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RESEARCH ARTICLE

The Portevin-Le Chatelier effect and dynamical systems

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Abstract

Let us consider Portevin-Le Chatelier (PLC) effect as a form of dynamic material instability. The tools of the investigation are the same as of the theory of dynamical systems. While at the PLC phenomenon negative rate dependence is coupled with the appearance of a self-sustained oscillatory behaviour in solid bodies, the results present us an interpretation of PLC as a flutter type of loss of material stability.

Keywords

Material instability · dynamical systems · flutter

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1 Introduction

The Portevin-Le Chatelier (PLC) effect is observed in alloys under certain conditions. It is considered to be a result from the dynamic interaction of mobile dislocations and solute atoms [8]. Mobile dislocations move by successive starts and stops between various obstacles. Solute atoms diffuse to and age mobile dislocations while they are temporarily arrested at these obstacles. This mechanism is referred as dynamic strain aging (SA) and leads to negative strain rate sensitivity (NRS). Our aim is to show that only NRS is sufficient to get the oscillatory behaviour of PLC as a type of material instability.

Even the first studies of material instability [1] distinguish two main types of it. These are called the divergence and the flutter. While divergence is treated as strain localization the nature and physical explanation of flutter remained an open question. In this work we show that in a special case NSR conditions are the same in both flutter and PLC phenomena [2, 3]. In the recent years there are several studies in this topic, but there are still several unsolved questions [4, 5]. To perform stability analysis the solid continuum will be studied as a dynamical sytem. For dynamical systems the basic definition of stability is formulated by Liapunov, which can be used for both finite and infinite dimensional systems.

The loss of stability can be performed on two basic ways. It can be a static or a dynamic bifurcation. Unfortunately, in the case of the rate independent constitutive equation

$$\sigma_{jk,k} = K^1_{jklm}\varepsilon_{lm},$$

where K^1_{jklm} is the tangent stiffness matrix, the dynamical system defined by the set of basic equations of the solid body show non-generic behaviour. That is, coexistent static and dynamic bifurcations may occur at the loss of stability. By introducing strain rate dependent terms into the constitutive equation the stability investigation can be performed as a generic stability investigation [6, 7]. Now we can study the real parts of the eigenvalues of differential operators defined by the fundamental equations of the continuum.

2 Dynamical System in Continuum Mechanics

Assume that state $S^0$of a solid continuum is studied, which can be identified by fixed values of the stress, strain or velocity fields $\sigma^0, \varepsilon^0, \upsilon^0$. The solid body can be described by a set of equations. By assuming small deformations ε these are:
the kinematic equations

$$\dot{\varepsilon}_{lm} = \upsilon_{lm} \tag{1}$$

where for all subscripts $l, m = 1, 2, 3$;
the equation of motion

$$\rho \ddot{\upsilon}_j = \sigma_{jk,k}, \tag{2}$$

where repeated indices denote summation; and the constitutive equation.

Assume that the constitutive has rate dependent terms

$$\sigma_{jk,k} = K^1_{jklm}\varepsilon_{lm} + K^2_{jklm}\varepsilon_{lm} \quad ; \tag{3}$$

where for coefficients we have

$$K^1_{jklm} = \frac{1}{2}\left[K_{jklm} + K_{jkml}\right] \quad \text{and} K^2_{jklm} = \frac{1}{2}\left[L_{jklm} + L_{jkml}\right]. \tag{4}$$

From Eq. 6 we define two differential operators for velocity field υ

$$\hat{K}_{jk}\upsilon_k := K_{jklm}\frac{\partial^2}{\partial x_m \partial x_l}\upsilon_k \text{ and } \hat{L}_{jk}\upsilon_k := L_{jklm}\frac{\partial^2}{\partial x_m \partial x_l}\dot{\upsilon}_k. \tag{5}$$

While (1), (2) and (3) form a set of basic equations of the solid continuum, they should be satisfied for values

$$\sigma^0_k, \quad \varepsilon^0_k, \quad \upsilon^0_k \tag{6}$$

at state S^0. Let us add arbitrary small perturbations to (6)

$$\sigma_k = \sigma^0_k + \Delta\sigma_k,\ \varepsilon_k = \varepsilon^0_k + \Delta\varepsilon_k,\ \upsilon_k = \upsilon^0_k + \Delta\upsilon_k. \tag{7}$$

From (1), (2), (3) and (5) a dynamical system can be formed as operator equation

$$\rho\ddot{\upsilon}_j = \hat{K}_{jk}\upsilon_k + \hat{L}_{jk}\dot{\upsilon}_k \tag{8}$$

can be derived. When the perturbed functions (7) are substituted into (8)

$$\rho\left(\ddot{\upsilon}^0_j + \Delta\ddot{\upsilon}_j\right) = \hat{K}_{jk}\left(\upsilon^0_k + \Delta\upsilon_k\right) + \hat{L}_{jk}\left(\dot{\upsilon}^0_k + \Delta\dot{\upsilon}_k\right)$$

is obtained. The system of (1), (2) and (3) is satisfied for functions (6), thus

$$\rho\ddot{\upsilon}^0_j = \hat{K}_{jk}\upsilon^0_k + \hat{L}_{jk}\dot{\upsilon}^0_k$$

is trivially valid. The remaining equation for the perturbations reads

$$\rho\Delta\ddot{\upsilon}_j = \hat{K}_{jk}\Delta\upsilon_k + \hat{L}_{jk}\Delta\dot{\upsilon}_k\ . \tag{9}$$

The stability investigation can be performed as usual in the theory of differential equations. Thus the eigenvalues λ of characteristic equations of (9)

$$\begin{aligned} y_{k+3} &= \lambda y_k\ , \\ \frac{1}{\rho}\left(\hat{K}_{jk}y_k + \hat{L}_{jk}y_{k+3}\right) &= \lambda y_{j+3} \end{aligned} \tag{10}$$

determine the stability-instability conditions. In (10) new variables

$$\begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \end{bmatrix} = \begin{bmatrix} \Delta\upsilon_1 & \Delta\upsilon_2 & \Delta\upsilon_3 & \Delta\upsilon_4 & \Delta\upsilon_5 & \Delta\upsilon_6 \end{bmatrix}$$

are used. There are homogeneous boundary conditions added to the system of partial differential equations (10), but no general analytic solution is possible. The two possibilities remained are: to perform numerical analysis, or to restrict the investigation to the uniaxial case.

3 The Stability Analysis

Let us consider a uniaxial solid of length l^0. Then the eigenvalue equation from its triaxial form (10) reads

$$\begin{aligned} y_4 &= \lambda y_1\ , \\ \lambda y_4 &= \tfrac{K_{1111}}{\rho}\tfrac{\partial^2 y_1}{\partial x_1^2} + \tfrac{L_{1111}}{\rho}\tfrac{\partial^2 y_4}{\partial x_1^2} \end{aligned} \tag{11}$$

with homogeneous boundaries

$$y_1(0) = y_4(0) = y_1\left(l^0\right) = y_4\left(l^0\right).$$

By substituting the first equation of (11) into the second one we get

$$\lambda^2 y_1 - \frac{K_{1111}}{\rho}\frac{\partial^2 y_1}{\partial x_1^2} - \lambda\frac{L_{1111}}{\rho}\frac{\partial^2 y_1}{\partial x_1^2} = 0 \tag{12}$$

as the characteristic equation. To obtain eigenfunctions of (12) at homogeneous boundaries functions

$$y_1(x_1) = Cexp\,(i\alpha x_1) \tag{13}$$

should be substituted into (12). Then the eigenvalue equation is

$$\lambda^2 + \lambda\frac{L_{1111}}{\rho}\alpha^2 + \frac{K_{1111}}{\rho}\alpha^2 = 0. \tag{14}$$

From the boundary conditions

$$\begin{aligned} y_1(0,t) &= 0 = A(t)\cos(0) + B(t)\sin(0)\ , \\ y_1\left(l^{0,}t\right) &= 0 = A(t)\cos\left(\alpha l^0\right) + B(t)\sin\left(\alpha l^0\right) \end{aligned}$$

are obtained, which implies

$$\alpha_k = \frac{k\pi}{l^0}. \tag{15}$$

The stability of state S^0depends on the real parts of solutions λ of (14). By substituting (15) into (14) we can easily derive solutions

$$\lambda_{k1,2} = \frac{-b\alpha_k^2 \pm \sqrt{b^2\alpha_k^4 - 4a\alpha_k^2}}{2}, \tag{16}$$

where notations

$$a = \frac{K_{1111}}{\rho} \text{ and } b = \frac{L_{1111}}{\rho} \tag{17}$$

are used.

The stability conditions of the theory of dynamical systems are:

– the state is stable (in the sense of Lyapunov), if for all eigenvalues

$$Re\lambda_j < 0,$$

– the state is unstable, if there is at least one unstable eigenvalue λ_u for which

$$Re\lambda_u > 0,$$

– the state is on the stability boundary, if there are only critical and stable eigenvalues:

$Re\lambda_c = 0$ and for the others $Re\lambda_j < 0, \quad j \neq c$.
The next part shows how the stability boundary can be studied.

4 Material Instability Modes and Dynamical Systems

In this part we concentrate on states of solids, at which stability boundary

$$Re\lambda_c = 0$$

is reached. Two possibilities are present.

The case called the static bifurcation. Then we have a zero eigenvalue

$$Re\lambda_c = 0, \quad Im\lambda_c = 0.$$

Now the uniqueness of the solution (state S^0) is lost. From (16) and (17) we find that the static bifurcation condition is

$$K_{1111} = 0,$$

which is exactly the same as the classical strain localization condition.

The other possible case on the stability boundary is to have a nonzero imaginary part, that is, a nonzero eigenvalue:

$$Re\lambda_c = 0, \quad Im\lambda_c \neq 0.$$

Under such conditions no loss of uniqueness is present. However, the Lyapunov stability of S^0 is lost. This phenomenon is known as the dynamic bifurcation. From (16) and (17) we find that the dynamic bifurcation condition is

$$L_{1111} = 0.$$

At dynamic bifurcation in (16) the most important part is the expression under square root

$$b^2\alpha_k^4 - 4a\alpha_k^2.$$

By substituting (15) and (17) into it at the first critical eigenvalue $k = 1$

$$L_{1111}^2\left(\frac{\pi}{l^0}\right)^4 - 4K_{1111}\left(\frac{\pi}{l^0}\right)^2,$$

that is,

$$K_{1111} = \frac{1}{4}\left(\frac{\pi}{l^0}\right)^2 L_{1111}^2. \tag{18}$$

The results and graph (18) could be presented in Fig. 1.

We can observe both static and dynamic bifurcation conditions and curve (18), which separates oscillatory and non-oscillatory behaviours. When the parameters determine a point in the positive quadrant, there may be a transient oscillatory behaviour with decreasing amplitude. It can be persistent by crossing the axis of the dynamic bifurcation. Here a negative rate dependence can be observed. These are exactly the properties of the Portevin-Le Chatelier effect. We may conclude that in the uniaxial case studied here PLC effect is caused by a dynamic bifurcation.

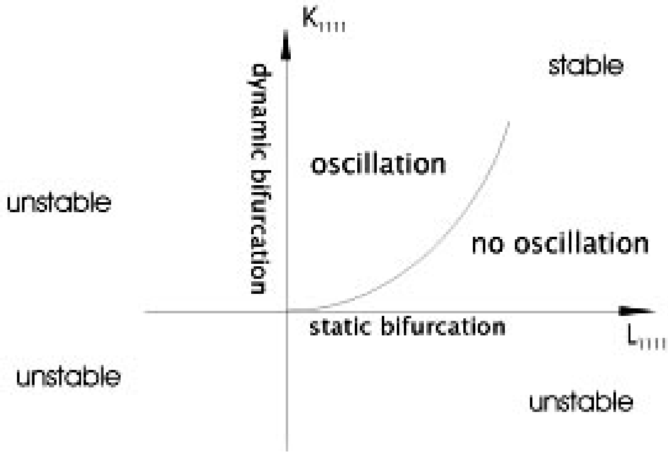

Fig. 1. The stability chart in uniaxial case

5 Summary

In this work the basic equations of solid continua were transformed into a generalized dynamical system. The stability of state S^0 is investigated. We find that two types of loss of stability are possible. The one, called the static bifurcation, proved to be the same as the strain localization. However, the other instability mode, the dynamic bifurcation, shows essential similarities to the Portevin-Le Chatelier effect. Both exhibit an oscillatory behaviour, and in both cases negative strain rate sensitivity is an essential condition. The main result of this study is that Portevin-Le Chatelier effect can be identified as a dynamic bifurcation at the negative strain rate sensitivity condition. Remark that the dynamic strain aging scenario of PLC phenomenon has some "sufficient" nature. Besides the negative strain rate sensitivity SA adds a micromechanical stop-and-go effect.

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