respectively, if $s + u > 2$. A *quasi-dual of a body* is the quasi-dual obtained with the process described above. For technicality, in case $s + u = 2$, we define the quasi-dual of the Gömböc as the path P_1 of length 1 (with one edge) connecting a stable and an unstable equilibrium (see Fig. 9). Obviously $n = s + u$. A *secondary equilibrium class* is an isomorphism class of quasi-duals, where the isomorphism is expected to preserve the colouring as well.

While the derivation process above requires the existence of the gradient field of the height function, we would also like to consider some reasonable body surfaces with no gradient, e.g. a polyhedra. As the construction of the Morse–Smale complex is already extended for some non-smooth functions as well [10], we believe the definition of the quasi-dual and the secondary class of a polyhedron could be understood by intuition and needs no rigorous theoretical background.

If a quadrangulation admits multiple non-isomorphic colourings, then its possible secondary classes form a partition of its map. It is easy to verify that a quadrangulation admits either two non-isomorphic colourings, i.e. switching the colours results in another secondary class, or only one colouring. In the latter case we call the secondary class a *self-dual*.

Assigning a quasi-dual to a convex body is a generalization of assigning a radial graph to a polyhedral skeleton. Radial graphs are all quadrangulations, and every quadrangulation is the radial graph of some spherical surface graph [5, 15]. Moreover, assigning to a generic connected plane graph G its 2-coloured radial graph $R(G)$ and colouring the images of the vertices of G "unstable" in $R(G)$, and the images of the faces of G "stable" defines a bijection [11]. For a fixed s, u, the bijection is between the set of secondary classes in the primary class $\{s, u\}$ and the set of maps of generic connected plane graphs with u vertices, s faces and $h = s + u - 2$ edges. E.g. in the case of the *minimal polyhedra*, the quasi-dual is actually the coloured radial graph of the skeleton, thus the Poincaré–Hopf theorem can be replaced by Euler's formula.

The *steps of Columbus' algorithm* are defined and their geometric feasibility were proved in [20], where feasibility means they transform any convex body to another convex one. Observing the change of the quasi-dual of the underlying body, the steps of Columbus' algorithm are combinatorial operators on quasi-duals we call the *coloured splittings*. A coloured splitting of a quasi-dual consists of a vertex splitting of the underlying quadrangulation followed by the proper colouring of the introduced vertex. We add that the coloured splittings corresponding to the original steps of Columbus' algorithm are all monotone. So each coloured splitting corresponds to two dual versions of a splitting of the quadrangulation: to one adding a new stable vertex, and to one adding a new unstable vertex.

For technicality, we add an auxiliary coloured splitting C_0 applicable only on the quasi-dual P_1 of the Gömböc, which is not literally a vertex splitting of a quadrangulation. Applying C_0 either results in the quasi-dual in the class $\{1, 2\}$, or in the quasi-dual in the class $\{2, 1\}$, see Fig. 9. So the definition of coloured splitting is relaxed to allow C_0 as well, considered a 1-splitting (i.e. $S_{1,1}$).

4.2 Generating secondary classes by monotone coloured splittings

While the monotone coloured splittings generate every primary class from P_1, they do not generate every secondary class, so they admit a nontrivial family of irreducible ancestors. We say a secondary class is *irreducible* if it cannot be created with a monotone coloured splitting. In other words its underlying quadrangulation with $n > 3$ is irreducible, or $n = 2$. So the secondary classes in $\{1, 2\}, \{2, 1\}$ are not considered irreducible (see Fig. 9), but P_1 in $\{1, 1\}$ is. By Proposition 1, we can characterise the irreducible secondary classes for $n > 3$ as bodies with a quasi-dual of minimum degree 3.

Fig. 10. Hierarchy of secondary classes. Notation: $\mathcal{I}^+$: all secondary classes, $\mathcal{G}^+$: generated from P_1, $\mathcal{F}$: small secondary classes s.t. $s + u < 8$, $\mathcal{I}$: irreducible ancestors, $\mathcal{M}$: minimal polyhedra, $\mathcal{M}^+$: generated from minimal polyhedra.
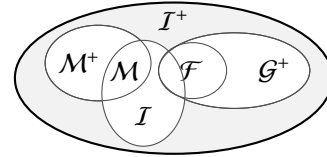
Tab. 1. Cardinalities $e(s, u)$ of the equilibrium classes.

|	$s = 1$	$s = 2$	$s = 3$	$s = 4$	$s = 5$	$s = 6$	$s = 7$	$s = 8$	$s = 9$
$u = 1$	1	1	1	2	3	6	12	27	65
$u = 2$	1	2	5	13	35	104	315	1021	
$u = 3$	1	5	20	83	340	1401	5809		
$u = 4$	2	13	83	504	2843	15578			
$u = 5$	3	35	340	2843	21420				
$u = 6$	6	104	1401	15578					
$u = 7$	12	315	5809						
$u = 8$	27	1021							
$u = 9$	65								

The results of this subsection are outlined in Fig. 10 which we reveal gradually. Let $\mathcal{G}$ denote the starting set consisting of the secondary class of the Gömböc, i.e. $\mathcal{G} = \{P_1\}$, and $\mathcal{G}^+$ the family generated from P_1 by monotone coloured splittings. Throughout this section, for any starting set of secondary classes X, let X^+ denote the family of secondary classes generated from X by the monotone coloured splittings. So let $\mathcal{I}$ denote the family of irreducible secondary classes, then $\mathcal{I}^+$ denotes the family of all possible secondary classes.

The first part of Corollary 1 states that $\mathcal{I}$ contains secondary classes besides P_1, which clearly follows from the fact that there is an infinite set of non-isomorphic irreducible quadrangulations. For the second part, let $\mathcal{F}$ denote the family of secondary classes in the primary classes $\{s, u\}$ such that $s + u < 8$. The second part states that $\mathcal{F} \subset \mathcal{G}^+$, which follows from Theorem 2 with the help of the auxiliary splitting C_0.

Note that $\mathcal{G}^+ \cap \mathcal{I}$ contains only P_1 by definition. Corollary 2 gives a geometric characterisation of a special family within $\mathcal{I}$ called the minimal polyhedra. Recall that a polyhedron is a *minimal polyhedron* if its every face contains one stable and its every vertex is an unstable equilibrium, and let $\mathcal{M}$ denote the family of their secondary classes. E.g. the Platonic solids or a right prism are all minimal polyhedra, however, an oblique prism may not be a minimal polyhedron: there may be a face which does not contain a stable point. Corollary 2 states that $\mathcal{M} \subset \mathcal{I}$ (see Fig. 10). As there are only three irreducible secondary classes until $s + u \leq 10$ according to our data set, we can safely say that every secondary class such that $s + u \leq 10$ are generated either from the Gömböc or from a minimal polyhedron, i.e. $\{S \in \mathcal{I}^+ : s + u \leq 10\} \subset \mathcal{G}^+ \cup \mathcal{M}^+$.

Nevertheless, there are other irreducible secondary classes as well, e.g. Fig. 11 shows a method to enumerate some of them with parallel edges. Clearly a minimal polyhedron cannot have parallel edges.

Finally, Theorem 4 states that the irreducible ancestor of a secondary class is unique, because the quadrangulations of the irreducible secondary classes and the irreducible quadrangula-
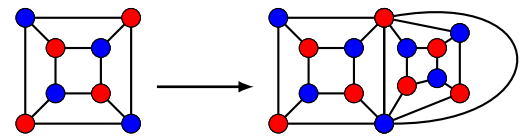
Fig. 11. Irreducible secondary class with parallel edges created by reflecting a simple one.

tions coincide for $n > 3$. Consequently, e.g. $\mathcal{G}^+$ and $\mathcal{M}^+$ are disjoint. Theorem 4 also results in an algorithm to determine the irreducible ancestor of a body: we simply need to apply monotone contractions iteratively to its quasi-dual, until we get the irreducible ancestor. This algorithm is trivial because we can choose the contractions arbitrarily: the resulting irreducible ancestor is unique hence is independent of the actual choices, as any contraction sequence would result in the same ancestor.

4.3 Generating primary classes by restricted splittings

The restricted coloured splittings $S_{1,2}$ (i.e. the monotone ones) can generate every primary class from the Gömböc [20], but not every secondary class by Corollary 1. We consider further restrictions in this subsection summarized by Corollary 3. In detail, the coloured splitting $S_{1,1}$ generates every primary class from P_1, and the coloured splitting $S_{2,2}$ generates every primary class from the starting set of the secondary classes of the singleton primary classes $\{\{1,1\}, \{2,1\}, \{3,1\}, \{1,2\}, \{1,3\}\}$. Note that according to Table 1 in Section 5 these are the only singleton primary classes.

Proof of Corollary 3. First we consider restricting the monotone coloured splittings to $S_{1,1}$. As 1-splitting is always applicable at any part of the graph if $n > 2$, any primary class can be reached with coloured 1-splittings from $\{1, 2\}$ or from $\{2, 1\}$. So the primary classes are generated from the starting set $\{\{1,1\}, \{1,2\}, \{2,1\}\}$, and this set can be generated from P_1 by C_0.

Now we consider restricting the monotone coloured splittings to $S_{2,2}$. The quadrangulations of the classes $\{2, 1\}, \{3, 1\}, \{1, 2\}, \{1, 3\}$ are 2-irreducible (see Fig. 5), so they has to be in the start-

Tab. 2. The number of multiquadrangulations (q), self-dual secondary classes (e_{SD}) and secondary classes ($\sum e$).

	$q(n)$	$e_{SD}(n)$	$\sum_s e(s, n-s)$
$n = 3$	1	-	2
$n = 4$	3	2	4
$n = 5$	7	-	14
$n = 6$	30	8	52
$n = 7$	124	-	248
$n = 8$	733	50	1416
$n = 9$	4586	-	9172
$n = 10$	33373	380	66366

Tab. 3. Ancestor secondary classes with respect to $S_{1,1}$ (third column), $S_{2,2}$ (second column), $S_{1,2}$ (fourth column) and non-ancestors (first column).

	1-2-contractible	2-irreducible	1-irreducible	irreducible
$n = 4$	0	3	1	0
$n = 5$	6	2	6	0
$n = 6$	32	4	16	0
$n = 7$	172	10	66	0
$n = 8$	1071	33	311	1
$n = 9$	7370	114	1688	0
$n = 10$	55766	474	10125	1

ing set. The 2-splitting is always applicable around a vertex v with $d(v) \geq 2$ introducing a vertex w of the same colour as v. It is clear that if $n > 3$, both independent sets of a bipartite quadrangulation contains vertices of degree at least 2. As C_4 in $\{2, 2\}$ is 2-contractible, the 2-splittings can be arbitrarily combined to generate some secondary class in any primary class $\{s, u\}$ such that $s + u > 4$. $\square$

In the next section we show statistics on the number of 1-irreducible and 2-irreducible secondary classes for a limited size in Table 3.

5 Computational results

This section presents some statistics on the data set attained by the computer program. As `plantri` supports dividing the computation into independent parts, we could perform this computation in parallel in a grid infrastructure using the Saleve framework [9]. Some of the numbers have been already presented in [13].

Table 1 shows the cardinalities of the classes $\{s, u\}$ with $s + u \leq 10$. The table is symmetric because a secondary class such that $s \neq u$ is clearly not self-dual. These numbers were already published as the cardinalities of the *unrooted and unsensed maps* by [21, 23, 24].

Table 2 shows the number of multiquadrangulations $q(n)$, which is the first publication of these numbers, up to our best knowledge. To compute $q(n)$ from the same program-generated data set, observe a relation between the number of secondary classes in the class $\{s, u\}$, the number of self-dual secondary classes and the number of quadrangulations of size $n = s + u$

and, denoted respectively by $e(s, u)$, $e_{SD}(s, u)$, $q(n)$:

$$2q(n) - e_{SD}(n/2, n/2) = \sum_{s=1}^{n-1} e(s, n-s). \qquad (1)$$

Table 3 shows the number of ancestor secondary classes with respect to splittings with different restriction criteria: $S_{2,2}$, $S_{1,1}$ and $S_{1,2}$.

References

1 **Archdeacon D, Hutchinson J, Nakamoto A, Negam S, Ota K**, *Chromatic numbers of quadrangulations on closed surfaces*, Journal of Graph Theory, **37**(2), (2001), 100–114, DOI 10.1002/jgt.1005.

2 **Arnold VI**, *Ordinary Differential Equations*, The MIT Press, 1978.

3 **Batagelj V**, *An inductive definition of the class of 3-connected quadrangulations of the plane*, Discrete Mathematics, **78**(1-2), (1989), 45–53, DOI 10.1016/0012-365X(89)90159-3.

4 **Bremer P-T**, *Topology-based Multi-resolution Hierarchies*, University of California, 2004.

5 **Brinkmann G, Greenberg S, Greenhill C, McKay BD, Thomas R, Wollan P**, *Generation of simple quadrangulations of the sphere*, Discrete Mathematics, **305**(1-3), (2005), 33–54, DOI 10.1016/j.disc.2005.10.005.

6 **Brinkmann G, McKay BD**, *Fast generation of planar graphs*, MATCH Communications in Mathematical and in Computer Chemistry, **58**(2), (2007), 323–357.

7 **Domokos G, Lángi Zs, Szabó T**, *The genealogy of convex solids*, (2012), http://arxiv.org/abs/1204.5494v1. unpublished.

8 **Dong S, Bremer P-T, Garland M, Pascucci V, Hart JC**, *Spectral surface quadrangulation*, ACM Transactions on Graphics, **25**(3), (2006), 1057–1066, DOI 10.1145/1141911.1141993.

9 **Dóbé P, Kápolnai R, Sipos A, Szeberényi I**, *Applying the Improved Saleve Framework for Modeling Abrasion of Pebbles*, In: Large-Scale Scientific Computing, Lecture Notes in Computer Science, Vol. 5910, 2010, pp. 467–474, DOI 10.1007/978-3-642-12535-5_55.

10 **Edelsbrunner H, Harer J, Zomorodian A**, *Hierarchical Morse complexes for piecewise linear 2-manifolds*, In: Proceedings of the seventeenth annual symposium on Computational geometry - SCG '01, 2001, pp. 70–79, DOI 10.1145/378583.378626.

11 **Fusy E**, *Counting unrooted maps using tree-decomposition*, Seminaire Lotharingien de Combinatoire, **B54A1**, (2007), 44.

12 **Gross JL, Yellen J**, *Handbook of graph theory*, Discrete mathematics and its applications, CRC Press, 2004.

13 **Kápolnai R, Domokos G**, *Inductive generation of convex bodies*, In: The 7th Hungarian-Japanese Symposium on Discrete Mathematics and Its Applications, 2011, pp. 170–178.

14 **Mohar B, Simonyi G, Tardos G**, *Local chromatic number of quadrangulations of surfaces*, (2010), http://arxiv.org/abs/1010.0133v1. Unpublished.

15 **Mohar B, Thomassen C**, *Graphs on Surfaces*, Johns Hopkins University Press; Baltimore, 2001.

16 **Nakamoto A**, *Generating quadrangulations of surfaces with minimum degree at least 3*, Journal of Graph Theory, **30**(3), (1999), 223–234, DOI 10.1002/(SICI)1097-0118(199903)30:3<223::AID-JGT7>3.0.CO;2-M.

17 **Nakamoto A**, *Diagonal transformations in quadrangulations of surfaces*, Journal of Graph Theory, **21**(3), (1996), 289–299, DOI 10.1002/(SICI)1097-0118(199603)21:3<289::AID-JGT3>3.0.CO;2-M.

18 **Negami S, Nakamoto A**, *Diagonal transformations of graphs on closed surfaces*, Science Reports of the Yokohama National University. Section I. Mathematics, Physics, Chemistry, **40**, (1993), 71–97, http://hdl.handle.net/10131/3593.

19 **Tutte WT**, *The non-biplanar character of the complete* 9-*graph*, Canadian Mathematical Bulletin, **6**, (1963), 319–330, DOI 10.4153/CMB-1963-026-x.

20 **Várkonyi P, Domokos G**, *Static Equilibria of Rigid Bodies: Dice, Pebbles, and the Poincare-Hopf Theorem*, Journal of Nonlinear Science, **16**(3), (2006), 255–281, DOI 10.1007/s00332-005-0691-8.

21 **Walsh TR**, *Generating nonisomorphic maps and hypermaps without storing them*, In: GASCom; Bordeaux, France, 2012, `http://www.info2.uqam.ca/~walsh_t/papers/GENERATING%20NONISOMORPHIC.pdf`.

22 **Walsh TR**, *A biased survey of map enumeration results*, WSEAS Transactions on Mathematics, **6**(1), (2007), 131–137.

23 **Walsh TR**, *Generating Nonisomorphic Maps without Storing Them*, SIAM Journal on Algebraic Discrete Methods, **4**(2), (1983), 161–178, DOI 10.1137/0604018.

24 **Wormald NC**, *Counting unrooted planar maps*, Discrete Mathematics, **36**(2), (1981), 205–225, DOI 10.1016/S0012-365X(81)80016-7.