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RESEARCH ARTICLE

On the Closed-Form Expression of Carson's Integral

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Abstract
Carson's integral is used in power systems analysis for evaluating the earth-return impedance of overhead conductors above homogeneous earth. Contrary to the widespread belief that it does not evaluate to a closed-form expression, a scrutiny of Carson's historical (1926) paper reveals that such an expression exists and was actually presented therein. Instead of using the original expression that involves the evaluation of just one Struve and one Bessel function, modern approaches wander between simplified but crude approximations of the integral and numerical evaluations by adaptive quadrature methods. In this paper we present the facts and show that neither of these approaches are necessary, since we can readily and rapidly evaluate the integral (to computer machine accuracy) by using the closed-form expression.

Keywords
Carson's integral, closed-form expression, earth-return impedance, Struve function

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1 Introduction

When studying the effect of earth in power transmission and related applications like transients in grounding systems or lightning phenomena, one resorts to electromagnetic field analysis. The results are frequently expressed in the form of integrals and their efficient computation is a matter of intense studies. Back in 1926 Carson studied the earth return impedance of conductors above homogeneous earth. His presented expression involved an improper integral, but a careful examination of his contribution reveals that he actually proceeded by giving a closed-form expression in the form of a Struve and a Bessel function of first order [1]. Due to the lack of mathematical and programming tools in those days, he had no option but to expand the acquired result in terms of infinite series. These series converge quickly at lower frequencies but as the frequency increases convergence decreases and truncation errors arise.

In mathematics, a solution is said to be of "closed-form" if it can be expressed analytically in terms of certain "well-known" functions. Hence, the choice of what to call closed-form or not is rather arbitrary and is based on the definition of the "well-known" function. Typically, the generally accepted set is that of elementary functions, but the definition can be extended to include special functions as well. This is quite acceptable if we take into account the capabilities of modern mathematical programming tools that compute most special functions like for example the gamma, the Bessel or even the Struve function to computer machine accuracy.

The fact that Carson did provide a closed-form expression for his improper integral is generally neglected [2-4]. As a result, research work is still devoted to the integral's calculation with the various approaches divided into two categories. The first involves the derivation of approximate expressions [5,6,8,10] and the second involves the numerical calculation of the integral using quadratures [4,11]. The first approach is quite handy but inherently non-universal with an error that varies according to the range of the input parameters. The second approach is universal but requires an extra effort and may become time-consuming at specific ranges of the input parameters due to the oscillating nature of the integrand.

In this paper we examine the closed-form solution of Carson's integral and show that this was actually the original expression presented by Carson himself. This is an important issue, up to the point of proposing the replacement of the term "Carson's integral" with the more appropriate "Carson's solution". We then provide the simple means to evaluate this solution both rapidly and exactly. In the calculations, we take into account the fact that after all Carson's integral is not valid above about 10 MHz since a TEM propagation was assumed prior to its derivation [8,9]. Nevertheless, the extra term in the wavenumber after Sunde's consideration of displacement current in earth [12] can be included without any change. In this work we use Matlab, but other mathematical programming tools like Mathematica or general programming languages can also be used.

2 Analysis

Consider Figure 1, which shows two conductors of a transmission line above and parallel to ground, which is homogeneous, with conductivity σ and magnetic permeability that of vacuum μ_0 .

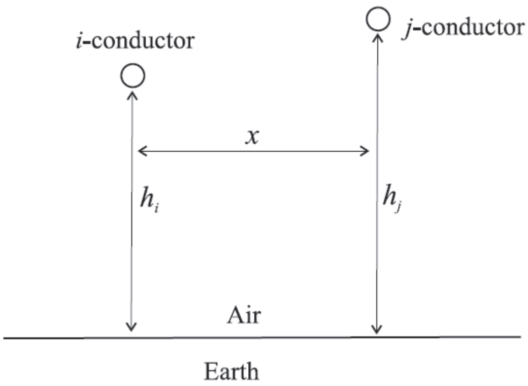

Fig. 1 Cross-section of the configuration with overhead conductors.

2.1 The closed-form expression

According to [1] and following the nomenclature in [8], for a TEM mode of propagation, the earth-return impedance is given by the formula

$$Z = \frac{j\omega\mu_0}{2\pi}\left(\ln\frac{D_2}{D_1} + I_C\right) \tag{1}$$

$$I_C = \int_0^\infty \frac{2e^{-H\lambda}}{\lambda + \sqrt{\lambda^2 + j\Omega}}\cos(x\lambda)d\lambda \tag{2}$$

where $\Omega = \omega\mu_0\sigma$ and ω is the angular frequency. Eqs (1) and (2) are used for both the self-impedance of a conductor and the mutual impedance between conductors. The second term in (1) $Z_C = (j\omega\mu_0/2\pi) I_C$ accounts for the finite conductivity earth correction term. For the self-impedance of, say, the i-conductor in Fig.1, we put $D_1 = r_i$, $D_2 = 2h_i$, $H = 2h_i$ and $x = 0$, with r_i denoting the conductor's radius. Sunde's extension of (2) involves the replacement of the term $j\Omega$ with $j\Omega - \omega^2 \mu_0\epsilon_0 \epsilon_r$ that contains the dielectric permittivities of vacuum ϵ_0 and the relative one of the ground ϵ_r . For the mutual impedance,we substitute $D_1 = \sqrt{(h_i - h_j)^2 + x_{ij}^2}$, $D_2 = \sqrt{H^2 + x_{ij}^2}$, $H = h_i + h_j$ and x as in Fig. 1. The integral term I_C in (2) can be evaluated analytically and the following closed-form expression is mentioned in many previous studies, like [8,13,14,15]

$$\begin{aligned} I_C = &\frac{\pi}{2u_1}\left[\mathbf{H}_1(u_1) - Y_1(u_1)\right] - \frac{1}{u_1^2} \\ &+ \frac{\pi}{2u_2}\left[\mathbf{H}_1(u_2) - Y_1(u_2)\right] - \frac{1}{u_2^2} \end{aligned} \tag{3}$$

where $\mathbf{H}_1$ and Y_1 denote the Struve function and the second kind Bessel function, respectively, of first order and $u_1 = \sqrt{j\Omega}(H - jx)$, $u_2 = \sqrt{j\Omega}(H + jx)$. Indeed, this closed-form result can also be verified by substituting the cosine function in (2) in terms of the complex exponentials that define it and then using the following analytical result found in [16] and designated therein as 3.368 in p. 345.

$$\int_0^\infty \frac{e^{-\mu x}}{x + \sqrt{x^2 + \beta^2}}dx = \frac{\pi}{2\beta\mu}\left[\mathbf{H}_1(\beta\mu) - Y_1(\beta\mu)\right] - \frac{1}{(\beta\mu)^2} \tag{4}$$

In (4), β and μ are general parameters under the constraints $|\arg\beta| < \frac{\pi}{2}, Re\{\mu\} > 0$, which are always satisfied for the analyzed problem of a transmission line.

2.2 Carson's original contribution

Now, how is this result connected with the one provided by Carson? In his original contribution [1], the integral in (2) was written as:

$$I_C = \int_0^\infty \frac{e^{-H\lambda}}{\lambda + \sqrt{\lambda^2 + j\Omega}}(e^{jx} + e^{-jx})d\lambda \tag{5}$$

which in turn is composed by the two integrals written in abbreviated form as

$$\begin{aligned} &\frac{1}{j\Omega}\int_0^\infty \left(\sqrt{\lambda^2 + j\Omega} - \lambda\right)e^{-(H\pm jx)\lambda}d\lambda = \\ &\frac{1}{j\Omega}\int_0^\infty \sqrt{\lambda^2 + j\Omega}e^{-(H\pm jx)\lambda}d\lambda - \frac{1}{j\Omega}\int_0^\infty \lambda e^{-(H\pm jx)\lambda}d\lambda \end{aligned} \tag{6}$$

Now the second term on the right side of (6) gives the third terms in the parentheses of (3) while the first term on the right side of (6) was given by Carson as the sum

$$\frac{1}{u_{1,2}}\left[K_1(u_{1,2}) + G(u_{1,2})\right] \tag{7}$$

with $u_{1,2}$ as defined previously. The first function in (7) is actually the Bessel function of the second kind and first order and following Heine's symbolism [17]

$$K_1(z) = -\frac{\pi}{2} Y_1(z) \tag{8}$$

In addition, according to Carson, the second function in (7) is expanded in the series

$$G(z) = \frac{z^2}{3} - \frac{z^4}{3^2 5} + \frac{z^6}{3^2 5^2 7} - \dots \tag{9}$$

which is actually the power series definition of the following function that involves the Struve function of the first order [17]

$$G(z) = \frac{2}{\pi} \mathbf{H}_1(z) \tag{10}$$

Hence, from (7), (8) and (10) we deduce that the closed-form expression (3) is equivalent to the result given by Carson as (7).

2.3 Exact evaluation

Evaluation of Carson's integral is therefore reduced to the computation of the Struve function of complex argument. As already mentioned, since in 1926 computing capabilities of special functions were limited, Carson proceeded with a further approximation of the integral with infinite series and asymptotic expansions and regretted that the given formulas appeared complicated. This fact is neglected mainly perhaps due to the supposed difficulty of computing the Struve function. Nevertheless, with today's numerical capabilities calculation of (3) is trivial not only to some prescribed accuracy but to computer machine precision. Although modern mathematical packages like Matlab or Mathematica can readily and instantly compute the aforementioned Bessel and Struve functions, the problem seems to lie in the larger argument range where both functions assume big complex values while the final result is a small complex number. Hence, a separate computation of the Struve and the Bessel function leads to severe cancelation errors. For the earth-return impedance this is equivalent to the case of higher frequencies and longer distances between conductors. The approach is to compute the two functions $\mathbf{H}_1(z)$ and $Y_1(z)$ separately for a specific range $|z| \le z_0$ when there is no problem with cancelation, and together $\mathbf{H}_1(z) - Y_1(z)$ for $|z| \ge z_0$, by using an asymptotic approximation. In both cases, we achieve computer machine accuracy. Note that there is no need to refer to the problem's input parameter ranges since these may assume any practical values that eventually will translate into one of the two regions of interest in the arguments of the special functions. This is quite an improvement over previous results (both approximate and numerical) the accuracy of which depended on these parameter ranges.

Based on a combination of Chebyshev expansions for $|z| \le 16$ and rational approximations for $|z| > 16$, the author has written and thoroughly tested a vectorized Matlab routine for rapidly evaluating the function $\mathbf{H}_1(z) - Y_1(z)$ to computer machine accuracy (14 significant digits). This extremely fast routine is available from the author in the Matlab File Exchange [20] and it can be called as **`StruveH1Y1(z)`**.

3 Results

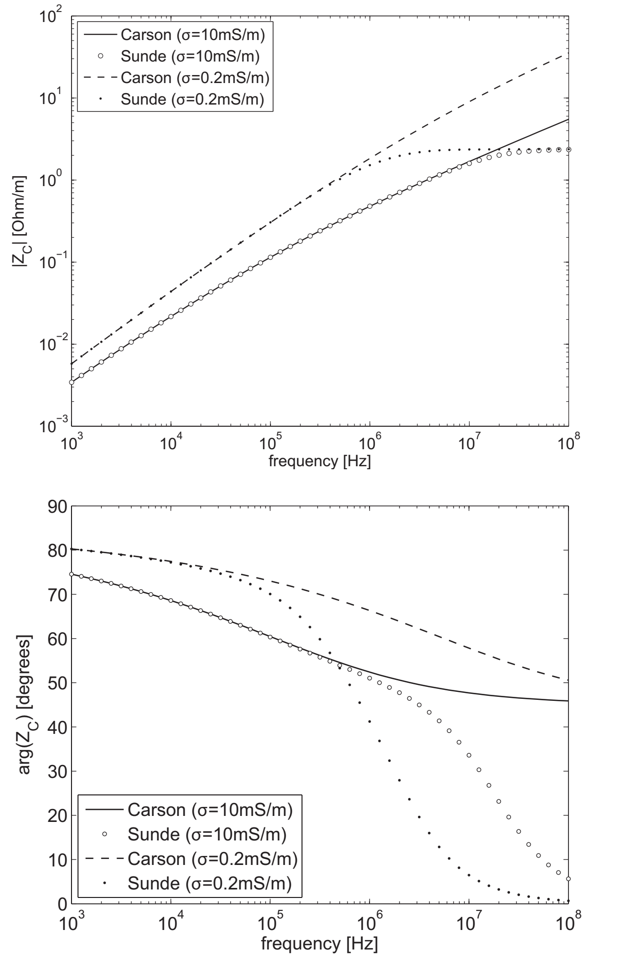

Fig. 2 Amplitude and phase of the impedance term $Z_C = (j\omega\mu_0 / 2\pi)I_C$ with (Sunde) and without (Carson) the displacement current. The Z_C term accounts for the finite conductivity correction term of the earth-return impedance.

The fact that the computation accuracy does not depend on particular parameter ranges makes the closed-form expression a universal tool. Up to now, it is interesting that these approximate solutions were compared not to the exact analytical expression but rather to numerical results valid for specific ranges or to the approximating series developed by Carson. Needless to say that the presented solution can now be the reliable reference to other solutions and numerical results.

Figure 2 repeats the calculation of Z_C and the results of Fig. 5 in [22]. What is shown is the self-impedance earth-correction term as a function of frequency for the depicted conductivities with h_i = 8m, and a relative dielectric permittivity ϵ_r = 10. Both Carson and Sunde calculations are shown and what is easily observed is the correct behavior of a constant amplitude at higher frequencies in the latter case. It is also obvious that as the frequency increases, the two expressions deviate since earth displacement currents start to become comparable to earth conduction currents.

Computation time is always of the order of a few milliseconds, comparable to the time required by the approximating expressions and orders of magnitude from the computation time required by the numerical calculations. For the latter the difference in computation time becomes even more pronounced as the number of parameter values increases. Note that the argument of the Matlab function `StruveH1Y1(z)` is a vector so that a parametric sweep (like the frequency sweep in Fig. 2) is facilitated. Routines for the computation of the self and mutual earth return impedance as well as for Carson's integral are now available from the author in the Matlab File Exchange [21].

4 Conclusion

We have shown that the closed-form solution of Carson's integral was presented by Carson himself and involves a Struve function of first order with complex argument. Up to now, the reference for comparing the accuracy of the various simplified expressions for calculating earth-return impedances were either the series of Carson or specially developed numerical integration schemes. The presented exact solution can now be the reference. The advantage is that now the integral value and hence the earth-return impedance is accurate for any practical value of the studied configuration parameters and the evaluation is rapid for even a very large number of such parameters. The numerical computation that is based on quadratures, can still be used for cases where the simple Carson solution cannot be applied, for example in multilayered earth and for earth with magnetic permeability different from that of vacuum.

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