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PERTURBATION AND WAVE DYNAMICAL METHODS IN MATERIAL INSTABILITY

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Abstract

This paper aims to propose a unified theoretical background for the investigation of unstable behaviour onset of inelastic solids. By using the theory of dynamical systems we present definitions and a method, which could be a general tool in classifying the ways of loss of stability. Then the main generic methods published in studies of the literature are investigated.

Keywords: material instability, localization, flutter, dynamical systems.

1. Introduction

The roots of the classical stability/instability concepts are generalizations of either experimental observations or physical concepts of conservative (i.e. elastic) systems. Such problems appeared at the mathematical formulation of inelastic deformation. The well-known DRUCKER's postulate [1] is a good example of the first approach. There is a three axial generalization given from the results of the uniaxial tensile test.

A similar case can be found as a starting point in the pioneering work of RICE [2]. The author builds the idea of strain localization on several shear band observations in fracture phenomena of rock mechanics. A similar and more general study was published by him and his co-authors [3, 4], which consider both the shear banding and necking (a generic phenomenon at standard tensile test of metals) types of localization.

The theory worked out concentrates on the propagation of harmonic waves in the field variables (for example in velocity field

$$v = v^* \exp\left(i\left(x_k n_k - ct\right)\right)$$

is applied) and uses the fourth order constitutive tangent stiffness tensor $\mathbf{H}$ being a generalization of the elasticity tensor (cf. LUBLINER's textbook [5] for details). Such tensors can define the acoustic tensor

$$\mathbf{A} = \mathbf{n} \cdot \mathbf{H} \cdot \mathbf{n},$$

where **n** denotes the normal vector of the wave front and its strong ellipticity (all the eigenvalues c^2 are positive) is used as stability condition. While this condition depends only on constitutive relations being the mathematical model of the material, the stability defined is called the material stability (or in case of 'loss of strong ellipticity' the material instability).

There are two basic ways to lose the stability of a certain material state at the quasi-static increase of loading. The one is the localization mentioned above (some authors call it the divergence instability), and the other is the flutter [2]. The difference is in the eigenvalue distribution of the acoustic tensor. In case of a divergence there is a zero eigenvalue of tensor **A**,

$$c^2 = 0,$$

while at flutter instability c^2 gets complex values.

In this paper we propose the use of the stability definitions of the theory of dynamical systems and Lyapunov's indirect methods for the investigations of the way of loss of stability. Instead of the classification into divergence and flutter instabilities we propose to use the basic classes of elementary bifurcation theory [6]. These are the static and dynamic bifurcation being essential in the stability analysis of dynamical systems.

2. The Basic Formulation of the Field Equations for Material Instability in a Uniaxial Case

Material instability problems concentrate on a state of the material denoted by S^0 satisfying the set of fundamental field equations of solid continua. These are the kinematic equation the Cauchy's equations of motion and the constitutive equation.

While stability is considered to be a local property, we may add infinitesimal perturbations to the basic variables and use small deformation theory for the set of basic equations written for the field of perturbations. For the sake of simplicity the study is restricted to uniaxial problems, but a quite similar approach is possible even for a general three-axial case.

We may denote as (small) perturbation variables of state S^0 by $\upsilon, \varepsilon, \sigma$ for the fields of velocity, strain and the symmetric stress. Then the basic equations in rate form are

$$\dot{\varepsilon} = \frac{1}{2}\left(\dot{\upsilon} \circ \nabla + \nabla \circ \dot{\upsilon}\right), \tag{1}$$

$$\rho \dot{\upsilon} = \sigma \nabla, \tag{2}$$

$$F\left(\dot{\varepsilon}, \ddot{\varepsilon}, \nabla^2 \dot{\varepsilon}, \dot{\sigma}, \ddot{\sigma}, \ldots\right) = 0, \tag{3}$$

where ρ is mass density as usual.

Eq. (3) is a general and indeterminate type of the constitutive equations. (Remark that such kind of formulation is called the rate-form of

$$\tilde{F}\left(\varepsilon, \dot{\varepsilon}, \nabla^2\varepsilon, \sigma, \dot{\sigma}, \dots\right) = 0.)$$

We use (3) to demonstrate the still unsolved problem of constitutive modelling of real engineering materials. It has two basic aspects. The one is the form of it being widely studied in literature, while the other is also an important one. This second difficulty is to select the basic functions (independent variables), on which the constitutive function depends. There are some conventions to get solutions like the use of standard rheological models, visco-(elasto-)plastic (models of) materials, or gradient dependent materials etc. The need of numerical consistence [7] (to exclude numerical instability like mesh sensitivity or cases of co-existent flutter and divergence) leads to the inclusion of rate or second gradient dependent terms [8] especially in numerical post-localization investigations.

To the set of perturbation field equations we should add homogeneous boundary conditions and the question of stability means an investigation of the behaviour of the boundary value problem formed by the set of *Eqs.* (1), (2) and (3).

3. Stability in Sense of the Theory of the Dynamical Systems

For performing a stability analysis let us define a dynamical system of infinite dimension [9]. For this reason we assume that *Eqs.* (1), (2) and a rate and second gradient dependent form of (3) can be transformed into the perturbation velocity field v satisfying homogeneous boundary conditions. Then

$$\begin{aligned} 2\rho\dddot{v} &= C^1\left(v\circ\nabla + \nabla\circ v\right)\nabla + C^2\left(\dot{v}\circ\nabla + \nabla\circ\dot{v}\right)\nabla \\ &\quad + C^3\nabla^2\left(v\circ\nabla + \nabla\circ v\right)\nabla + C^4\rho\ddot{v} \end{aligned} \tag{4}$$

is obtained, where piecewise constant coefficients C^i, $i = 1, \dots, 4$ are determined by the partial derivatives of the constitutive functional (see the left hand side of (3))

$$C^1 = \left.\frac{\partial F}{\partial\dot{\varepsilon}}\right|_{S^0}, \dots, C^2 = \left.\frac{\partial F}{\partial\ddot{\varepsilon}}\right|_{S^0}, \dots, C^3 = \left.\frac{\partial F}{\partial\nabla^2\dot{\varepsilon}}\right|_{S^0}, \dots, C^4 = \left.\frac{\partial F}{\partial\dot{\sigma}}\right|_{S^0}$$

at the state S^0 under consideration. Remark that because of the use of the rate-form (3), quantity C^1 is not a rate dependent coefficient and C^3, C^4 are (pure) gradient dependent coefficients.

Now an infinite dimensional dynamical system

$$\dddot{v} = F^1 v + F^2\dot{v} + F^3\ddot{v} \tag{5}$$

is defined by (5), where (differential) operators F^i, $i = 1, 2, 3$ acting on field v are defined. Then by introducing new variables

$$y_1 = v,$$
$$y_2 = \dot{v},$$
$$y_3 = \ddot{v}$$

a formal mathematical transformation may result a first order abstract dynamical system

$$\dot{y}_1 = y_2, \tag{6}$$
$$\dot{y}_2 = y_3, \tag{7}$$
$$\dot{y}_3 = F^1 y_1 + F^2 y_2 + F^3 y_3. \tag{8}$$

By applying Lyapunov's indirect method [6] we may derive the characteristical equation of the system of (6), (7) and (8)

$$\lambda y_1 = y_2,$$
$$\lambda y_2 = y_3,$$
$$\lambda y_3 = F^1 y_1 + F^2 y_2 + F^3 y_3.$$

After proper rearrangements this equation has the form

$$\lambda^3 y_1 - \lambda^2 F^3 y_1 - \lambda F^2 y_1 - F^1 y_1 = 0 \tag{9}$$

and the stability condition is

- $\mathrm{Re}\,\lambda_i < 0$, $i = 1\ldots$ for all λ_i satisfying (9).

The stability boundary (a loss of stability may be possible) is

- there exists a critical λ_{cr}, for which $\mathrm{Re}\,\lambda_{\mathrm{cr}} = 0$, while $\mathrm{Re}\,\lambda_i < 0$, for all $\lambda_i \neq \lambda_{\mathrm{cr}}$ satisfying (9).

Two possible generic ways of loss of stability are called

- the *static bifurcation* (SB), when

$$\mathrm{Re}\,\lambda_{\mathrm{cr}} = 0; \tag{10}$$

- the *dynamic bifurcation* (DB), when

$$\mathrm{Re}\,\lambda_{cr} \neq 0. \tag{11}$$

Then from (9) and (10) the (necessary) condition of the static bifurcation type of material instability is

$$F^1 y_1 = 0, \tag{12}$$

while (9) and (11) imply the (necessary) condition for the dynamic bifurcation

$$F^3 \left(F^2 y_1\right) + F^1 y_1 = 0. \tag{13}$$

Note that conditions (12) and (13) are partial differential equations with homogeneous boundary conditions. Having done all the necessary substitutions and rearrangements the condition of (12) implies

$$C^1 \left(v \circ \nabla + \nabla \circ v\right) \nabla + C^3 \nabla^2 \left(v \circ \nabla + \nabla \circ v\right) \nabla = 0 \tag{14}$$

and (13) implies

$$\begin{aligned} &\rho C^4 \left(C^2 \left(\dot{v} \circ \nabla + \nabla \circ \dot{v}\right) \nabla\right) + C^1 \left(v \circ \nabla + \nabla \circ v\right) \nabla \\ &\quad + C^3 \nabla^2 \left(v \circ \nabla + \nabla \circ v\right) \nabla = 0, \end{aligned} \tag{15}$$

where the physical variable v is applied again.

Remark that the same bifurcation conditions ((12) and (13)) can be obtained for the three axial cases. Unfortunately, in such a general case the implied boundary value problem is a very complicated one and no analytic method can be found to evaluate it. Then we could do a more or less restrictive weak formulation or search for some numerical methods.

4. Stability Investigation for a Simplified Case

For the sake of less technical problems in rearrangements of mathematical formulae we simplify the constitutive equation to a so-called second gradient dependent one

$$\hat{F}\left(\varepsilon, \dot{\varepsilon}, \nabla^2 \varepsilon, \sigma, \ldots\right) = 0,$$

being widely used in numerical post-localization studies. Its main benefit is that such constitutive formulation eliminates mesh sensitivity effects [7, 8].

While we have homogeneous boundary conditions for the perturbation velocity field of a body of length L,

$$v = v^* \exp\left(i \alpha_k x\right) \tag{16}$$

should substituted into a modified form of (4)

$$\rho \ddot{v} = C^1 \left(v \circ \nabla + \nabla \circ v\right) \nabla + C^2 \left(\dot{v} \circ \nabla + \nabla \circ \dot{v}\right) \nabla + C^3 \nabla^2 \left(v \circ \nabla + \nabla \circ v\right) \nabla. \tag{17}$$

Here the derivation of C^i is the same as in the previous part and in (16) $\alpha_k = \exp\left(i \alpha_k x\right)$, and $\alpha_k = \frac{k\pi}{L}$, $k = 1, 2, 3, \ldots$.

The characteristic equation of (17) can be solved to eigenvalues λ_k

$$\lambda_{k,12} = \frac{-\frac{1}{\rho}c_2\alpha_k^2 \pm \sqrt{\frac{1}{\rho^2}c_2^2\alpha_k^4 - \frac{4}{\rho}\alpha_k^2\left(c_3\alpha_k^2 + c_1\right)}}{2},$$

where constants c_i, $i = 1, \ldots, 3$ are determined by the material parameters C^i. The stability conditions are applied to λ_k and the conditions of the SB or DB types of loss of stability are

- SB, if $\left(c_3\alpha_k^2 + c_1\right) = 0$;
- DB, if$c_2 = 0$.

When we concentrate on static bifurcation (SB), the first unstable eigenvalue determines the value of the critical α_{cr}. By using the boundary conditions

$$\alpha_{\text{cr}} = \sqrt{-\frac{c_1}{c_3}}, \text{ and } v_{\text{cr}} = \exp\left(i\sqrt{-\frac{c_1}{c_3}}x\right)$$

are obtained, where v_{cr} denotes the critical eigenfunction.

5. Conclusions

Then conventional wave dynamical material instability investigations concentrate on (harmonic) wave speeds and the acoustic tensor defined by the fourth order constitutive tangent stiffness tensor. In some cases (for example at a co-existent flutter and divergence instability) the results of it lead to a lot of difficulties.

In this work we propose an alternative method based on the theory of dynamical systems. Apart from the beauty to have a unified treatment for both finite and infinite dimensional mechanical systems, we have some real benefits. The one is a clearer insight into the way of loss of stability. The other is the possibility to get an explanation of the popularity of the 'second gradient-dependent materials'. There we could find a finite dimensional critical eigenspace (spanned by functions v_{cr}) to the unstable eigenvalue.

At last we remark that all methods can be called as 'perturbation methods'. However, for the classical studies all the perturbations are restricted. They may be harmonic waves for example. In dynamical systems the only restrictive condition is that the perturbations are small (infinitesimal), but no specific class of functions is prescribed.

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