NUMERICAL ANALYSIS OF TWO DIMENSIONAL HEAT CONDUCTIVITY IN STEADY STATE REGIME

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Abstract

Solving the differential equation of the heat conduction the temperature in each point of the body can be determined. However, in the case of bodies with boundary surface of sophisticated geometry no analytic method can be used. In this case the use of numerical methods becomes necessary. The finite element method is based on the integral equation of the heat conduction. This is obtained from the differential equation using variation calculus. The temperature values will be calculated on the finite elements. Then, based on these partial solutions, the solution for the entire volume will be determined. Using this method we can divide into elements also fields with any border. In this paper the temperature distribution in an orthotropic body and in a pipe insulation is analysed using a software developed by the author.

Keywords: finite element, heat conduction, mathematical model, variation calculus.

1. Introduction

To establish the temperature distribution in a solid body, in many cases numerical methods should be used in heat transfer calculations. The most used methods are the finite difference and the finite element method.

The finite difference method is based on the differential equation of the heat conduction, which is transformed into a numerical one. The temperature values will be calculated in the nodes of the network. Using this method convergence and stability problem can appear.

The finite element method is based on the integral equation of the heat conduction. This is obtained from the differential equation using variation calculus. The temperature values will be calculated on the finite elements. Then, based on these partial solutions, the solution for the entire volume will be determined. Using this method we can divide into elements also fields with unregulated border.

In this paper the temperature distribution is analysed in a solid body, with linear variation of the properties, using the finite element method.

2. The Analytical Model of the Heat Conduction

The temperature in a solid body is a function of the time and space coordinates. The points corresponding to the same temperature value belong to an isothermal surface. This surface in a two dimensional Cartesian system is transformed into an isothermal curve.

The *heat flow rate Q* represents the heat quantity through an isothermal surface S in the time unit:

$$Q = \int_S q \, \mathrm{d}s, \tag{1}$$

where the density of heat flow rate q is given by the Fourier law:

$$q = -\lambda \frac{\partial t}{\partial n} = -\lambda \operatorname{grad} t, \tag{2}$$

where λ is the thermal conductivity of the material.

The thermal conductivity of the building materials is the function of the temperature and the variation can accordingly be expressed as:

$$\lambda = \lambda_0[1 + b(t - t_0)], \tag{3}$$

where: λ_0 is the thermal conductivity corresponding to the t_0 temperature; b – material constant.

If there is heat conduction within an inhomogeneous and anisotropy material, considering the heat conductivity constant in time, the temperature variation in space and time is given by the Fourier equation:

$$\rho c \frac{\partial t}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda_x \frac{\partial t}{\partial x}\right) + \frac{\partial}{\partial x}\left(\lambda_y \frac{\partial t}{\partial y}\right) + \frac{\partial}{\partial z}\left(\lambda_z \frac{\partial t}{\partial z}\right) + Q_0, \tag{4}$$

where: t is the temperature; τ – time; ρ – density of the material; c – specific heat of the material; λ_x, λ_y, λ_z – thermal conductivity in the directions x, y and z; Q_0 – power of the internal sources.

To solve the differential equations it is necessary to have supplementary equations. These equations contain the geometrical conditions of the interpretation field, the starting conditions (at $\tau = 0$) and the boundary conditions. The boundary conditions describe the interaction between the studied field and the surroundings. In function of these interactions different conditions are possible:

- the *Dirichlet* (type I) boundary conditions give us the temperature values on the boundary surface S_t of the studied field (*Fig. 1*) like a space function constant or variable in time:

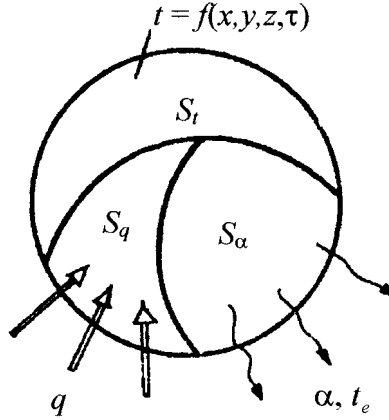$$t = g(x, y, z, \tau) \tag{5}$$

$$t = f(x,y,z,\tau)$$

Fig. 1. Boundary conditions

- the *Newton* (type II) boundary condition gives us the value of the density of heat flow rate trough the S_q boundary surface of the studied field:

$$q = \lambda_x \frac{\partial t}{\partial x} n_x + \lambda_y \frac{\partial t}{\partial y} n_y + \lambda_z \frac{\partial t}{\partial z} n_z, \qquad (6)$$

where n_x, n_y, n_z are the cosine directors corresponding to the normal direction on the S_q boundary surface.

- the *Cauchy* (type III) boundary condition gives us the external temperature value and the convective heat transfer coefficient value between the S_α boundary surface of the body and the surrounding fluid:

$$\alpha(t - t_a) = \lambda_x \frac{\partial t}{\partial x} n_x + \lambda_y \frac{\partial t}{\partial y} n_y + \lambda_z \frac{\partial t}{\partial z} n_z, \qquad (7)$$

where: α is the convective heat transfer coefficient from S_α to the fluid (or inversely); t_a – the fluid temperature.

The analytical model described by the *Eqs.* (4)…(7) can be completed with the material equations which provide us information about variation of the material properties depending on temperature. In the case of materials with linear physical properties, this equations ($\lambda = $ const.) are not used in the model.

Solving the differential equation of the heat conduction (4) we can determine the temperature values in each point of the body. However, in the case of bodies with boundary surface of sophisticated geometry, the *Eq.* (4) cannot be solved using analytical methods. In this case numerical methods should be applied. The increasing availability of computers has also lead into the direction of more frequent use of these methods.

To use the finite element method, the transformation of the *Eqs.* (4)…(7) into integral model is necessary. To realize this transformation we can use *variation calculus*.

3. The Finite Element Method

The temperature $t(x, y, z, \tau)$ which represent a solution for the differential heat conduction *Eq.* (4) and for *Eqs.* (5)…(7) conditions, also represents a solution for the steady state equation of the V field:

$$\delta\Gamma = 0 \tag{8}$$

which is equivalent, from mathematical point of view, with the *Eqs.* (4)…(7) and where Γ is the functional of the heat conduction.

Consequently, in the integration of the *Eq.* (4) with *Eqs.* (5)…(7) boundary conditions are equivalent with the minimization of the Γ functional:

$$\Gamma = \int_V \frac{1}{2}\left[\lambda_x\left(\frac{\partial t}{\partial x}\right)^2 + \lambda_y\left(\frac{\partial t}{\partial y}\right)^2 + \lambda_z\left(\frac{\partial t}{\partial z}\right)^2\right]\mathrm{d}V + \int_V\left(\rho c\frac{\partial t}{\partial \tau} - Q_0\right)t\,\mathrm{d}V$$
$$- \int_{S_q} qt\,\mathrm{d}S + \int_{S_\alpha} \alpha t\left(\frac{1}{2}t - t_a\right)\mathrm{d}S \tag{9}$$

Q_0 is positive when the internal sources produce heat and negative when these sources absorb heat; q is positive when the body receives heat and negative when the body yields heat to the surrounding fluid; α is positive on the surfaces where the heat transfer happens from the body to the fluid and it is negative inversely.

If we assume a steady state heat transfer and the body does not content internal heat sources the *Eq.* (9) can be expressed as:

$$\Gamma = \int_V \frac{1}{2}\left[\lambda_x\left(\frac{\partial t}{\partial x}\right)^2 + \lambda_y\left(\frac{\partial t}{\partial y}\right)^2 + \lambda_z\left(\frac{\partial t}{\partial z}\right)^2\right]\mathrm{d}V$$
$$- \int_{S_q} qt\,\mathrm{d}S + \int_{S_\alpha} \alpha t\left(\frac{1}{2}t - t_a\right)\mathrm{d}S. \tag{10}$$

Additionally, if the process is done within an isotropic and homogenous material, we obtain:

$$\Gamma = \int_V \frac{\lambda}{2}\left[\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2\right]\mathrm{d}V$$
$$- \int_{S_q} qt\,\mathrm{d}S + \int_{S_\alpha} \alpha t\left(\frac{1}{2t} - t_a\right)\mathrm{d}S. \tag{11}$$

Taking into account the Dirichlet boundary conditions the *Eq.* (11) can accordingly expressed as:

$$\Gamma = \int_V \frac{1}{2}\left[\left(\frac{\partial t}{\partial x}\right)^2 + \left(\frac{\partial t}{\partial y}\right)^2 + \left(\frac{\partial t}{\partial z}\right)^2\right]\mathrm{d}V. \tag{12}$$

The minimization of the functional is done correspondingly to each finite element. The solution for the entire field is obtained joining the partial solutions.

4. Two-Dimensional Steady State Heat Conduction

Though the heat conduction is carried out within three-dimensional bodies, the temperature distribution variation is significant only in certain directions. Thus, the analysis of temperature distribution in bars, plain or cylindrical walls is done using a two-dimensional model.

In the steady state heat transfer processes the temperature does not depend on the time, thus in the *Eq.* (9) $\partial t/\partial \tau = 0$. In addition, at two-dimensional problems, the temperature does not vary on z direction, thus $\partial t/\partial z = 0$.

4.1. General Equations of the Finite Element Method

In our case the *Eq.* (9) can be expressed as:

$$\Gamma = \int_V \left\{\frac{1}{2}\left[\lambda_x\left(\frac{\partial t}{\partial x}\right)^2 + \lambda_y\left(\frac{\partial t}{\partial y}\right)^2 + \lambda_z\left(\frac{\partial t}{\partial z}\right)^2\right] - Q_0 t\right\}\mathrm{d}V$$
$$- \int_{S_q} qt\,\mathrm{d}S + \int_{S_\alpha} \alpha t\left(\frac{1}{2}t - t_a\right)\mathrm{d}S. \tag{13}$$

Taking into account that the temperature function is not continuous on the entire field, the *Eq.* (13) can be integrated only on the finite elements. On the entire field the functional Γ can be written as a sum of m functionals Γ^e, where m is the number of finite elements:

$$\Gamma = \sum_{e=1}^{m} \Gamma^e \tag{14}$$

$$\Gamma = \sum_{e=1}^{m}\left\{\int_{V_e}\frac{1}{2}\left[\lambda_x\left(\frac{\partial t^e}{\partial x}\right)^2 + \lambda_y\left(\frac{\partial t^e}{\partial y}\right)^2\right]\mathrm{d}V\right.$$
$$\left. - \int_{V_e} Q_0 t^e\,\mathrm{d}V - \int_{S_{qe}} qt^e\mathrm{d}S + \int_{S_{\alpha e}} \alpha t^e\left(\frac{1}{2}t^e - t_a\right)\mathrm{d}S\right\}. \tag{15}$$

In the *Eq.* (15) the 'e' exponent refers to a finite element.

For a given finite element the temperature t^e can be calculated based on the temperature values in the nodes:

$$t^e = N_1 t_1 + N_2 t_2 + \cdots + N_n t_n = [N]\{t\}_e \tag{16}$$

where: n is the number of the finite element nodes; $[N]$ – form matrix of the finite element; $\{t\}_e$ – vector of the temperature values in the nodes.

In the expression (15) appear the partial derivates of the temperature, therefore the *Eq.* (16) should be derived:

$$\{B\} = \left\{ \begin{array}{c} \dfrac{\partial t^e}{\partial x} \\[2mm] \dfrac{\partial t^e}{\partial y} \end{array} \right\} = \left[\begin{array}{cccc} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial x} & \cdots & \dfrac{\partial N_n}{\partial x} \\[2mm] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_2}{\partial y} & \cdots & \dfrac{\partial N_n}{\partial y} \end{array} \right] \left\{ \begin{array}{c} t_1 \\ t_2 \\ t_n \end{array} \right\} = [J]\{t\}_e. \tag{17}$$

If the thermal conductivities are written in matrix form:

$$[D] = \left[\begin{array}{cc} \lambda_x & 0 \\ 0 & \lambda_y \end{array} \right] \tag{18}$$

then the *Eq.* (15) can accordingly be expressed as:

$$\Gamma^e = \int_{V_e} \frac{1}{2}([J]\{t\}_e)^T [D][J]\{t\}_e \, dV - \int_{V_e} Q_0 [N]\{t\}_e \, dV - \int_{S_{qe}} q[N]\{t\}_e \, dS$$
$$+ \int_{S_{\alpha e}} \frac{\alpha}{2}([N]\{t\}_e)^2 dS - \int_{S_{\alpha e}} \alpha t_a [N]\{t\}_e \, dS. \tag{19}$$

Because

$$([J]\{t\}_e)^T = \{t\}_e^T [J]^T$$
$$([N]\{t\}_e)^2 = ([N]\{t\}_e)^T ([N]\{t\}_e) = \{t\}_e^T [N]^T [N]\{t\}_e \tag{20}$$

the *Eq.* (19) can be expressed as:

$$\Gamma^e = \int_{V_e} \frac{1}{2}\{t\}_e^T [J]^T [D][J]\{t\}_e \, dV - \int_{V_e} Q_0 [N]\{t\}_e \, dV - \int_{S_{qe}} q[N]\{t\}_e \, dS$$
$$+ \int_{S_{\alpha e}} \frac{\alpha}{2}\{t\}_e^T [N]^T [N]\{t\}_e \, dS - \int_{S_{\alpha e}} \alpha t_a [N]\{t\}_e dS. \tag{21}$$

If we derive the matrix *Eq.* (21) the further equation is obtained:

$$\frac{\partial \Gamma^e}{\partial \{t\}_e} = \left(\int_{V_e} [J]^T [D][J] \, dV + \int_{S_{\alpha e}} \alpha [N]^T [N] \, dS \right) \{t\}_e - \int_{Ve} Q_0 [N]^T \, dV$$
$$- \int_{S_{qe}} q[N]^T \, dS - \int_{S_{\alpha e}} \alpha t_a [N]^T \, dS. \tag{22}$$

Because $dV = h\,dA$ and $dS = h\,dL$, where h is the thickness of the finite element, dA – aria of the finite element, dL – length of the finite element side

$$\frac{\partial \Gamma^e}{\partial \{t\}_e} = h\left(\int_{A_e} [J]^T[D][J]\,dA + \int_{L_e} \alpha[N]^T[N]\,dL\right)\{t\}_e - h\int_{A_e} Q_0[N^T\,dA$$

$$-h\int_{L_e} q[N]^T\,dL - h\int_{L_e} \alpha t_a[N]^T\,dL. \tag{23}$$

The finite element thickness h is considered constant and equal with 1 m.

The *Eq.* (23), can be written as in compressed form:

$$\frac{\partial \Gamma^e}{\partial \{t\}_e} = [k]\{t\}_e - \{f\}, \tag{24}$$

where:

$$[k] = h\int_{A_e} [J]^T[D][J]\,dA + h\int_{L_{\alpha e}} \alpha[N]^T[N]\,dL \tag{25}$$

$$\{f\} = h\int_{A_e} Q_0[N]^T\,dA + h\int_{L_{qe}} q[N]^T\,dL + h\int_{L_{\alpha e}} \alpha t_a[N]^T\,dL, \tag{26}$$

where: $[k]$ is the matrix of the heat conduction corresponding to a finite element, the first term is related to conduction and the second term to convection on the $L_{\alpha e}$ side of the $S_{\alpha e}$ boundary surface; $\{f\}$ – vector of heat sources containing the internal sources Q_0, the density of heat flow rate q on the S_{qe} boundary surface and convection on the S_α boundary surface.

The minimization of the Γ functional supposes the equality with zero of the first derivate in each point of the studied field.

$$\frac{\partial \Gamma}{\partial \{t\}} = \frac{\partial}{\partial \{t\}} \sum_{e=1}^m \Gamma^e = \sum_{e=1}^m \frac{\partial \Gamma^e}{\partial \{t\}_e}. \tag{27}$$

Introducing *Eq.* (24) in (27) we obtain the equation system corresponding to the entire field:

$$[K]\{t\} = \{F\}, \tag{28}$$

where:

$$[K] = \sum_1^m [k]; \qquad \{F\} = \sum_1^m \{f\} \tag{29}$$

where: $[K]$ is matrix of heat conduction of the entire field; $\{F\}$ – vector of heat sources corresponding to studied field; $\{t\}$ – vector of unknown temperatures.

The *Eq.* (28) represents the form with finite element of the differential equation of heat conduction, which contains a number of equations equal to the number of the nodes with unknown temperature values.

4.2. Matrix of the Heat Conduction

If we use finite elements with triangle form in a certain point of the finite element, using the relation (16) the t^e temperature (*Fig. 2*), can be written as:

$$t^e = N_i t_i + N_j t_j + N_k t_k = [N_i \ N_j \ N_k] \left\{ \begin{array}{c} t_i \\ t_j \\ t_k \end{array} \right\} = [N]\{t\}_e, \qquad (30)$$

where: t_i, t_j, t_k are the temperatures in i, j, k nodes (nodes of triangle finite element); $[N]$ – form matrix of the finite element [6].

The conduction matrix of a finite element is:

$$[k] = [k_1] + [k_2] \qquad (31)$$

where:

$$[k_1] = h \int_{A_e} [J]^T [D][J] \, dA; \qquad [k_2] = h\alpha \int_{L_{\alpha e}} [N]^T [N] \, dL. \qquad (32)$$

The $[J]$ matrix, using the relation (17) can be expressed as:

$$\{B\} = \left\{ \begin{array}{c} \dfrac{\partial t^e}{\partial x} \\[2mm] \dfrac{\partial t^e}{\partial y} \end{array} \right\} = \left[\begin{array}{ccc} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\[2mm] \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{array} \right] \left\{ \begin{array}{c} t_i \\ t_j \\ t_k \end{array} \right\} = [J]\{t\}_e. \qquad (33)$$
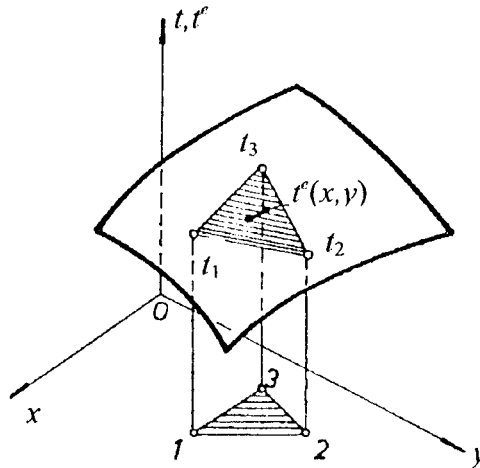

Fig. 2. Finite element with triangle form

If we derive the elements of the form matrix:

$$[J] = \begin{vmatrix} \dfrac{\partial N_i}{\partial x} & \dfrac{\partial N_j}{\partial x} & \dfrac{\partial N_k}{\partial x} \\[2mm] \dfrac{\partial N_i}{\partial y} & \dfrac{\partial N_j}{\partial y} & \dfrac{\partial N_k}{\partial y} \end{vmatrix} = \frac{1}{2A_e} \begin{bmatrix} b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix}, \tag{34}$$

where A_e is the area of the finite element, and the b respective c can be written as [1]:

$$b_i = y_i - y_k; \qquad b_j = y_k - y_i; \qquad b_k = y_i - y_j$$
$$c_i = x_k - x_j; \qquad c_j = x_i - x_k; \qquad c_k = x_j - x_i. \tag{35}$$

Consequently, the $[J]$ matrix is constant.

Because the λ_x and λ_y thermal conductivities do not vary for a finite element, the $[D]$ matrix is also constant, thus:

$$[k_1] = h \int_{A_e} [J]^T [D][J] \, dA = h[J]^T [D][J] A_e. \tag{36}$$

Introducing the expression of $[J]$ matrix from the *Eq.* (34) and the expression of $[D]$ matrix from the *Eq.* (18):

$$[k_1] = \frac{h}{4A_e} \begin{bmatrix} \lambda_x b_i b_i + \lambda_y c_i c_i & \lambda_x b_i b_j + \lambda_y c_i c_j & \lambda_x b_i b_k + \lambda_y c_i c_k \\ \lambda_x b_j b_i + \lambda_y c_j c_i & \lambda_x b_j b_j + \lambda_y c_j c_j & \lambda_x b_j b_k + \lambda_y c_j c_k \\ \lambda_x b_k b_i + \lambda_y c_k c_i & \lambda_x b_k b_j + \lambda_y c_k c_j & \lambda_x b_k b_k + \lambda_y c_k c_k \end{bmatrix} \tag{37}$$

The matrix $[k_2]$ from the *Eq.* (31) can be written as:

$$[k_2] = h\alpha \int_{L_{\alpha e}} \begin{bmatrix} N_i N_i & N_i N_j & N_i N_k \\ N_j N_i & N_j N_j & N_j N_k \\ N_k N_i & N_k N_j & N_k N_k \end{bmatrix} dL. \tag{38}$$

Using the L – natural coordinates and considering that convective heat transfer exists on the jk side of the finite element, we obtain:

$$[k_2] = h\alpha \int_{L_{\alpha e}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & L_j L_j & L_j L_k \\ 0 & L_j L_k & L_k L_k \end{bmatrix} dL. \tag{39}$$

To solve the *Eq.* (39), the following relation should be used:

$$\int_{X_i}^{X_j} L_i^\alpha L_j^\beta \, dx = \frac{\alpha! \beta! (X_j - X_i)}{(\alpha + \beta + 1)!}. \tag{40}$$

Consequently for products with the same indices j or k is obtained:

$$\int_{L_{\alpha e}} L_j L_j \, dL = \int_{L_{\alpha e}} L_k L_k \, dL = \int_{L_{\alpha e}} L_j^2 \, dL = \frac{2!0!}{(2+0+1)!} L_{\alpha e} = \frac{L_{\alpha e}}{3} \tag{41}$$

and for products with different indices j and k is obtained:

$$\int_{L_{ae}} L_j L_k \, dL = \int_{L_{ae}} L_k L_j \, dL = \frac{1!1!}{1+1+1)!} L_{ae} = \frac{L_{ae}}{6}. \tag{42}$$

Introducing into *Eq.* (39):

$$[k_2] = \frac{h\alpha L_{ae}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}. \tag{43}$$

If convective heat transfer exists the ij or ki sides of the finite element are:

$$[k_2] = \frac{h\alpha L_{ae}}{6} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \qquad [k_2] = \frac{h\alpha L_{ae}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}. \tag{44}$$

The matrix $[k_2]$ exists only in the case when at least, on one side of the finite element heat transfer is realized by convection.

4.3. *Vector of the Heat Sources*

This vector is based on the *Eq.* (26) from three terms, which can be calculated using the L – natural coordinates. Supposing that Q_0 is constant for a finite element, using the followed relation:

$$\int_{A_e} L_i^\alpha L_j^\beta L_k^\gamma \, dA = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} 2A_e \tag{45}$$

we obtain that:

$$\{f_Q\} = h \int_{A_e} Q_0 [N]^T \, dA = h Q_0 \int_{A_e} \begin{Bmatrix} N_i \\ N_j \\ N_k \end{Bmatrix} dA = h Q_0 \int_{A_e} \begin{Bmatrix} L_i \\ L_j \\ L_k \end{Bmatrix} dA$$

$$= \frac{h Q_0 A_e}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}. \tag{46}$$

The second term, for a certain density of heat flow rate, corresponds to the heat transfer on the boundary surface of the studied field. Supposing that the body receives the heat flow through $L_{ki} = L_{qe}$ side of the finite element, using the relation (40) we obtain:

$$\{f_q\} = h \int_{L_{qe}} q [N]^T \, dL = hq \int_{L_{qe}} \begin{Bmatrix} N_i \\ 0 \\ N_k \end{Bmatrix} dL = hq \int_{L_{qe}} \begin{Bmatrix} L_i \\ 0 \\ L_k \end{Bmatrix} dL$$

$$= \frac{hq L_{qe}}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}. \tag{47}$$

The third term, from the *Eq.* (26), corresponds to convective heat transfer on the jk ($L_{jk} = L_{\alpha e}$) side of the finite element. Using the relation (40) we obtain

$$\{f_\alpha\} = h \int_{L_{\alpha e}} \alpha t_a [N]^T \, \mathrm{d}L = h\alpha t_a \int_{L_{\alpha e}} \left\{ \begin{array}{c} 0 \\ N_j \\ N_k \end{array} \right\} \mathrm{d}L = h\alpha t_a \int_{L_{\alpha e}} \left\{ \begin{array}{c} 0 \\ L_j \\ L_k \end{array} \right\} \mathrm{d}L$$

$$= \frac{h\alpha t_a L_{\alpha e}}{2} \left\{ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right\}. \tag{48}$$

It could be observed that the element zero in the vector (47) and (48) can occupy any position, corresponding to the side of finite element with heat transfer.

Based on the equation systems obtained for the finite elements, they can realize the equation system for the entire studied field. This system can be solved using analytical or iterative methods.

In practice there are many situations where it is indispensable to know the temperature distribution in a body (e.g. in different mechanic and electronic components). In civil engineering it is important to analyse the temperature distribution in thermal bridges, in pipe walls, in insulation materials. In present there are different programms on the software market which permit numerical analysis of the temperature distribution (e.g. WAEBRU) but these programms are too expensive and our department cannot buy them. In this context to analyze the temperature distribution in a solid body under steady state heat transfer regime using the mathematical model presented above the ATEFS software has been developed by authors of this article. The equation system is solved using the Gauss method.

5. Application

5.1. *Temperature Distribution in an Orthotropic Body*

The temperature distribution is analysed in a solid body 500×400 mm sectional dimensions (*Fig. 3*). The body receives heat flow on two sides: $q_x = 2320$ W, $q_y = 928$ W. On the other two sides the body transmit heat by convection $\alpha_x = \alpha_y = 23.2$ W/(m²·K). The material of the body has orthotropic properties with the following values of the thermal conductivities: $\lambda_x = 11.6$ W/(m·K), $\lambda_y = 5.8$ W/(m·K).

The studied field is divided into 40 finite elements with 30 nodes.

Running the ATEFS program the following values of temperatures in the nodes have been obtained:

The temperature distribution in the body is presented in *Fig. 4*.

Wood is the only one orthotropic material which is used in civil engineering, and this property should be taken into account at heat loss determination of the buildings (e.g. heat flow direction perpendicular or parallel on the fiber).
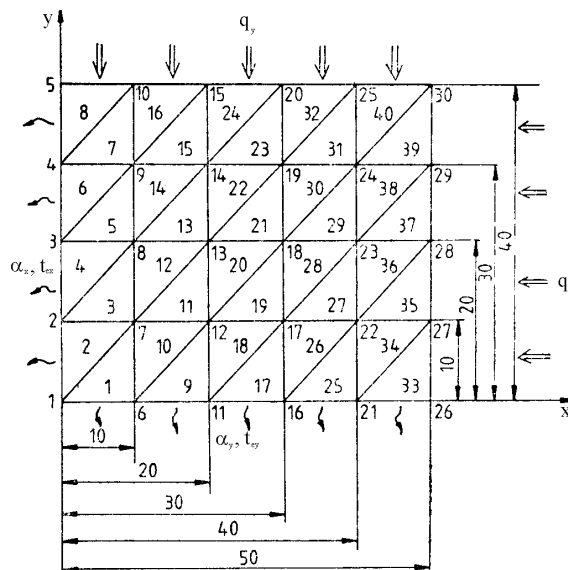
Fig. 3. The studied field

Table 1. Temperature values in the nodes

NN	t [°C]	NN	t [°C]	NN	t [°C]
0	1	2	3	4	5
1	49.06	11	67.58	21	90.67
2	65.54	12	89.6	22	119.10
3	79.95	13	109.02	23	142.15
4	93.39	14	126.45	24	161.68
5	107.70	15	142.75	25	178.73
6	58.15	16	78.09	26	107.37
7	77.09	17	103.4	27	137.49
8	94.15	18	124.89	28	161.19
9	109.88	19	143.58	29	181.01
10	125.44	20	160.35	30	198.15

5.2. *Temperature Distribution in Pipe Insulation*

Using the ATEFS programme the temperature distribution in pipe insulation was analysed (*Fig. 5*). The calculus was made for a pipe with 800 mm nominal diam-
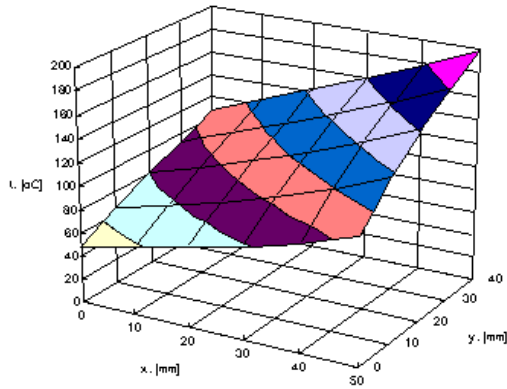
Fig. 4. The temperature distribution in the studied body

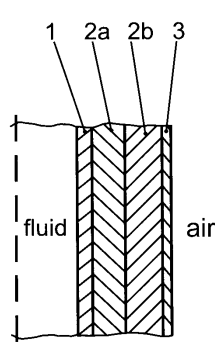eter and the warm water temperature was 150 °C. The ambient temperature was considered 1 °C.

Fig. 5. Structure of the analysed insulation: 1 – pipe wall; 2a, 2b – insulation layers; 3 – protection coat

To obtain results which describe the real situation as exactly as possible the convective heat transfer coefficient on the external insulation surface was considered variable with values between 10 and 25.6 W/(m²·K).

In *Figs. 6* and *7* the analysed field and the temperature distribution are presented in the pipe section. It can be observed that due to the variable boundary conditions on the insulation surface the isotherm curves are not circular curves which are obtained when the classical calculus is used.
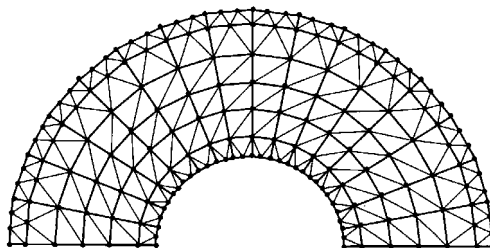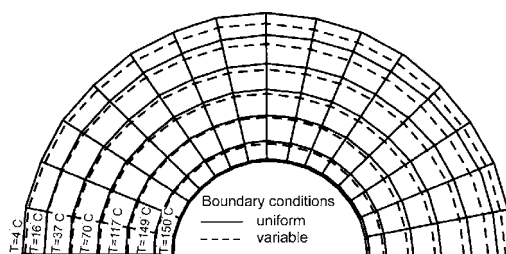
Fig. 6. The analysed field

Fig. 7. Temperature distribution in pipe insulation

6. Conclusions

The numerical analysis with the finite element method represents an efficient way to obtain the temperature distributions in steady state or unsteady state conductive heat transfer processes. Using the presented method, different simulation programms could be realized what makes it possible to effectuate a lot of different numerical experiments of practical problems.

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