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ON THE ANALYTICAL METHODS APPROXIMATING THE TIME PERIOD OF THE NONLINEAR PHYSICAL PENDULUM

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Abstract

This paper gives a survey on the computation of the time period of the nonlinear physical pendulum. The analytical methods, based on the third-order normal forms and on Krylov's asymptotic technique are compared to each other. It is shown that these procedures can yield the same result. Though in case of Krylov's technique, it depends on the power series expansion of the angular eigenfrequency. Finally, the analytical formulas are compared to the results of numerical computations.

Keywords: nonlinear dynamics, third-order normal forms, asymptotic technique.

1. Introduction

Let us consider the model of a single physical pendulum shown in *Fig.1*. The mass of the pendulum is m and its length is l. To be more or less general, the support can move vertically according to the function $r(t)$, and let $M(t)$ denote a possible moment excitation. These are taken into consideration till the derivation of the equation of motion and will be neglected in the main analytical investigations.

The model does not consist of damping. The generalized coordinate q denotes φ angle of the pendulum measured from the vertical direction. With the known base excitation $r(t)$, this clearly describes the position of the pendulum. Thus, this is a one-degree-of-freedom system.

2. Deriving the Equation of Motion

The position of the centre of gravity S is given by

$$\mathbf{r}_S(t) = \begin{bmatrix} \frac{l}{2}\sin\varphi(t) \\ r(t) - \frac{l}{2}\cos\varphi(t) \end{bmatrix} \tag{1}$$

and hence, its velocity:

$$\mathbf{v}_S = \dot{\mathbf{r}}_S \equiv \begin{bmatrix} \frac{l}{2}\dot{\varphi}\cos\varphi \\ \dot{r}(t) + \frac{l}{2}\dot{\varphi}\sin\varphi \end{bmatrix} \tag{2}$$

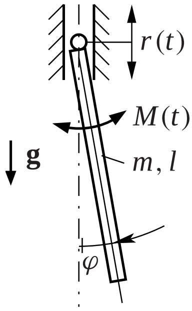

Fig. 1. The sketch of the pendulum.

(the formula $\mathbf{v}_S = \mathbf{v}_A + \dot{\varphi}\mathbf{k} \times \mathbf{r}_{AS}$ yields the same expression).

The kinetic energy of the pendulum:

$$\mathcal{T} = \frac{1}{2} m \mathbf{v}_S^2 + \frac{1}{2}\Theta_{S,z}\dot{\varphi}^2 \equiv \frac{1}{2} m \left(\frac{l^2}{4}\dot{\varphi}^2 + \dot{r} l \dot{\varphi} \sin\varphi + \dot{r}^2 + \frac{l^2}{12}\dot{\varphi}^2 \right). \quad (3)$$

The potential energy of the gravitational field ($\mathbf{g} = -g\mathbf{j}$):

$$\mathcal{U} = -m\mathbf{g} \cdot \mathbf{r}_S \equiv mg(r - \frac{l}{2}\cos\varphi). \quad (4)$$

The virtual power of the excitation:

$$\delta\mathcal{P} = M(t)\delta\dot{\varphi}. \quad (5)$$

Substituting *Eqs.* (3)–(5) into the LAGRANGIAN-equation of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{T}}{\partial\dot{q}} - \frac{\partial\mathcal{T}}{\partial q} + \frac{\partial\mathcal{U}}{\partial q} = \frac{\partial}{\partial\delta\dot{q}}\delta\mathcal{P} \quad (6)$$

we obtain:

$$\frac{1}{3}ml^2\ddot{\varphi} + \frac{m}{2}(\ddot{r}l\sin\varphi + \dot{r}l\dot{\varphi}\cos\varphi) - \frac{m}{2}\dot{r}l\dot{\varphi}\cos\varphi + mg\frac{l}{2}\sin\varphi = M(t).$$

After simplification and rearrangement (with $r(t) = r_0 \sin\omega t$):

$$\ddot{\varphi} + \alpha^2\left(1 - \omega^2\frac{r_0}{g}\sin\omega t\right)\sin\varphi = \alpha^2\frac{2}{mgl}M(t), \quad (7)$$

where $\alpha = \sqrt{3g/2l}$ is the angular eigenfrequency of the linearized system. Thus, the time period of the free oscillation is

$$T_0 = \frac{2\pi}{\alpha} \equiv 2\pi\sqrt{\frac{2l}{3g}}.$$

3. The Unexcited Nonlinear System

In the following, we investigate the self-excited vibrations of the nonlinear system without the additive and parametric external excitations, $M(t)$ and $r(t)$, respectively:

$$\ddot{\varphi} = -\alpha^2 \sin\varphi. \tag{8}$$

After multiplying it with $2\dot{\varphi}$, it yields

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\varphi})^2 = -2\alpha^2\dot{\varphi}\sin\varphi \equiv 2\alpha^2\frac{\mathrm{d}}{\mathrm{d}t}\cos\varphi,$$

which can be integrated and we obtain

$$\dot{\varphi} = \alpha\sqrt{2\cos\varphi + C_1}. \tag{9}$$

Separating the variables, it can be written as

$$t = \int_0^{\varphi_t} \frac{\mathrm{d}\varphi}{\alpha\sqrt{2\cos\varphi + C_1}},$$

if $\varphi_t = 0$ at $t = 0$.

Applying the variable substitution $\sin\frac{\varphi}{2} = k\sin\psi$, we get

$$t = \int_0^{\psi_t} \frac{2k\cos\psi\,\mathrm{d}\psi}{\alpha\sqrt{C_1 + 2(1 - 2k^2\sin^2\psi)}\sqrt{1 - k^2\sin^2\psi}}$$
$$\equiv \frac{1}{\alpha}\int_0^{\psi_t} \frac{\mathrm{d}\psi}{\sqrt{1 - k^2\sin^2\psi}} \equiv \frac{F(k, \psi_t)}{\alpha}$$

when $k = \sqrt{2 + C_1}/2$. The last integral expression $F(k, \psi_t)$ is the LEGENDRE normal form of the elliptic integral of the first kind and can be given in closed form using JACOBI's elliptic function [1]. Furthermore, $C_1 = -2\cos\varphi_{T/4}$ if $\dot{\varphi}_{T/4} = 0$ in *Eq.* (9), i.e. $\varphi_{T/4}$ is the amplitude of the oscillation. Hence,

$$k = \sin\frac{\varphi_{T/4}}{2} \quad \text{and} \quad \psi_{T/4} = \frac{\pi}{2}$$

and the time period of the oscillation is

$$T = 4\frac{F(k, \frac{\pi}{2})}{\alpha}. \tag{10}$$

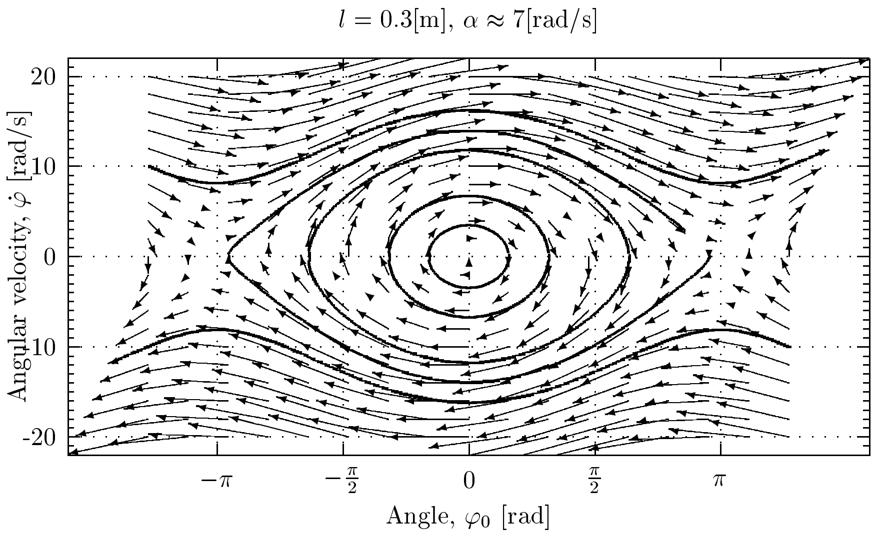

Fig. 2. Phase portrait of the free vibrations of a nonlinear pendulum.

Investigating the fixed points of *Eq.* (8), we found infinite many equilibrium points given by

$$\varphi^\star = k\pi, \quad k \in \{0, \pm 1, \pm 2, \dots\}.$$

However, there are only two physically different equilibrium points: the LIAPUNOV stable $\varphi^\star = 0$ for the 'normal' pendulum ($\sin\varphi \approx \varphi$ yields pure imaginary characteristic roots $\lambda_{1,2} = \pm i\alpha$) and the unstable $\varphi^\star = \pi$ for the inverted pendulum ($\sin\varphi \approx -\varphi \Rightarrow \lambda_{1,2} = \pm\alpha$). The phase portrait in the vicinity of these equilibrium points can be seen in *Fig. 2*.

From this point, only the third degree nonlinearities are taken into account which can be obtained by Taylor series expansion around the equilibrium $\varphi = 0$:

$$\ddot{\varphi} + \alpha^2\varphi = \frac{\alpha^2}{6}\varphi^3 + O(\varphi^5). \tag{11}$$

Considering only the linear part, the equilibrium $\varphi = 0$ can only be *stable in the* LIAPUNOV *sense* (the solutions for small initial conditions do not diverge, but do not even converge to the equilibrium, simply saying), since the characteristic polynomial has a pair of conjugate pure imaginary roots.

However, in the case of large angular displacements, the solutions are only orbitally stable (the paths stay close together in space, but it is not true 'in time'), since the time period of the oscillation depends on the initial conditions.

3.1. *The* JORDAN *Canonical Form*

Let us rewrite *Eq.* (11) into first order (CAUCHY) form introducing the variables $x_1 = \varphi$, $x_2 = \dot{\varphi}$:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\alpha^2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ \frac{\alpha^2}{6} x_1^3 \end{bmatrix} + O(x_1^5). \tag{12}$$

The **A** coefficient matrix of the linear part has complex eigenvectors because of the complex eigenvalues. However, these eigenvectors are conjugates of each other, since the elements of the coefficient matrix **A** are real numbers.:

$$\begin{aligned} \mathbf{A}(\mathbf{u} + \mathrm{i}\mathbf{v}) &= \mathrm{i}\alpha(\mathbf{u} + \mathrm{i}\mathbf{v}) \\ \Rightarrow \mathbf{A}\mathbf{u} &= -\alpha\mathbf{v} \\ \Rightarrow \mathbf{A}\mathbf{v} &= \alpha\mathbf{u} \\ \Rightarrow \mathbf{A}(\mathbf{u} - \mathrm{i}\mathbf{v}) &= -\alpha\mathbf{v} - \mathrm{i}\alpha\mathbf{u} \equiv -\mathrm{i}\alpha(\mathbf{u} - \mathrm{i}\mathbf{v}) \end{aligned}$$

and from the so-called generalized eigenvectors **u** and **v**, a transformation matrix **T** can be built. In our case e.g.:

$$\mathbf{T} = [\ \mathbf{u} \quad \mathbf{v}\ ] \equiv \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix}$$

and its inverse:

$$\mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix}.$$

Thus, the $\mathbf{x} = \mathbf{T}\mathbf{y}$ linear transformation brings *Eq.* (12) to the following form:

$$\dot{\mathbf{y}} = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ \frac{\alpha}{6} y_1^3 \end{bmatrix} + O(y_1^5), \tag{13}$$

or generally

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y} + \mathbf{f}(\mathbf{y}) + O(\|\mathbf{y}\|^4). \tag{14}$$

The vector $\mathbf{f}(\mathbf{y})$ contains the nonlinearities of at most the third degree. In details, its elements have the form as follows:

$$\mathbf{f}(\mathbf{y}) = \begin{bmatrix} f_1^{(20)} y_1^2 + f_1^{(11)} y_1 y_2 + f_1^{(02)} y_2^2 + f_1^{(30)} y_1^3 + f_1^{(21)} y_1^2 y_2 + f_1^{(12)} y_1 y_2^2 + f_1^{(03)} y_2^3 \\ f_2^{(20)} y_1^2 + f_2^{(11)} y_1 y_2 + f_2^{(02)} y_2^2 + f_2^{(30)} y_1^3 + f_2^{(21)} y_1^2 y_2 + f_2^{(12)} y_1 y_2^2 + f_2^{(03)} y_2^3 \end{bmatrix}.$$

3.2. The Nonlinear Near-Identity Transformation

The effect of the (at most third-degree) nonlinearity becomes obvious through the third-order normal form. Hence, we transform *Eq.* (13) to such a form:

$$\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + (z_1^2 + z_2^2)\begin{bmatrix} \delta & \beta \\ -\beta & \delta \end{bmatrix}\mathbf{z} + O(\|\mathbf{z}\|^4). \tag{15}$$

Let us introduce a radius with $r^2 = \|\mathbf{z}\|^2$. The previously defined r is proportional to the amplitude of the oscillation. Thus, we can derive the differential equation determining $r(t)$ multiplying *Eq.* (15) with $\mathbf{z}$:

$$r\dot{r} = \mathbf{z}^\top \dot{\mathbf{z}} \equiv \alpha(z_1 z_2 - z_2 z_1) + \mathbf{z}^2(\delta \mathbf{z}^2 + \beta(z_1 z_2 - z_2 z_1)) \\ + O(\|\mathbf{z}\|^5) \equiv r^4\delta + O(r^5),$$

that is

$$\dot{r} = \delta r^3 + O(r^4). \tag{16}$$

Hence, if $\delta < 0$, r tends to zero, i.e. the nonlinear part stabilizes 'in third order'. Of course, in the case of the investigated physical pendulum, we expect $\delta = 0$ since it is a Hamiltonian system.

The time period of the oscillation can be approximated from the differential equation derived for the angular polar-coordinate ϑ using the transformations $z_1 = r\cos\vartheta$ and $z_2 = -r\sin\vartheta$:

$$r^2\dot{\vartheta} = \dot{z}_1 z_2 - z_1 \dot{z}_2 + O(\|\mathbf{z}\|^5) \equiv \alpha z_2^2 + r^2(\delta z_1 z_2 + \beta z_2^2) \\ + \alpha z_1^2 - r^2(-\beta z_1^2 + \delta z_1 z_2) + O(r^5),$$

that is

$$\dot{\vartheta} = \alpha + \beta r^2 + O(r^3). \tag{17}$$

The third-order normal form defined by (15) can be obtained using the near-identity transformation as follows:

$$\mathbf{y} = \mathbf{z} + \mathbf{g}_2(\mathbf{z}) + \mathbf{g}_3(\mathbf{z}), \tag{18}$$

where $\mathbf{g}_2(\mathbf{z})$ and $\mathbf{g}_3(\mathbf{z})$ contain only second and third-degree nonlinearities, respectively.

Differentiating *Eq.* (18) with respect to the time and substituting it back into *Eq.* (14), we get

$$\left(\mathbf{I} + \frac{\partial \mathbf{g}_2}{\partial \mathbf{z}} + \frac{\partial \mathbf{g}_3}{\partial \mathbf{z}}\right)\dot{\mathbf{z}} = \mathbf{J}\mathbf{z} + \mathbf{J}\mathbf{g}_2(\mathbf{z}) + \mathbf{J}\mathbf{g}_3(\mathbf{z}) \\ + \mathbf{f}(\mathbf{z} + \mathbf{g}_2(\mathbf{z})) + O(\|\mathbf{z}\|^4), \tag{19}$$

where only the third-degree terms were taken into consideration (see also $\mathbf{f}(\mathbf{y})$). The inverse of matrix $\mathbf{I}+\mathbf{X}$ expanded into power series to the second degree is

$$(\mathbf{I}+\mathbf{X})^{-1} \approx \mathbf{I}-\mathbf{X}+\mathbf{X}^2.$$

Thus, the inverse of the coefficient matrix on the left hand-side of *Eq.* (19) can be approximated as

$$\left(\mathbf{I}+\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}+\frac{\partial \mathbf{g}_3}{\partial \mathbf{z}}\right)^{-1} \approx \mathbf{I}-\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}-\frac{\partial \mathbf{g}_3}{\partial \mathbf{z}}+\left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}\right)^2,$$

with which multiplying *Eq.* (19) and holding only the at most third degree terms we get:

$$\begin{aligned}\dot{\mathbf{z}} \approx\ & \mathbf{J}\mathbf{z}+\mathbf{J}\mathbf{g}_2(\mathbf{z})-\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}\mathbf{J}\mathbf{z}+\mathbf{J}\mathbf{g}_3(\mathbf{z})+\left(\left(\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}\right)^2-\frac{\partial \mathbf{g}_3}{\partial \mathbf{z}}\right)\mathbf{J}\mathbf{z}\\ & -\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}\mathbf{J}\mathbf{g}_2(\mathbf{z})+\left(\mathbf{I}-\frac{\partial \mathbf{g}_2}{\partial \mathbf{z}}\right)\mathbf{f}(\mathbf{z}+\mathbf{g}_2(\mathbf{z})).\end{aligned}$$

Collecting the terms of this equation according to the power of $z_i z_j$, the coefficients of the second-degree terms have to vanish. Furthermore, the coefficients of the third-degree terms have to correspond with *Eq.* (15). With these conditions, the coefficients of the nonlinear transformation given by *Eq.* (18) can be determined, and thus δ and β can also be obtained. The generally derived formulas to calculate δ and β are the following:

$$\begin{aligned}\delta =\ & \frac{1}{8}\left(3 f_1^{(30)}+f_2^{(21)}+f_1^{(12)}+3 f_2^{(03)}\right)\\ & +\frac{1}{8\alpha}\left(f_2^{(11)}(f_2^{(20)}+f_2^{(02)})-f_1^{(11)}(f_1^{(20)}+f_1^{(02)})\right.\\ & \left.-2(f_1^{(20)} f_2^{(20)}+f_1^{(02)} f_2^{(02)})\right)\end{aligned} \tag{20}$$

and

$$\begin{aligned}\beta =\ & -\frac{1}{8}\left(3 f_2^{(30)}-f_1^{(21)}+f_2^{(12)}-3 f_1^{(03)}\right)\\ & -\frac{1}{24\alpha}(\dots)\end{aligned} \tag{21}$$

For the case of the pendulum, according to *Eq.* (13) only $f_2^{(30)} \neq 0$:

$$\begin{aligned}\delta &= 0,\\ \beta &= -\frac{3}{8} f_2^{(30)} \equiv -\frac{3\alpha}{48}.\end{aligned}$$

That is, the third-degree nonlinearity does not influence the amplitude of the oscillation (and neither do the terms of higher degree, surely, this is a conservative Hamiltonian system).

The time period of the oscillation can be determined substituting β back into the expression of angular velocity $\dot{\vartheta}$ given in *Eq.* (17). Thus, with

$$r^2 = \mathbf{z}^2 \approx \mathbf{y}^2 \equiv (\mathbf{T}^{-1}\mathbf{x})^2 \equiv x_1^2 + x_2^2/\alpha^2 \equiv \varphi_0^2 + (\dot{\varphi}_0/\alpha)^2,$$

we obtain

$$\dot{\vartheta} = \alpha - \frac{\alpha}{16}\varphi_0^2 + O(\varphi_0^3),$$

where for the sake of simplicity, the initial angular velocity was assumed to be zero: $\dot{\varphi}_0 = 0$. Hence,

$$T \approx \frac{2\pi}{\dot{\vartheta}} = \frac{2\pi}{\alpha\left(1 - \frac{1}{16}\varphi_0^2\right)}. \tag{22}$$

3.3. *Approximating the Time Period Using* Krylov*'s Asymptotic Technique*

Following the method described by Krylov [3], [2], let us search the solution $\varphi(t)$ of *Eq.* (11) belonging to the initial conditions $\varphi(0) = \mu$, $\dot{\varphi}(0) = 0$ in the following form:

$$\varphi(t) \approx \mu\varphi_1(t) + \mu^2\varphi_2(t) + \mu^3\varphi_3(t), \tag{23}$$

and let the initial conditions be satisfied in the following manner:

$$(\varphi_1(0), \dot{\varphi}_1(0)) = (1, 0) \tag{24}$$

$$(\varphi_i(0), \dot{\varphi}_i(0)) = (0, 0) \qquad (i = 2, 3) \tag{25}$$

Furthermore, let γ denote the angular eigenfrequency of the solution $\varphi(t)$. Its power series expansion, with respect to the initial angular displacement, is

$$\gamma = \alpha + \mu h_1 + \mu^2 h_2 + O(\mu^3). \tag{26}$$

Let us take the square of *Eq.* (26), and express α^2 from it and put it back into *Eq.* (11):

$$\ddot{\varphi} + \gamma^2\varphi = \left(2\alpha h_1\mu + h_1^2\mu^2 + 2\alpha h_2\mu^2 + O(\mu^3)\right)\varphi + \frac{\alpha^2}{6}\varphi^3 + O(\varphi^5). \tag{27}$$

Let us substitute the power series expansion of $\varphi(t)$ according to *Eq.* (23) into *Eq.* (27) and collect the power of μ:

$$\mu : \ddot{\varphi}_1 + \gamma^2\varphi_1 = 0 \tag{28}$$

$$\mu^2 : \ddot{\varphi}_2 + \gamma^2\varphi_2 = 2\alpha h_1\varphi_1 \tag{29}$$

$$\mu^3 : \ddot{\varphi}_3 + \gamma^2\varphi_3 = \left(h_1^2 + 2\alpha h_2\right)\varphi_1 + 2\alpha h_1\varphi_2 + \frac{\alpha^2}{6}\varphi_1^3 \tag{30}$$

$$\mu^4 : \ldots$$

The solution of *Eq.* (28) satisfying the initial condition (24) is $\varphi_1(t) = \cos\gamma t$. After substituting it into *Eq.* (29), we obtain a differential equation with resonant right hand-side, which has an *aperiodic* particular solution. This can only be avoided with $h_1 = 0$. However, in this case, *Eq.* (29) will be a homogeneous differential equation and its solution for the initial conditions (25) is $\varphi_2(t) \equiv 0$.

This simplifies *Eq.* (30) as follows:

$$\ddot{\varphi}_3 + \gamma^2\varphi_3 = \left(2\alpha h_2 + \frac{3\alpha^2}{24}\right)\cos\gamma t + \frac{\alpha^2}{24}\cos 3\gamma t,$$

where the trigonometric formula $4\cos^3\gamma t = \cos 3\gamma t + 3\cos\gamma t$ was applied. Again, the resonant case can be avoided by eliminating $\cos\gamma t$, i.e.

$$h_2 = -\frac{\alpha}{16}$$
$$\Rightarrow T \approx \frac{2\pi}{\gamma} = \frac{2\pi}{\alpha(1 - \frac{1}{16}\mu^2)}, \qquad (31)$$

if h_1 and h_2 are substituted back into *Eq.* (26).

Remarks Instead of the power series equation of the angular eigenfrequency γ given by (26) the following formula could also be applied

$$\gamma^2 = \alpha^2 + \mu h_1 + \mu^2 h_2 + O(\mu^3).$$

However, this yields the following result instead of (30):

$$\ddot{\varphi}_3 + \gamma^2\varphi_3 = \left(h_2 + \frac{3\alpha^2}{24}\right)\cos\gamma t + \frac{\alpha^2}{24}\cos 3\gamma t,$$

from which $h_2 = -\alpha^2/8$ and thus the time period of the oscillation is

$$T \approx \frac{2\pi}{\alpha\sqrt{1 - \frac{1}{8}\mu^2}}.$$

However, this approximation is worse than the formula at (31). The latter shows a quite good agreement with the real value of the time period in the interval $0 \le \mu < 2.7$ [rad] of the initial angular displacement. This can be verified by numerical simulation, as shown in *Fig. 3*.

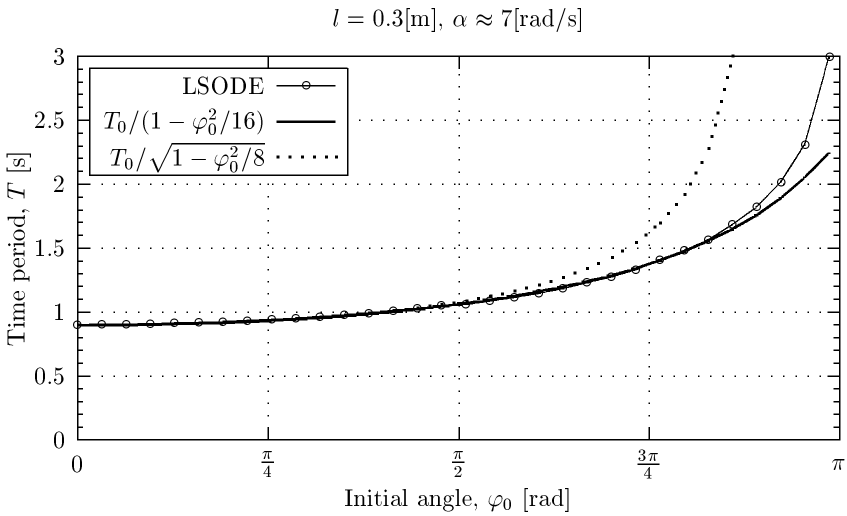

Fig. 3. The approximations of the time period of a single pendulum.

4. Conclusions

The equation of motion of a generally excited single pendulum was derived. After the known analysis of the linear forced system, the free vibrations of the nonlinear system were investigated: we show that the method based on the third-order normal forms and Krylov's asymptotical technique can yield the same formula of the time period. However, the way of application of small parameters in Krylov's method can influence the order of the formula.

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