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FRACTAL DIMENSION AS MEASURE OF CONTROL TIME

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Abstract

Nonlinearity caused by the application of digital control may lead to chaotic behaviour. There are several cases when these chaotic oscillations disappear suddenly. This phenomenon is referred to as transient chaos. In the present paper, we analyse a simple model of a digitally controlled mechanical system which may perform transient chaotic vibrations, and propose a new procedure for the estimation of the duration of these transients. The relation between the mean lifetime and the so-called escape rate is also examined. As a result, a new formula is introduced, whose reliability is validated with the help of the new lifetime estimation method.

Keywords: digital control, transient chaos, micro-chaos, escape rate, Pesin's identity.

1. Introduction

Parallel with the fast development of computer technology, more and more commercial products are operated using digital control loops.

Unfortunately, engineers are usually not aware of the effects of the discretization in time (sampling) and the discretization of the measured data (round-off error), however, these effects may lead to chaotic behaviour [1, 2]. In several cases, chaotic oscillations disappear suddenly – this phenomenon is referred to as transient chaos [3, 4]. The occurrence of transient chaotic behaviour increases the control time since the control is not effective until the system reaches its steady state. Moreover, control time in non-chaotic cases is usually negligible compared to the lifetime of chaotic transients.

The transient chaotic behaviour cannot be described as a motion on a strange attractor. There is another invariant set in the phase-space of transient chaotic systems, responsible for the irregular vibrations, the so-called chaotic repeller or chaotic saddle [3]. The chaotic repeller can be considered as a dense but measure zero set of repelling (hyperbolic) points, which form a strange structure, a 'maze' for the solution curves. If a solution arrives at the neighbourhood of the repeller, a very long time might be needed for it to leave. The observed law, governing the escape of the solutions from the neighbourhood of the repeller is quite simple: the probability that a solution has not escaped yet decays exponentially with time in an asymptotic sense. Thus, if we start solutions from N_0 different initial points, the number of solutions remaining in the neighbourhood of the repeller until time t is

$$N(t) = N_0 e^{-\kappa t}, \tag{1}$$

where the exponent κ is called escape rate.

In the present paper, we analyse a simple mathematical model of a digitally controlled mechanical system. This system will be introduced in the next section, and the expected duration of a – non-chaotic – transient process will be calculated. Section 3 is devoted to the elucidation of the relation between the escape rate and the mean lifetime of chaotic transients, while our new lifetime estimation method will be presented in Section 4.

2. Mechanical and Mathematical Models

The mechanical model of a polishing machine can be seen in *Fig. 1*. It consists of a revolving cylinder sliding on the rough surface of a fixed block. The velocity of the shaft is denoted by v, while the circumferential velocity of the polishing tool equals $v_0 = R\omega_0$. Besides the dry friction force C, a mixed dry-viscous friction force acts on the polishing tool of mass m. This mixed friction force acting between the fixed block and the cylinder depends on the velocity difference $v_0 - v$.

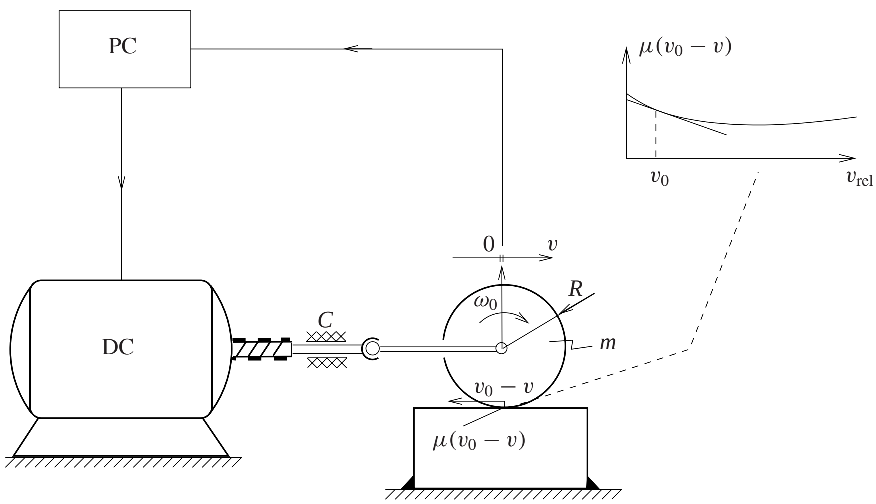

Fig. 1. The mechanical model of the computer-controlled polishing system

At low relative speeds $v_{rel} = v_0 - v$, the combined dry and viscous friction force acting on the cylinder is locally decreasing as the relative velocity $v_0 - v$ increases (see *Fig. 1*). In case of $v = 0$, this friction force equals $mg\mu(v_0)$.

The shaft of the polishing tool is driven by a DC motor, which exerts a control force $Q = -mg\mu(v_0) - Dv$. D denotes a differential gain, with which the strength of the control can be tuned. The linearized, first order equation of motion of the system is

$$m\dot{v} + mg\mu'(v_0)v = mg\mu(v_0) - C\,\mathrm{sgn}(v)\underbrace{-mg\mu(v_0) - Dv}_{Q}. \tag{2}$$

Since the computer samples the velocity with sampling time τ, and the velocity measurement has a finite resolution h, the solution of this equation can be expressed by the following piecewise linear mapping:

$$F_m:\ x_{j+1} = \begin{cases} ax_j - b\,\mathrm{Int}(x_j) - S & \text{if } S/a < x_j, \\ 0 & \text{if } -S/a \le x_j \le S/a, \text{ and} \\ ax_j - b\,\mathrm{Int}(x_j) + S & \text{if } x_j < -S/a, \end{cases} \tag{3}$$

where

$$x_j = v(j)/h, \qquad a = e^{fg\tau} > 1,$$
$$b = \frac{D}{fmg}\left(e^{fg\tau} - 1\right) > 0, \qquad S = \left(e^{fg\tau} - 1\right)\frac{C}{fmgh} > 0,$$
$$f = |\mu'(v_0)| = -\mu'(v_0) > 0,$$

and $v(j)$ denotes the velocity at the j^{th} sampling instant. This map is the generalized version of the so-called micro-chaos map [1]. The details of the derivation of map (3) can be found in [5].

Since there is an attracting domain $[-S/a, S/a]$ in the domain of definition of the map (3), the existence of transient chaos is expected in this case. This attracting domain near the origin corresponds to the static dry friction that captures the shaft at low velocities.

In *Fig.* 2, the graph of the map (3) can be seen for the parameters $a = 3/2$, $b = 6/5$, and $S = 2/15$ (thick lines), with a quite long trajectory (thin lines), eventually arriving at the domain of attraction of the origin. It can be seen that there is a sub-interval I_0 which is directly reachable from the right by a trajectory – see dashed lines in the figure. If a solution arrives at I_0, the transient chaotic behaviour is over. The size of I_0 can be given as

$$|I_0| = a\,\frac{S}{a-1} - (a - b). \tag{4}$$

We introduce $|I_0|$ to be a new parameter instead of S, because it characterizes the system better than the friction parameter S, and it is uniquely related to that.

It is clearly seen in *Fig.* 2 that there is a domain which is densely occupied by the trajectory. The corresponding interval $I_{\text{rep}} = (x^*, a - S)$ contains the repeller of the system. Since the fixed point x^* can be given as

$$x^* = \frac{S}{a-1} = \frac{a - b + I_0}{a}, \tag{5}$$

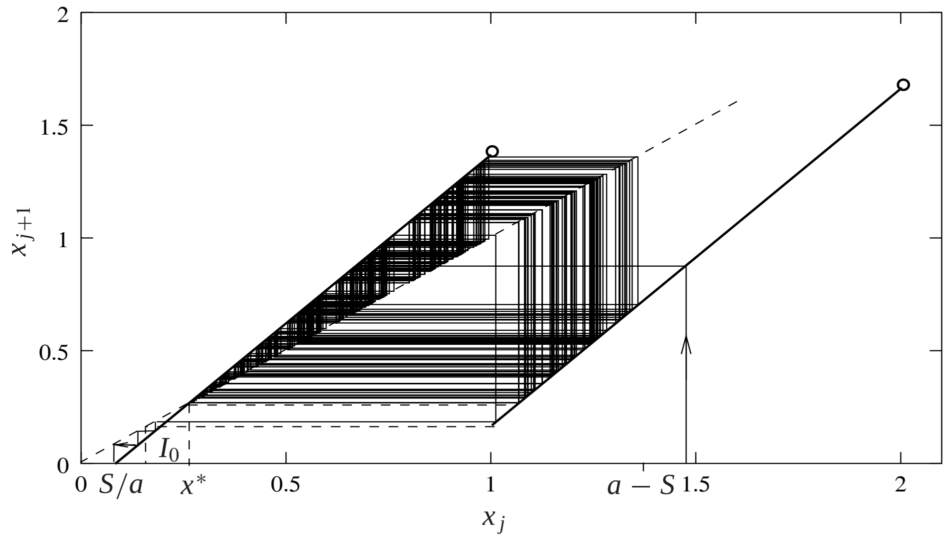

Fig. 2. A long trajectory at $a = 1.5$, $b = 1.2$, and $|I_0| = 0.1$

the size of the repeller is $|I_{\mathrm{rep}}| = b - |I_0|$.

Fig. 3 presents the structure of the domain of definition of map (3). The number of steps needed to escape – i.e. to reach the interval I_0 – is naturally known in case of the pre-images of I_0. Some of these numbers are shown in the figure.

I_0 has only one first pre-image, which is denoted by I_1 in *Fig. 3*. I_1 has two pre-images, one in the interval A – it is denoted by I_2 – and another one in the interval B. As it is shown in the figure, two steps are needed to escape from these intervals. I_2 has two pre-images as well, one in A and another one in B. The structure of A, B, and C can partially be explored this way. All three intervals contain an infinity of sub-intervals of size $|I_0|/a^j$, from where j steps are necessary to escape. These intervals and numbers will be referred to as *fundamental escape intervals* and *kickout numbers*, respectively. The intervals between the fundamental escape intervals are denoted by shaded columns in the figure and will be referred to as *fundamental fractal intervals*.

If we search for the images of fundamental fractal intervals, we find that sooner or later all of them are mapped onto the fundamental fractal interval denoted by F. On the other hand, the pre-images of almost all sub-intervals of B are found in F. Thus, F has the property of self-similarity. So, F contains an infinity of fractal and escape sub-intervals, similarly to intervals A, B, and C.

Since all fractal intervals are mapped onto F, this happens with the fractal sub-intervals of F, too. Thus, these intervals are stretched, and some parts of them are mapped onto the *escape* sub-intervals of F. By exploiting the self-similarity of F, the overall size of the intervals that are mapped onto escape sub-intervals of

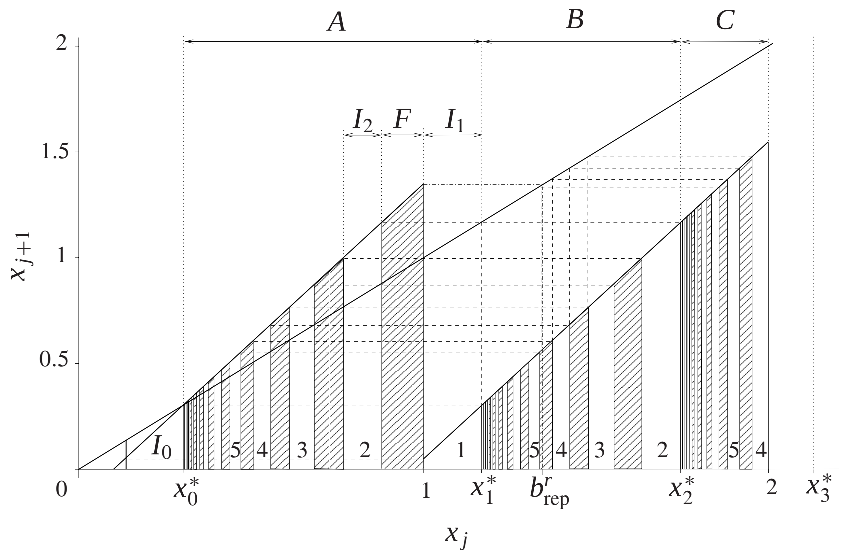

Fig. 3. The structure of the domain of definition at $a = 1.5$, $b = 1.3$, and $|I_0| = 0.25$

F can be calculated, weighted by the appropriate kickout numbers. This way, the mean kickout number – the mean number of steps necessary to escape – can be calculated. This procedure is usually very complex since the eventually escaping intervals are typically mapped into fractal sub-intervals of F before being mapped onto an escape interval [5].

Now, consider one of the most simple cases, namely, when the fundamental fractal intervals shrink to discrete points. The size of fundamental fractal intervals is $|F_j| = |F|/a^j$, where $|F| = (ab - a|I_0| - b)/a^2$ denotes the size of the greatest fractal interval F. $|F| = 0$ fulfils at the critical value $|I_0|_{\mathrm{crit}} = b - b/a$. Over this value, the pre-images of the fixed point x^* form the borders between the fundamental escape intervals, and the behaviour of the solutions cannot be referred to as chaotic. For example, at $a = 1.5$, and $b = 1.3$, $|I_0|_{\mathrm{crit}} = 0.43\dot{3}$. The structure of the domain of definition is shown in *Fig. 4*, at $|I_0| = 0.5$.

In this case, $I_{\mathrm{rep}} = (x^*, a - S) = (0.46\dot{6}, 1.26\dot{6})$. The size of the interval, from which one step is necessary to escape is $|I_1| = a - S - 1 = 0.26\dot{6}$, while the size of the other escape intervals can be given as $|I_1|/a^i$. The mean kickout number can be obtained by calculating the corresponding weighted sum:

$$K_m = |I_1| \frac{\sum_{i=1}^{\infty} i a^{1-i}}{|I_{\mathrm{rep}}|} = \frac{a}{a-1} = 3, \tag{6}$$

thus, the result does not depend on $|I_0|$ or b.

In this case, the behaviour is not chaotic, so the escape rate cannot be defined. However, as it can be seen in *Fig. 4*, the sizes of the pre-images of I_1 decrease

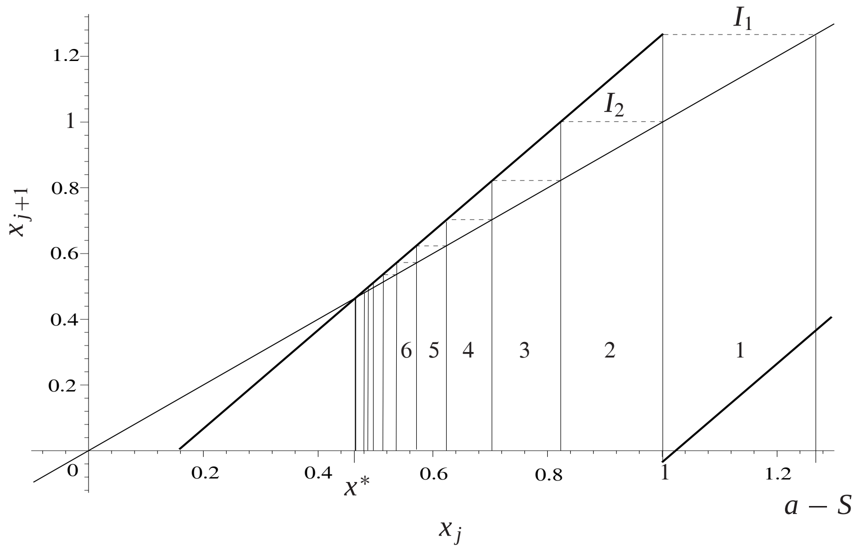

Fig. 4. The structure of the domain of definition at $a = 1.5$, $b = 1.3$, and $|I_0| = 0.5$

exponentially. Thus, a local escape rate κ_{loc} can be introduced in the neighbourhood of the fixed point x^*. Note that the escape rate is usually considered as the measure of the global instability of the system, while local instability is measured by the Lyapunov exponents. Consequently, the local escape rate is equivalent to the Lyapunov exponent. However, the introduction of this new quantity enables us to find a corresponding escape rate to the mean kickout number (6). This way, the exact relation between these quantities can be found.

3. Relation Between the Escape Rate and the Mean Lifetime

The usual escape rate calculations are based on the assumption that the probability that a solution has not escaped yet from a set Γ containing the repeller, decays exponentially with time (see (1)). The reciprocal of the exponent κ is usually considered to be the duration of the transient chaotic phenomenon [3, 4]. This approach has several deficiencies:

- First, the exponential decay is only asymptotically fulfilled. For example, *Fig.* 5 shows the results of a numerical examination of the generalized micro-chaos map at parameters $a = 1.5$, $b = 1.2$, and $|I_0| = 0.17$ (see *Eq.* (3)). We started 150000 iterations from the interval I_{rep} containing the repeller, and detected the number of solutions with certain kickout numbers. As a comparison, an exponentially decreasing curve is also presented, showing the asymptotic nature of the exponential decay. Although the convergence is usually quite fast – especially in case of great escape rates –, this systematic error cannot always be neglected.

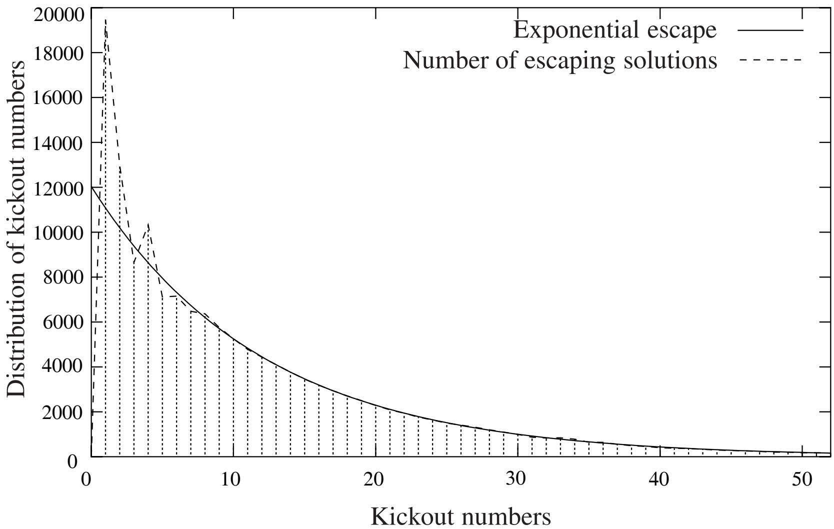

Fig. 5. The exponential decay is only asymptotically fulfilled

- Second, in case of maps, the mean lifetime of transient chaotic behaviour is not equal to the reciprocal of the escape rate. Even in the hypothetical case of exact exponential decay, a different relation is fulfilled between these quantities. In case of continuous flows with exponential rate of escape, the mean lifetime t_m can be calculated according to (1):

$$t_m = \frac{\int_0^\infty t N(t)\,\mathrm{d}t}{\int_0^\infty N(t)\,\mathrm{d}t} = \frac{N_0 \int_0^\infty t e^{-\kappa t}\,\mathrm{d}t}{N_0 \int_0^\infty e^{-\kappa t}\,\mathrm{d}t} = \frac{1}{\kappa}. \tag{7}$$

This result corresponds to the usually used relation. However, in case of maps, we obtain another expression for the mean number of iteration steps K_m necessary to escape. In case of exponential escape not only the number N_n of solutions remaining in the neighbourhood of the repeller decays exponentially, but also the number of escaping trajectories. Since $N_n = N_{n-1}\, e^{-\kappa}$, the number of solutions escaping in the n^{th} step is

$$N_{n-1} - N_n = N_{n-1}\left(1 - e^{-\kappa}\right) = N_0\, e^{-\kappa(n-1)}\left(1 - e^{-\kappa}\right) = N_0\,(e^{\kappa} - 1)\; e^{-\kappa n}. \tag{8}$$

Thus, the so-called mean kickout number K_m can be given as:

$$K_m = \frac{\sum_{n=1}^\infty n N_0\,(e^\kappa - 1)\, e^{-\kappa n}}{\sum_{n=1}^\infty N_0\,(e^\kappa - 1)\, e^{-\kappa n}} = \frac{\sum_{n=1}^\infty n e^{-\kappa n}}{\sum_{n=1}^\infty e^{-\kappa n}} = \frac{1}{1 - e^{-\kappa}}. \tag{9}$$

Note that the sums in the numerator and in the denominator are the upper Darboux sums of the corresponding intergrals in (7). In case of small escape rate and long lifetime, $1/(1 - e^{-\kappa}) \approx 1/\kappa$, indeed. However, in this case, the convergence to the so-called conditionally invariant distribution [3, 4] is slower, which means that the exponential decay of the escape probability is fulfilled only after a significantly long time.

- Third, the calculation of the escape rate provides a value for the mean lifetime of the motion in the neighbourhood of the repeller. However, if the repeller and the set of possible initial conditions do not coincide, another calculation has also to be made to estimate the time needed by the trajectories to reach the repeller.
- Fourth, in some cases, it is not easy to verify whether a certain system exhibits chaotic behaviour, or just another type of complex motion. The methods published in the literature – except for the direct lifetime estimation – exploit the fact that the trajectories are chaotic, thus their results may be questionable when the chaoticity of the system is not proved. In several cases, the hyperbolicity of the system is also assumed. In case of maps, it means that the eigenvalues of the map must be strictly bounded away from 1. However, physical systems are usually non-hyperbolic, and the escape is not exponential in such cases but follows a power law, instead [4].

4. Lifetime Estimation

The analytical calculation of the mean lifetime is usually very complex in case of the micro-chaos map. We propose another procedure in this section for the estimation of the mean kickout number. This method is based on Pesin's identity [3]:

$$\kappa = (1 - D_1)\lambda, \tag{10}$$

where λ is the Lyapunov exponent and D_1 denotes the information dimension of the repeller.

In the case examined in Section 2, we considered only the neighbourhood of the fixed point. Thus, a local version of (10) must be used:

$$\kappa_{\text{loc}} = (1 - D_1)\lambda. \tag{11}$$

In this case, the Lyapunov exponent can be calculated as the logarithm of the slope of the map: $\lambda = \log(a)$. Since the Lyapunov exponent is constant in the domain of definition, the dimension D_1 is equal to the fractal dimension D_0 [3]. Thus, according to (9) and (11), the fractal dimension can be considered as the measure of the mean lifetime – and consequently – the control time.

In the example, there is a countable infinity of measure zero 'fractal' intervals, thus, the fractal dimension of the considered domain is zero, too: $D_0 = 0$. By substituting this value into *Eq.* (11), we obtain:

$$\kappa_{\text{loc}} = (1 - D_0)\log(a) = \log(a). \tag{12}$$

Using *Eq.* (9), the mean kickout number can be calculated as

$$K_m = \frac{1}{1 - e^{-\log(a)}} = \frac{a}{a-1}. \tag{13}$$

So, we obtained the result of the exact elementary calculation (6), again. This way, the reliability of relation (9) is verified.

Note that the above described method can be extended to other, not so trivial cases, too. As we have already mentioned in Section 2, each fractal interval contains an infinity of smaller copies of itself. Thus, they have a multi-scale fractal structure with the scaling factors $r_j = Ca^{-j}$, where a is the slope of the map and C is a coefficient depending on a, b, and $|I_0|$. The fractal dimension D_0 of multi-scale fractals can be calculated by solving the equation

$$\sum_{j=0}^{N} r_j^{D_0} = \sum_{j=0}^{\infty} C^{D_0} a^{-jD_0} = 1, \tag{14}$$

where $N+1$ is the number of different scales [6]. By finding appropriate estimations for the coefficient C, the fractal dimension and the kickout number could also be estimated.

The search for satisfactory estimations of this coefficient plays a central role in our recent research work.

Acknowledgements

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