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RESEARCH ARTICLE

Calculation of the temperature of boundary layer beside wall with time-dependent heat transfer coefficient

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Abstract

This paper proposes to investigate the changes in the temperature of external wall boundary layers of buildings when the heat transfer coefficient reaches its stationary state in time exponentially. We seek the solution to the one-dimensional parabolic partial differential equation describing the heat transfer process under special boundary conditions. The search for the solution originates from the solution of a Volterra integral equation of the second kind. The kernel of the Volterra integral equation is slightly singular therefore its solution is calculated numerically by one of the most efficient collocation methods. Using the Euler approach an iterative calculation algorithm is obtained, to be implemented through a programme written in the Maple computer algebra system. Changes in the temperature of the external boundary of brick walls and walls insulated with polystyrene foam are calculated. The conclusion is reached that the external temperature of the insulated wall matches the air temperature sooner than that of the brick wall.

Keywords

heat conduction · slightly singular Volterra integral equation · numerical solution · heat transfer coefficient · relaxation time

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1 Introduction

The heat transfer and thermal conductivity of the wall boundary layers of buildings ensure the temperature potentials inside and outside the wall boundaries of buildings. The majority of the relevant models focus on temperature changes at internal points in the wall boundary depending on the various insulation layers. It is assumed that the temperature of the external wall boundary and the average temperature of the outside air surrounding the wall are the same i.e. the outside temperature becomes constant and the temperature of the internal wall boundary is also constant. This problem is not discussed by this paper. We investigate how quickly the external wall boundary adopts the temperature of the air boundary layer under a constant temperature of the internal wall boundary. This only requires the examination of the simplified model of the wall boundary presented in Fig. 1.

In the mathematical model the wall is regarded as one of infinite thickness. Its temperature is denoted by $u(x, t)$ at position $x > 0$ and time $t > 0$. It is sufficient to deal with the one dimensional case as it can be assumed that along the length of the wall the temperature is distributed equally at each vertical cross-section. In accordance with the denotations $u(0, t)$ denotes the temperature of the external boundary wall layer.

The external layer of the wall is in contact with the ambient air. The internal layer of the wall mixes with the air of the room, assumed to have a constant temperature. The wall resists the incoming heat flow (perpendicular to the wall) therefore in the "air -wall" boundary layer the temperature of the air is different than the surface temperature of the wall. The temperature of the external wall boundary takes some time to adopt the average temperature of the outdoor air.

Θ denotes the average temperature of the air flow. In winter it is typically $\Theta < \mu(0, t)$ i.e. the outdoor air cools the wall. In summer $(0, t) < \Theta$, i.e. the outdoor air warms the wall. The model can be applied to both cases but simplified criteria $u(0, 0) = 0$ is used. The initial temperature $t = 0$ (sec) of the wall layer is thus 0°C and $0 < \Theta$, i.e. the air flow gradually warms the external wall boundary. Then, if t tends to infinity the temperature of the external wall boundary reaches air tem-

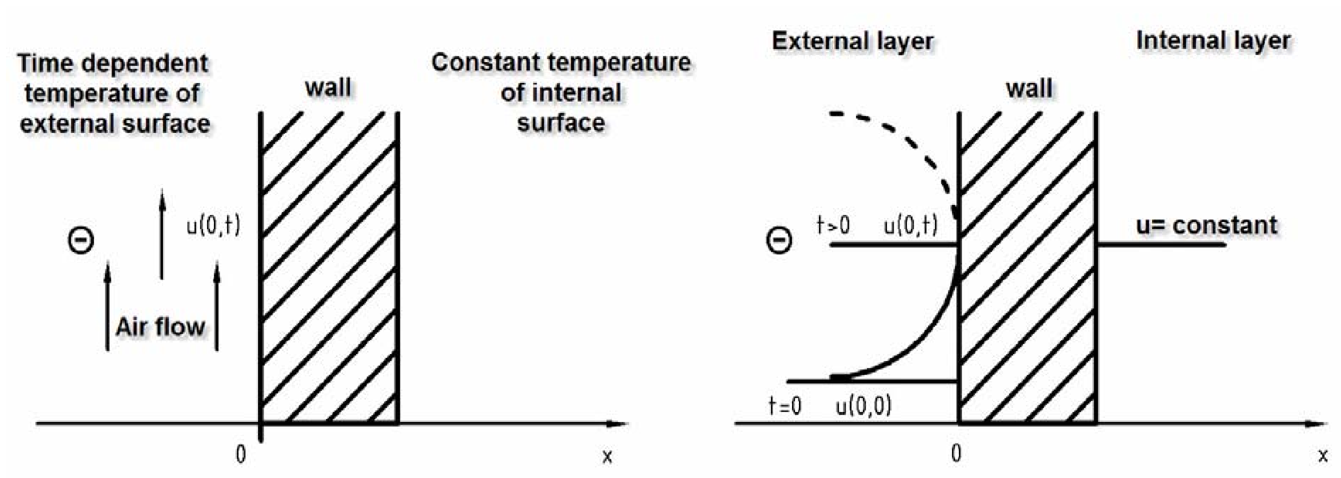

Fig. 1. Simplified model of wall boundary

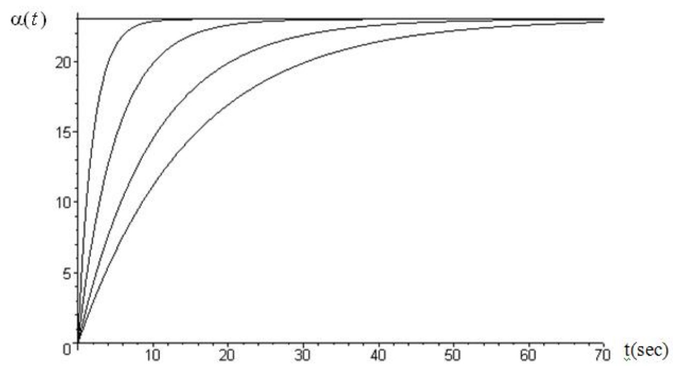

Fig. 2. Curves of the α heat transfer coefficient in the boundary layer with relaxation times T= 2, 5, 10, 15 (sec) (downwards from top)

perature Θ. This change will be calculated numerically.

The thermal impact of the air on the wall is characterized by convective heat transfer coefficient α. It is assumed that the thermal transmittance coefficient $\alpha(t)$ of the wall changes in time

$$\alpha(t) = \alpha_0 \left(1 - e^{\left(-\frac{t}{T}\right)}\right) \tag{1}$$

exponentially [1] where
α_0 is the constant convective heat transfer coefficient of the external side,
α_0= 23 W/m^{2}K and T >0 is the so-called relaxation time with a dimension in*sec*. The change of the heat transfer coefficient in time ispresented in Fig. 2.

In Fig. 2 the graphs show the change of the convective heat transfercoefficient for relaxation times $T = 2, 5, 10$ and 15. As T increases the graphs go downwards i.e. the function reaches a state of balance later if T is higher. Time t required to reach the same degree of balance is in direct proportion to the length of T. For example time t required to reach 0,95 times balance α_0 is attained by solving equation $0,95 = 1 - e^{\left(-\frac{t}{T}\right)}$ where the solution is $t = 2.9957322274T$.

Our objective is to determine by continual approaches function $u(0, t)$ which shows in time the change of temperature of the external wall boundary layer $x = 0$, taking into account that the air flow parallel to the wall modifies in time the $\alpha(t)$ heat transfer factor of the wall-air boundary layer according to exponential equation presented in formula (1). The subject of this paper therefore belongs to the topic of boundary layer theory.

The partial equation is written up and will be solved using the Volterra equation.

In section 2 of this paper the partial differential equation determining temperature $u(x, t)$ of the wall will be presented together with the necessary boundary conditions. To seek the solution the Laplace transformation of the solution in time t will be used. To attain the result we need to solve a Volterra integral equation of the second kind.

Section 3 will present the formulas determining the iterative solution of the obtained Volterra integral equation. Using these formulas a Maple computer algebra programme will be written calculating the approaching values of the solution in an iterative way. Issues relating to the numerical precision and stability of the approaching values can be consulted in the bibliography (see [1,3] and [4]).

Section 4 deals with two types of wall: brick wall and wall insulated with polystyrene. The necessary parameters are calculated for these two cases and the numerical solutions are written up. Finally a conclusion will be drawn about the behaviour of the boundary layer of the two types of walls.

2 The mathematical model of the problem and deduction of the Volterra integral equation

2.1 Governing equations and solution procedure

Temperature $u(x, t)$ of the wall satisfies the regular heat conduction equation

$$\frac{\partial}{\partial t} u(x, t) = a \left(\frac{\partial^2}{\partial x^2} u(x, t) \right) \quad (\text{x} > 0; \text{t} > 0) \tag{2.a}$$

with the following boundary conditions:

$$u(x, 0) = 0 \tag{2.b}$$

$$u(\infty, t) = 0 \tag{2.c}$$

$$\lambda \left(\frac{\partial}{\partial x} u(x,t) \right) \Big|_{x=0} = \alpha_0 \left(1 - e^{\left(-\frac{t}{T}\right)} \right) (u(0,t) - \Theta) \quad (2.d)$$

where:

λ is the thermal conductivity of the wall and the insulation material, W/mK,

α_0 is the stationary convective heat transfer coefficient, W/m^2K,

$a = \frac{\lambda}{\rho c}$ thermal diffusivity, m^2/s (if the temperature is not evenly distributed it defines the speed of equalization)

ρ density, kg/m^3,

c specific heat capacity, J/(kg K).

Dynamic equation (2.d) plays a crucial role in the process. The speed of the change in temperature perpendicularly impacting the wall is proportional to the difference in the temperature of the wall surface and the air and the proportion factor is heat transfer coefficient α_0 obtained by formula (1).

2.2 Steps of deducing the Volterra integral equation

Temperature $u(0,t)$ of the external wall boundary layer is obtained from Eqs. (2.a-d) and it is not necessary to calculate temperature $u(x,t)$ $(x > 0)$ of the internal points of the wall. We need to deduce from Eqs. (2.a-d) a so-called Volterra integral equation only containing function $u(0,t)$, which has a slightly singular kernel of the second kind. From the theory of Volterra integral equations of the second kind the method of numerical solutions is used where the Euler approach is applied to calculate the value of the integral. This scheme is the simplest version of the collocation method.

The integral equation to calculate temperature $u(0,t)$ of the wall-air boundary layer is derived through the following steps. (working of this problem Garbai [2] too).

Step 1

Variables t, x and $u(x,t)$ in Eqs. (2.a-d) are replaced (transformation) $X = \frac{x}{\sqrt{Ta}}$; $\tau = \frac{t}{T}$; $U(X,t) = \frac{u(x,t)}{\Theta}$

by new dimensionless variables X, τ and dependent variable U(X,τ). These variables turn Eqs. (2.a-d) into the following simpler version:

$$\frac{\partial}{\partial t} U(X,\tau) = \frac{\partial^2}{\partial X^2} U(X,\tau) \quad (X > 0; \tau > 0) \quad (3.a)$$

$$U(X,0) = 0 \quad (3.b)$$

$$U(\infty,\tau) = 0 \quad (3.c)$$

$$\left(\frac{\partial}{\partial X} U(X,\tau) \right) \Big|_{x=0} = A \left(1 - e^{(-\tau)} \right) (U(0,\tau) - 1) \quad (3.d)$$

where $A = \frac{\alpha_0 \sqrt{aT}}{\lambda}$.

In the following section we will deal with the solution to Eqs. (3.a-d) but variables X, τ and $U(X,\tau)$ are replaced by the original variables x, t and $u(x,t)$.

As in Eqs. (3.a-d) boundary layer U(0,τ) started from the initial τ = 0˚C and following the transformation the mean air temperature went up to 1 ˚C we will investigate the curve along which the function of temperature U(0,τ) increases from 0 to 1. If we wish to return to the original temperature Θ [˚C] the calculated $U(0,\tau)$ value needs to be multiplied by Θ because $u(x,t) = \Theta U(X,\tau)$

After the transformation the variable value of τ shows how many times it exceeds relaxation time T. $U(0, \tau = r)$ means the temperature that the boundary layer adopts at point of time $t = r \times T$, i.e. r times of relaxation time T.

Step 2

To find solution u(x,t) to parabolic partial differential equations (3.a-d) the Laplace transformation is applied to variable t. The Laplace transform of w(x,s) for t of function (x,t) is defined by formula

$$L_t(u(x,t))(s) = \int_0^\infty e^{(-st)} u(x,t)\, dt = w(x,s)$$

The Laplace transform of partial differential Eq. (3.a) turns into

$$L_t\left(\frac{\partial}{\partial t} u(x,t) \right) = L_t\left(\frac{\partial^2}{\partial x^2} u(x,t) \right)$$

$$s w(x,s) - u(x,0) = \frac{\partial^2}{\partial x^2} w(x,s)$$

$$s w(x,s) = \frac{\partial^2}{\partial x^2} w(x,s) \quad (4)$$

ordinary differential equation where the uniformity for the derived Laplace transform and initial condition (3.b) $u(x,0) = 0$ are used.

Step 3

With regard to variable x Eq. (4) is an ordinary linear differential equation of the second kind with a constant coefficient where s is one of the parameters. Its general solution is

$$w(x,s) = F1(s) e^{(\sqrt{s}x)} + F2(s) e^{(-\sqrt{s}x)}$$

where $F1$ and $F2$ are the discretionary function of parameter s. As in the case of x$\to \infty$ $u(x,t)$ is zero according to (3.c) therefore $\lim_{x\to\infty} w(x,s) = 0$ also applies to the Laplace transform of w(x,s) for t. This will only be satisfied if F1(s)=0 because the wall is ∞ therefore

$$w(x,s) = F2(s) e^{(-\sqrt{s}x)} \quad (5)$$

Step 4

We derive the Laplace transform of $w(x,s)$ from (5) by x and re-substitute $w(x,s)$ from Eq. (5) into the obtained equation

$$\frac{\partial}{\partial x} w(x,s) = -F2(s) \sqrt{s} e^{(-\sqrt{s}x)}$$

$$\frac{\partial}{\partial x} w(x,s) = -\sqrt{s} w(x,s) \quad (6)$$

Step 5

Since we are interested in the temperature of the boundary layer substitution x=0 needs be performed in Eq. (6).

$$\left(\frac{\partial}{\partial x} w(x,s)\right)\Bigg|_{x=0} = -\sqrt{s}w(0,s) \tag{7}$$

Step 6

If the Laplace transformation is used in variable t for boundaryEq. (3.d)

$$\left(\frac{\partial}{\partial x} w(x,s)\right)\Bigg|_{x=0} = -\frac{A}{s(1+s)} + AL_t\left(\left(1-e^{(-t)}\right)u(0,t)\right)(s) \tag{8}$$

is obtained where the Laplace transformation equality was used

$$L_t\left(1-e^{(-t)}\right) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)} \tag{9}$$

As the left sides of Eqs. (7) and (8) are the same the right sides are also equal

$$-\sqrt{s}w(0,s) = -\frac{A}{s(1+s)} + AL_t\left(\left(1-e^{(-t)}\right)u(0,t)\right)(s)$$

Both sides of the equation should be divided by expression $(-\sqrt{s})$ and it should be taken into account that $w(0,s) = L_t(u(0,t))$

$$L_t(u(0,t)) = \frac{A}{s(1+s)\sqrt{s}} - \frac{AL_t\left(\left(1-e^{(-t)}\right)u(0,t)\right)(s)}{\sqrt{s}} \tag{10}$$

Step 7

Eq. (10) includes the Laplace transform $L_t(u(0,t))$ of temperature function $u(0,t)$ of the wall boundary layer and the Laplace transform of function $\left(1-e^{(-t)}\right)u(0,t)$. In function Eq. (10) $\dfrac{A}{s(1+s)\sqrt{s}}$ can be brought to zero if $u(0,t)$ is substituted by

$$u(0,t) = g(t) + 1 \quad (0 \le t) \tag{11}$$

Based on the linearity of Laplace transformation and correlation (9)

$$L_t(g(t)+1) = \frac{A}{s(1+s)\sqrt{s}} - \frac{AL_t\left(\left(1-e^{(-t)}\right)(g(t)+1)\right)}{\sqrt{s}}$$

$$L_t(g(t)+1) = \frac{AL_t\left(\left(1-e^{(-t)}\right)g(t)\right)}{\sqrt{s}} \tag{12}$$

Step 8

The inverse Laplace transformation should be used for both sides of Eq. (12) and the inverse Laplace transformation correlation $\left(L_s^{(-t)}\right)\left(\frac{1}{\sqrt{s}}\right) = \dfrac{1}{\sqrt{\pi t}}$ should be applied as well as the convolution formula written up with inverses

$$g(t_i) = -1 + \frac{A\int_0^{t_i} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t_i-s}}ds}{\sqrt{\pi}} = -1 + \frac{A\left(\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t_i-s}}ds\right)}{\sqrt{\pi}}$$

where $F(s) = L_t(f(t))$ and and $H(s) = L_t(h(t))$. After arranging the equation

$$g(t) = -1 - \frac{A\int_0^{t} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t-s}}ds}{\sqrt{\pi}} \tag{13}$$

Volterra integral equation of the second kind is obtained in which kernel $K(t,s) = \frac{1}{\sqrt{t-s}}$ is not interpreted at the $s = t$ end point of the integral interval.

3 Numerical solution to Volterra integral equation using the Euler approach

We have seen that function $g(t) = u(0,t) - 1$ satisfies the slightly singular Volterra integral equation of the second kind (13) where function $u(x,t)$ is the solution to group of Eqs. (3.a)-(3.d).

Let's see 0< t <H time interval and its distribution to N>1 equal parts

$$(B_N)\ t_0 = 0 < t_1 < t_2 < t_3 < \cdots < t_i < t_{i+1} < \ldots < t_N = H$$

where $t_i = i\delta$ is the i-th division point and $\delta = \frac{H}{N}$ is the spacing $(i = 0, 1, 2, \ldots, N)$.

We seek the approaching values $g(t_1), g(t_2), \ldots, g(t_N)$ to solution g(t) of the Volterra integral Eq.(13) at (B_N) division points.

We know the initial g(0)=-1 value. It is assumed that approaching values $g(t_1), g(t_2), \ldots, g(t_{i-1})$, are known. To obtain the value of $g(t_i)$ we need to break up the integral from the integral equation into the sum of integrals from partial intervals (B_N)

$$g(t_i) = -1 + \frac{A\int_0^{t_i} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t_i-s}}ds}{\sqrt{\pi}} = -1 + \frac{A\left(\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t_i-s}}ds\right)}{\sqrt{\pi}} \tag{14}$$

where it may be: $i = 1, 2, 3, \ldots, N$.

In integral from $[t_j, t_{j+1}]$ partial interval multiplying factor $f(s) = \left(1-e^{(-s)}\right)g(s)$ should be approached by constant value $f(t_j)$ recorded in the left end point $s = t_j$ while singular kernel $\frac{1}{\sqrt{t_i-s}}$ should be integrated by s in the given integral. This gives us the following

$$\int_{t_j}^{t_{j+1}} \frac{\left(1-e^{(-s)}\right)g(s)}{\sqrt{t_i-s}}ds = \left(1-e^{(-t_j)}\right)g(t_j)\int_{t_j}^{t_{j+1}} \frac{1}{\sqrt{t_i-s}}ds = \left(1-e^{(-t_j)}\right)g(t_j)\,2\left(\sqrt{t_i-t_j} - \sqrt{t_i-t_{j+1}}\right) \tag{15}$$

for all $j = 0, 1, 2, ..., (i-1)$.

Therefore to obtain the approaching value for $g(t_i)$ the following recursive calculation formula is used

$$g\left(t_i\right) = -1 + \frac{2A\left(\sum_{j=0}^{i-1}\left(1-e^{\left(-t_j\right)}\right) g\left(t_j\right)\left(\sqrt{t_i-t_j}-\sqrt{t_i-t_{j+1}}\right)\right)}{\sqrt{\pi}} \tag{16}$$

On the right side of recursive formula (16) values $g\left(t_0\right) = -1, g\left(t_1\right), g\left(t_2\right), ..., g\left(t_{i-1}\right)$ are presented therefore the values of function g can be calculated step by step. To calculate $g(t_2)$ is only necessary to know $g\left(t_0\right) = -1$. To calculate $g\left(t_2\right)$the previously calculated $g\left(t_1\right)$ is also required. To calculate $g\left(t_3\right)$ both $g\left(t_1\right)$ and $g\left(t_2\right)$ previously calculated are needed. The calculation of the next temperature figure always uses the previously obtained values of temperature.

4 Numerical results for different wall surfaces

There is a single parameter in Volterra integral Eq. (13) which can be calculated using correlation $A = \frac{\alpha_0\sqrt{aT}}{\lambda}$ with the earlier mentioned data of the surface material of the wall. In the case of a B30 brick wall λ=0.647, ρ=1460 and c=0.88. If relaxation time $T = 1$ sec is used in the calculation then A= 0.7977 is obtained.

For a B30 brick wall relaxation time $T = 2$ sec A=1.128165798 is obtained.

For insulating material polystyrene foam the same data are λ=0.047 with parameters ρ=15, c=1.46. If $T = 1$ then A=22.67027819 while for $T = 2$ the parameter is A=32.06061487.

It can be thus stated that the value of coefficient A in the Volterra integral Eq. (13) increases if

- better insulting materials are used
- relaxation time T is increased.

The following numerical and graphical results were attained through the Maple computer algebra system. A Maple function was written for recursive correlation (16). Using elements g_0=-1, g_1, g_2, ..., g_{i-1}in a list it calculates the next g_i element through formula

$$\text{iteration} := i \rightarrow -1 - \frac{2A\left(\sum_{j=0}^{i-1}\left(1-e^{(-j\delta)}\right) g_j\left(\sqrt{(i-j)\,\delta}-\sqrt{(i-j-1)\,\delta}\right)\right)}{\sqrt{\pi}} \tag{17}$$

Choosing parameter A=0.8 which corresponds to brick wall B30 with relaxation time $T = 1$ [sec] the graph shows the approaching value for surface temperature $u(0,t) = g(t) + 1$.

The Fig. 3 presents the curve along which the external layer of the brick wall increases from the initial temperature of 0˚C to

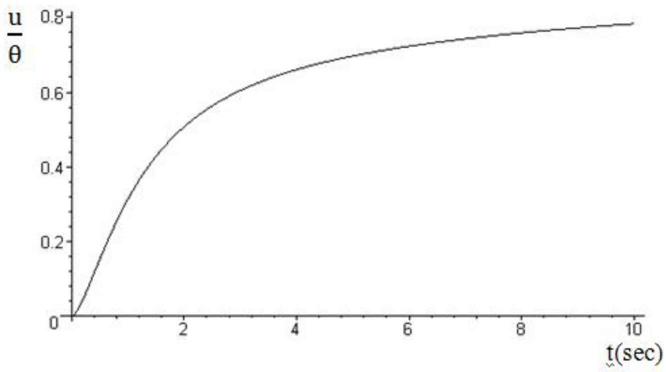

Fig. 3. Surface temperature function $u(0,t)$ in the case of B30 brick wall

1˚C outdoor air temperature of 0˚C to 1˚C outdoor air temperature Θ=1˚C. The horizontal line is the t-time axis measured in sec and the vertical line is wall surface temperature u(0,t) measured in ˚C.

In Fig. 4 the graph for function $u(0,t)$ has also been drawn in a coordinate system where logarithm formula $x = \log_{10}\left(A^2Tt\right)$ was used on the horizontal axis. This figure illustrates better the change of the temperature function for small t time.

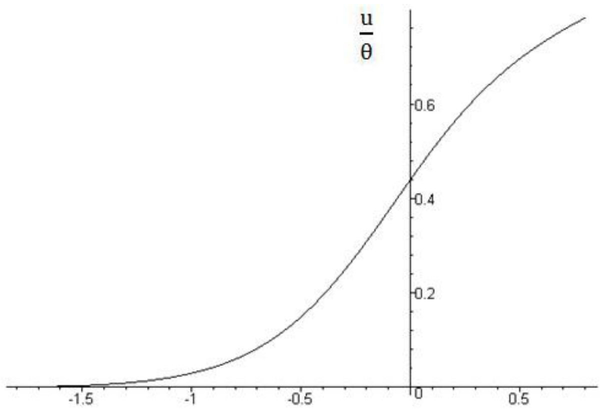

Fig. 4. Function u(0,t) with parameter A=0.8 and horizontal scale $x = \log_{10}\left(A^2Tt\right)$

The approaching calculations are repeated for parameters corresponding to A=22 polystyrene insulation. The result is presented in Fig. 5.

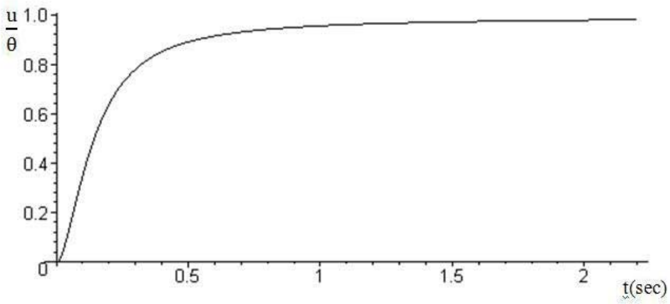

Fig. 5. Surface temperature function u(0,t) if polystyrene foam (A=22) is used

The graph for function $u(0,t)$ with parameter value A=22 was written up according to the (Fig. 6 where the horizontal axis is logarithmically scaled.

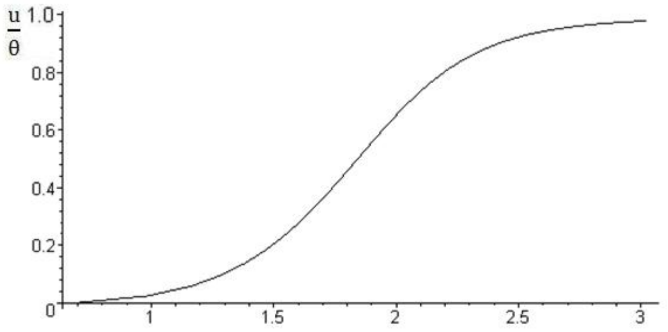

Fig. 6. Function u(0,t) with parameter A=22 and horizontal scale $x = \log_{10}\left(A^2 T t\right)$

Drawing up the graphs in Figs. 3 and 5 in a common coordinate system the difference between the two graphs is highly visible as shown in Fig. 7. The higher graph shows how the external wall boundary layer adapts to air temperature Θ=1°C in the case of a B30 brick wall with polystyrene insulation and the lower graph shows the curve for a wall without insulation.

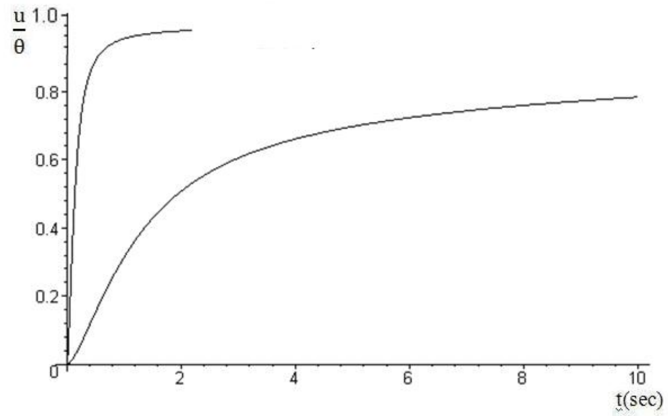

Fig. 7. Surface temperatures u(0,t) for insulated and uninsulated B30 brick wall

Fig. 6 shows that the temperature of the external surface of the insulated wall almost immediately adopts the temperature of the outdoor air while this process takes longer for an uninsulated brick wall. The explanation for this fact is that insulation does not carry the outdoor temperature inside but forms an obstacle to the spreading of the heat so there is no equalization between the temperatures of the external and internal layers in terms of thermal transmittance. The brick wall, however, transmits the external volume of heat better and a kind of equalization process starts between the indoor and outdoor temperatures. As a result the temperature of the external surface of the B30 brick wall is impacted by indoor temperature 0 °C i.e. the brick wall carries the volume of heat inside and reaches outdoor temperature 1°C later than the external surface of the insulated wall.

The graphs of the temperature functions presented in Figs. 4 and 6 were put into the same coordinate system (Fig. 8) where the horizontal axis is logarithmically scaled.

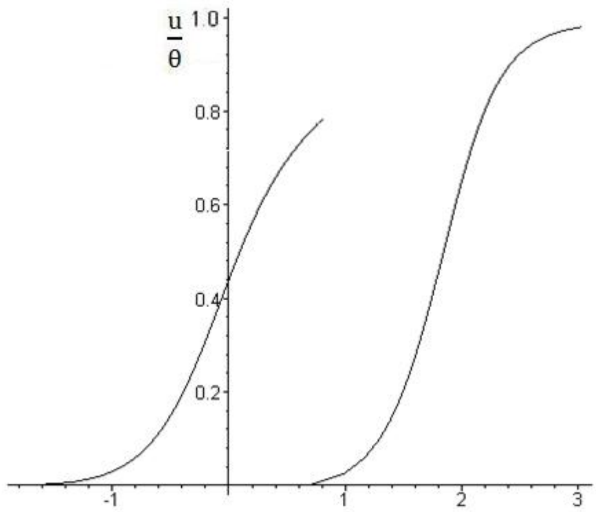

Fig. 8. Surface temperatures $u(0, t)$ shown on a logarithmic scale

The logarithmic scale used on the horizontal axis reverses the place of the two graphs compared to Fig. 7. In Fig. 8 the left graph applies to the B30 brick wall while the right graph applies to the polystyrene insulation.

5 Conclusion

Based on Fig. 7 the following conclusion can be drawn about the adaptation capability of the external wall surface: the temperature of the wall surface adapts more quickly in time to the air temperature if the wall is insulated. This phenomenon is explained by the fact that bricks in an uninsulated wall carry the heat inside and thus the temperature of the external wall surface adapts more slowly to the air temperature. In this case the indoor temperature influences the outdoor surface temperature. If the wall is insulated, however, the indoor temperature cannot impact the temperature of the external wall surface to the same extent as the insulation provides a thermal barrier between the external and internal temperatures and adopts the outdoor air temperature almost immediately.

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