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RESEARCH ARTICLE

Time delay and anticipatory effects in modeling dynamical sytems

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Abstract

In applied mechanics several papers concentrate on the comparison of delayed and non-delayed approaches of controlled machines. We may study both continuous and discrete time systems. The principal points of interest in the following work are how continuous time systems differ from its representation as some discrete time system in stability and robustness and how the discretisation of a continuous time subsystem acts on the stability properties of the coupled system.

Keywords

discretisation · delayed differential equations · simulation

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1 Introduction

The stability of controlled mechanical systems is a key aspect. In numerous problems of mechanical engineering a machine is controlled by a digital device to perform some task. Such system has two essentially different parts. The one is the machine in the sense of mechanical engineering. It is usually described as a continuous time system by using one of the traditional methods of applied mechanics. The other subsystem is the discrete controller. Generally we have a complex nonlinear system of a continuous time and a discrete time subsystems. Instability may arise from either the continuous or the time discrete parts. For example, in balancing the unstable continuous time system should be stabilised by digital control. An obvious problem in such problems is the sampling delay effect. When it is neglected an anticipatory model is obtained. The properties of it may and may not be different from the original one.

In our previous paper [1] the equation of motion was derived for a simple controlled inverted pendulum with length l and mass m (see Fig. 1). The pendulum was attached to a cart with a hinge and its stability was achieved by applying force F to the cart.

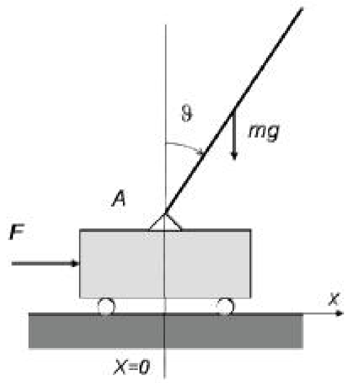

Fig. 1. Inverted pendulum

By using Lagrangian formalism the equation of motion is

$$\frac{1}{3}ml^2\ddot{\vartheta} + \frac{1}{2}ml\ddot{x}\cos\vartheta - \frac{1}{2}mgl\sin\vartheta = 0 \tag{1}$$

$$m\ddot{x} + \frac{1}{2}ml\ddot{\vartheta}\cos\vartheta - \frac{1}{2}ml\dot{\vartheta}^2\sin\vartheta = Q(t) \tag{2}$$

where the two generalised coordinates are the position x of the cart and the angular position ϑ of the pendulum measured from the upwards vertical. On the right hand side of (Eq. 2) the generalised force Q is equal to the control force

$$(Q(t) \equiv)\, F(t) = c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau), \tag{3}$$

where PD control gains are denoted by c_0, c_1. In (Eq. 3) the output of the controller is delayed expressing the fact that in most cases there is a time delay τ between the measurement or sampling and the action of the controller.

2 Discrete dynamical systems

There are two possible ways of approach in controlled mechanical systems. Firstly, we may express $\ddot{x}$ from (Eq. 2) and substitute into (Eq. 1) then

$$\ddot{\vartheta} = -\frac{3\sin\vartheta\cos\vartheta}{4-3\cos^2\vartheta}\dot{\vartheta}^2 + \frac{6g\sin\vartheta}{(4-3\cos^2\vartheta)\,l} - \frac{\cos\vartheta}{(4-3\cos^2\vartheta)\,ml}F(t) \tag{4}$$

$$F(t) = c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau). \tag{5}$$

The other possibility is to keep the two generalised coordinates and force F as unknown functions from (Eq. 3) and then

$$\begin{aligned} \ddot{\vartheta} &= \frac{3l\dot{x}^2\sin\vartheta\cos\vartheta - 6g\sin\vartheta}{(-4+3\cos^2\vartheta)\,l} + \frac{6F\cos\vartheta}{(-4+3\cos^2\vartheta)\,ml} \\ \ddot{x} &= \frac{3g\sin\vartheta\cos\vartheta - 2l\dot{x}^2\sin\vartheta}{(-4+3\cos^2\vartheta)} - \frac{4F}{(-4+3\cos^2\vartheta)\,m} \\ F(t) &= c_1\dot{\vartheta}(t-\tau) + c_0\vartheta(t-\tau). \end{aligned} \tag{6}$$

A detailed derivation of the continuous time dynamical systems (Eq. 4), (Eq. 5) and (Eq. 6) can be found in [1] and it is followed by a linear stability investigation of the upright position. Then the behaviour of the systems with delayed and non-delayed control is compared by numerical analysis, which requires discretisation.

When delay is omitted ($\tau = 0$) in the control law (Eq. 5) an incursive feed-in-time system is obtained [3]

$$F(t) = c_1\dot{\vartheta}(t) + c_0\vartheta(t),$$

which is more obvious for a discrete time $t \in [t_0,\ t_1, \ldots, t_i, \ldots]$ system

$$F(t_i) = c_1\dot{\vartheta}(t_i) + c_0\vartheta(t_i), \tag{7}$$

while any nonzero delay should be interpreted as some recursive form. Assume for the sake of simplicity that

$$t_i = t_0 + i\Delta t, \quad \text{where} \quad i = 1, 2, \ldots,$$

and $\tau = \Delta t$, where Δt is a small positive time step. Then (Eq. 5) results recursion

$$F(t_i) = c_1\dot{\vartheta}(t_{i-1}) + c_0\vartheta(t_{i-1}).$$

When numerical simulation is needed, we should form a set of difference equations instead of (Eq. 5), (Eq. 4) and (Eq. 6). Let us introduce new variables:

$$y_1 = \vartheta,\ y_2 = \dot{\vartheta},\ y_3 = x,\ y_4 = \dot{x},\ y_5 = F$$

and simplifying notation

$$y_k(i) = y_k(t_i) \qquad (k = 1, 2, ..., 5).$$

Then in the feed-in-time ($\tau = 0$) case from (Eq. 4) and (Eq. 7)

$$\begin{aligned} y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\ y_2(i+1) &= y_2(i) + \left(-\frac{3\sin y_1(i)\cos y_1(i)}{4-3\cos^2 y_1(i)}(y_2(i))^2 \right. \\ &\quad \left. + \frac{6g\sin y_1(i)}{(4-3\cos^2 y_1(i))\,l} - \frac{6\cos y_1(i)}{(4-3\cos^2 y_1(i))\,ml} y_5(i)\right)\Delta t \\ y_5(i+1) &= c_1 y_2(i+1) + c_0 y_1(i+1) \end{aligned} \tag{8}$$

or simply

$$\begin{aligned} y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\ y_2(i+1) &= y_2(i) + \\ &+ \left(-\frac{3\sin y_1(i)\cos y_1(i)}{4-3\cos^2 y_1(i)}(y_2(i))^2 + \frac{6g\sin y_1(i)}{(4-3\cos^2 y_1(i))\,l} \right. \\ &\left. - \frac{6c_1\cos y_1(i)}{(4-3\cos^2 y_1(i))\,ml} y_2(i) - \frac{6c_0\cos y_1(i)}{(4-3\cos^2 y_1(i))\,ml} y_1(i)\right)\Delta t \end{aligned} \tag{9}$$

is obtained. From (Eq. 6) the recursive discrete dynamical system reads

$$\begin{aligned} y_1(i+1) &= y_1(i) + \Delta t y_2(i) \\ y_2(i+1) &= y_2(i) + \\ &+ \left(\frac{3y_4^2(i)\,l\sin y_1(i)\cos y_1(i) - 6g\sin y_1(i)}{(-4+3\cos^2 y_1(i))\,l} \right. \\ &\left. + \frac{6\cos y_1(i)\,y_5(i)}{(-4+3\cos^2 y_1(i))\,ml}\right)\Delta t \\ y_3(i+1) &= y_3(i) + \Delta t y_4(i) \\ y_4(i+1) &= y_4(i) + \\ &+ \left(\frac{3g\sin y_1(i)\cos y_1(i) - 2y_4^2(i)\,l\sin y_1(i)}{(-4+3\cos^2 y_1(i))} \right. \\ &\left. - \frac{4y_5(i)}{(-4+3\cos^2 y_1(i))\,m}\right)\Delta t \\ y_5(i+1) &= c_1 y_2(i) + c_0 y_1(i) \end{aligned} \tag{10}$$

3 Numerical analysis of discretised systems

Assumed that the origin of delay is the sampling Δt=0.0005s. Remark that setting is a serious restriction. The reason to use it is

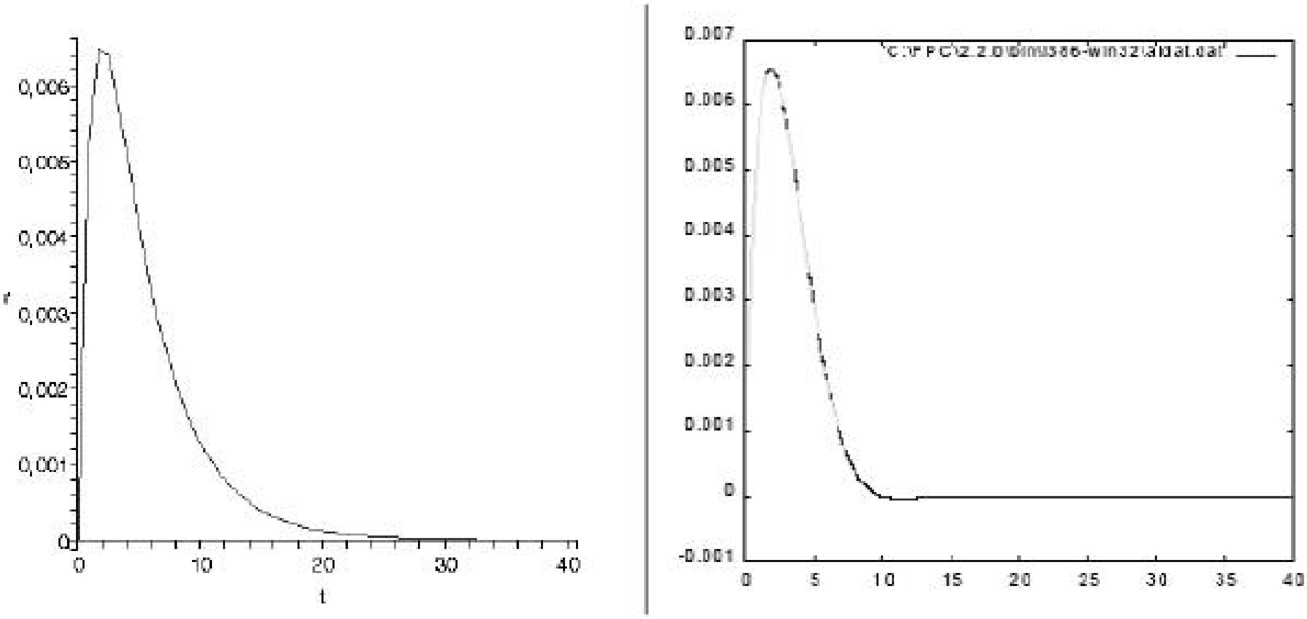

Fig. 2. Aperiodic case

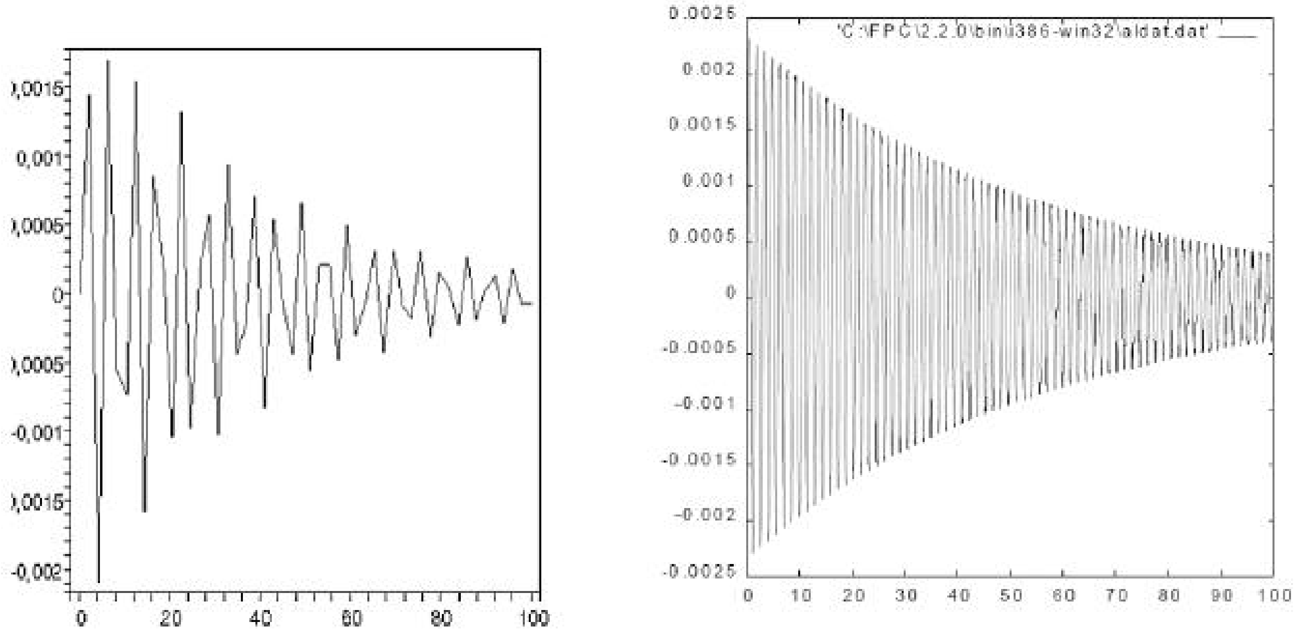

Fig. 3. Similarity.

to study the simplest possible case and to remain at the original time continuous problem of Stépán [2]. However, we should investigate in further work the general case , which seems to be a much more realistic assumption. The initial values are

$$y_1(0) = 0,\ y_2(0) = 0.001,\ y_3(0) = 0,\ y_4(0) = 0,\ y_5(0) = 0.$$

Control gains are varied. Time histories for feed-in-time and feed-back are plotted.

Fig. 2 shows that in the stable non-periodic region (Eq. 9) and (Eq. 10) leads almost to the same results. However, when control parameters are selected near to the stability boundary of the delay equation, the difference is more obvious. In Fig. 3 we find damped oscillations, while in Fig. 4. the feed-in-time system remains near to the origin, while in case of feedback an obvious instability can be detected. In the first graph of Fig. 4 the system remains in a interval, which can be considered as a stable (but not asymptotically stable) behaviour. The other observation is that feedback results a much higher frequency oscillation that feed-in-time.

4 Conclusions

For a digitally controlled system sampling delay plays an important role as it was already published in the literature. Numerical simulation has shown that effect is almost negligible away

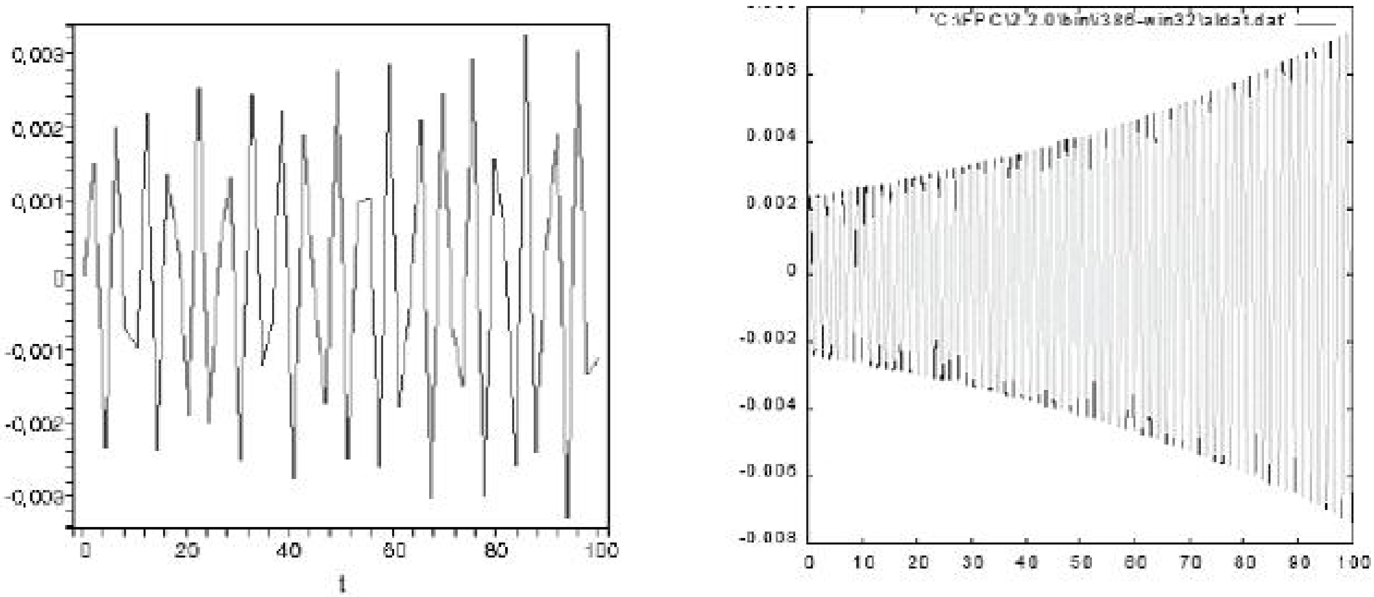

Fig. 4. Qualitative difference.

from stability boundaries and bifurcation values. In addition we have found that the resulting oscillation is of higher frequency for a feed-back then for a feed-in-time. On the other hand an interesting behaviour is detected: the feed-in-time case seems to be less regular. While the second graphs of both Fig. 3 and Fig. 4 shows increasing or decreasing harmonic oscillations, the first graphs show certain irregularity. We feel a contradiction because a chaotic system may be "regularized" by anticipative effects. Another remarkable fact is that the delay of continuous time system may disappear at the discretization.

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