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RESEARCH ARTICLE

Thermomechanical stress from conditional Lagrange derivative

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Abstract
A constititive equation taking into account thermodynamical processes can be derived as a conditional Lagrange derivative, which uses the first law of thermodynamics as condition. In case of the resulting constitutive equation the seconlaw of thermodynamics is also satisfied.

Keywords
stress · conditional Lagrange derivative · entropy production

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1 Introduction

The work of internal virtual forces is given by expressions

$$\int_{t_0}^{t_1}\int_V \sigma\cdot\cdot\delta\varepsilon\, dV\, dt,$$

On portion of the virtual work can be expressed as a variaton of a functional, but we do not know too much about the others. In forming the variaton of a functional we often get expressions

$$\int_{t_0}^{t_1}\int_V (\quad)\cdot\cdot\delta\varepsilon\, dV\, dt,$$

where the quantity in bracket ( ) is called the Lagrange derivative of the basic functions of the functional [2]. In the uniaxial case for basic function u

$$\pounds(u) \equiv \frac{\partial(u)}{\partial\varepsilon} - \left(\frac{\partial(u)}{\partial\dot{\varepsilon}}\right)^{\bullet} - \left(\frac{\partial(u)}{\partial\varepsilon'}\right)'$$

Here overdot denotes time derivative and prime denotes derivative with respect to the spatial coordinate. As we have already mentioned the terms, which cannot be written as variatons of functionals are unknow, but stress tensor should satisfy certain equations. Such equations could be attached to the functional as additional conditions. Then stress tensor can be obtained as a conditional Lagrange derivative of the basic functions of the functional [3].

2 The basic equation of thermodynamics

Let us use the following notations:

- internal energy e
- heat flux vector $\mathbf{h}$,
- thermodynamical tempreture ϑ,
- heat source intensity r,
- mechanical stress and strain $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$,
- mass density ρ,
- entropy S,
- entropy production s,

- mechanical potential a [1].

The first law of thermodynamics

$$u_1 \equiv \rho\dot{e} - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} + \operatorname{div}\mathbf{h} - \rho r = 0 \tag{1}$$

The second law of thermodynamics

$$\rho\left(\dot{e} - \vartheta\dot{S}\right) + \frac{1}{\vartheta}\mathbf{h}\cdot\operatorname{grad}\vartheta - \rho r - s = 0 \qquad s > 0 \tag{2}$$

from (1) and (2)

$$u_2 \equiv \rho\vartheta\dot{S} + \operatorname{div}\mathbf{h} - \frac{1}{\vartheta}\mathbf{h}\cdot\operatorname{grad}\vartheta - \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} = 0 \qquad s > 0 \tag{3}$$

Such equations can also be written in uniaxial case. In the following we restrict ouselves to the uniaxial poblem, while the multiaxial generalisation of that can easily be performed.

3 The conditional Lagrange derivative

Assume that the basic function in the functional mentioned earlier is u_0 and the additional condition is $u_1 = 0$ (the first law of thermodynamics), then the new basic function is

$$u_0 + \lambda u_1,$$

where λ is called the Lagrange multiplier and we should remember that $u_1 = 0$. Stress is the Lagrange derivative of the new basic function,

$$\sigma = \pounds\left(u_0 + \lambda u_1\right) = \pounds\left(u_0\right) + \lambda\pounds\left(u_1\right)$$

We need the Lagrange derivative of u_1. Assume that $u_1$is a function of ε and ϑ. then

$$\pounds\left(u_1\right) = \sigma_\vartheta\dot{\vartheta}.$$

Here and in the following index denotes partial derivative, that is, $\sigma_\vartheta = \frac{\partial\sigma}{\partial\vartheta}$, etc.

The stress reads

$$\sigma = \frac{\partial u_0}{\partial\varepsilon} + \lambda\sigma_\vartheta\dot{\vartheta} \equiv u_{0\varepsilon} + \lambda\sigma_\vartheta\dot{\vartheta}.$$

To determine λ we should substitute σ into equation $u_1 = 0$, then

$$\lambda = \frac{\rho\dot{e} - u_{0\varepsilon}\dot{\varepsilon} + h_x - \rho r}{\sigma_\vartheta\dot{\vartheta}\dot{\varepsilon}},$$

where $h_x = \frac{\partial h}{\partial x}$. At last the stress tensor is

$$\sigma = \rho\frac{\partial a}{\partial\varepsilon} + \frac{\rho\vartheta\dot{S} + h_x - \rho r}{\dot{\varepsilon}}. \tag{4}$$

Assume that all the quantities are functions of ε and ϑ, the

$$\sigma = \rho\left(\frac{\partial a}{\partial\varepsilon} + \vartheta\frac{\partial\dot{S}}{\partial\varepsilon}\right) + \frac{\rho\vartheta\frac{\partial\dot{S}}{\partial\varepsilon}\dot{\vartheta} + h_\varepsilon\varepsilon_x}{\dot{\varepsilon}} + \frac{h_\vartheta\vartheta_x - \rho r}{\dot{\varepsilon}}$$

is obtained.

We have assumed that the first derivative is the highest for all state variables, which appears in the expressions.

4 The form of entropy productions

The second law of thermodynamics should be satisfied and it may effect form (4) of the thermo-mechanical stress. For this reason we use

$$s = \sigma\dot{\varepsilon} - \rho\frac{\partial a}{\partial\varepsilon}\dot{\varepsilon} - \frac{h}{\vartheta}\vartheta_x > 0. \tag{5}$$

To obtain (5) we use (3) and add to another form of the second law (3) multiplied by $-\frac{1}{\dot{\varepsilon}}$. The physical meaning of (5) is that the dissipative power is positive and heat propagates from places of higher temperature to places of lower temperatures.

5 Conclusions

The use of conditional Lagrange equation makes possible to take thermodynamical processes into consideration in finding constitutive equations for the stress tensor. The expression of σ can be formally generalised to multiaxial stress states and it satisfies inequality (5). Remark that other pairs of variables could be used instead of ε and ϑ, but the second law of thermodynamcs should be satisfied in all cases.

References

1 **Eringen AC**, *Continuoum Physics*, Academic Press, New York, London, 1975.

2 **Schouten IA**, *Tensor Analysis forb Physicists*, Oxford University Press, Oxford, 1951.

3 **Béda Gy**, *Generalized Mindlin's method for the determination of constitutive equations of solids*, J. of Computational and Applied Mechanics **6** (2005), no. 2, 153–158.