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HOPF BIFURCATION CALCULATIONS IN DELAYED SYSTEMS

Gábor OROSZ

Bristol Centre for Applied Nonlinear Mathematics
Department of Engineering Mathematics, University of Bristol
Queen's Building, University Walk, Bristol BS8 1TR, United Kingdom
e-mail: g.orosz@bristol.ac.uk

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Abstract

A formal framework for the analysis of Hopf bifurcations in delay differential equations with a single time delay is presented. Closed-form linear algebraic equations are determined and the criticality of bifurcations is calculated by normal forms.

Keywords: delay-differential equation, Hopf bifurcation.

1. Introduction

The aim of this paper is to outline a formal framework for the analytical bifurcation analysis of Hopf bifurcations in delay differential equations with a single fixed time delay. We give a general formalization of these calculations and determine closed-form algebraic equations where the stability and amplitude of periodic solutions close to bifurcation can be calculated. The given algorithm may be implemented in symbolic algebra packages (such as Maple or Mathematica).

The bifurcation theory of ordinary differential equations (ODEs) can be generalized to delay-differential equations (DDEs) through the investigation of retarded functional differential equations (RDFEs); see HALE & VERDUYN LUNEL [7] for details. A review of bifurcations in DDE systems is available in the book of DIEKMANN et al. [2]. Furthermore, the theorem of normal form calculations has been recently published by HALE et al. [6]. The first closed-form Hopf bifurcation calculation was executed by HASSARD et al. [8] in the case of a simple scalar first order DDE, while Stépán presented such calculations first [14, 15] for vector DDEs.

The result of the Hopf bifurcation algorithm is a first Fourier approximation of stable or unstable periodic solutions which can be derived analytically as a function of the bifurcation parameters. However, it is acceptable only for bifurcation parameters close enough to the critical point, since a third degree Taylor series expansion of the non-linearity is used in the DDE. These calculations are very complicated, particularly in systems where the centre-manifold reduction is required. However, in some simple cases it is possible to use computer algebra packages; e.g., see CAMPBELL & BÉLAIR [1].

The analytical estimate is useful in many applications, especially when the periodic solutions are unstable. Analytical studies of Hopf bifurcations in delayed systems are carried out, for example, on machine tool vibrations by KALMÁR-NAGY et al. [9] and on voltage oscillations of neuron systems by SHAYER & CAMPBELL [12].

We note that ENGELBORGHS et al. have recently constructed a Matlab package called DDE-BIFTOOL [3, 4], which can follow branches of stable and unstable periodic solutions against a chosen bifurcation parameter. This semi-numerical method uses the exact form of the non-linearities, hence it provides reliable results even when bifurcation parameter is far away from its critical value at the bifurcation point. This has been applied for extensive investigation of semiconductor laser systems; see GREEN et al. [5].

2. Retarded Functional Differential Equations

In dynamical systems with memory the rate of change of the present state depends on the past state of the system. Time development of these systems can be described by the retarded functional differential equation

$$\dot{x}(t) = G(x_t; \mu), \tag{1}$$

where dot refers to the derivative with respect to the time t, the state variable is $x\colon \mathbb{R} \to \mathbb{R}^n$, while the function $x_t\colon \mathbb{R} \to \mathbb{X}_{\mathbb{R}^n}$ is defined by the shift $x_t(\vartheta) = x(t+\vartheta)$, $\vartheta \in [-r, 0]$. Here the length of the delay $r \in \mathbb{R}^+$ is assumed to be finite. The non-linear functional $G\colon \mathbb{X}_{\mathbb{R}^n} \times \mathbb{R} \to \mathbb{R}^n$ acts on the function space $\mathbb{X}_{\mathbb{R}^n}$ of $\mathbb{R} \to \mathbb{R}^n$ functions. For the sake of simplicity, we consider a scalar bifurcation parameter, that is, $\mu \in \mathbb{R}$, and assume that G is a near-zero functional in x_t for any μ:

$$G(0; \mu) = 0. \tag{2}$$

Thus RFDE (1) possesses the trivial solution

$$x(t) \equiv 0, \tag{3}$$

which exists for all the values of the bifurcation parameter μ. Since the function space $\mathbb{X}_{\mathbb{R}^n}$ is infinite-dimensional, the dimension of the phase space of RFDE (1) also becomes infinite.

For example, one may use a particular form for the functional G and obtain the equation

$$\dot{x}(t) = g\left(\int_{-r}^{0} \mathrm{d}\gamma(\vartheta)\rho_\vartheta\big(x(t+\vartheta)\big); \mu\right), \tag{4}$$

where $g, \rho_\vartheta\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $g(0; \mu) = 0$, and the $n \times n$ matrix $\gamma\colon [-r, 0] \to \mathbb{R}^{n\times n}$ is a function of the variation ϑ.

The measure γ can be concentrated on some particular values:

$$\gamma(\vartheta) = \left(\delta(\vartheta) + \sum_{i=1}^{m} \delta(\vartheta + \tau_i)\right)\mathrm{I}, \tag{5}$$

where $\tau_i \in (0, r]$, $i = 1, \ldots, m$, $m \in \mathbb{N}$, and the non-delayed term is formally separated from the delayed terms. Here and henceforward the $n \times n$ identity matrix is indicated by I. Substituting measure (5) into example (4) results in

$$\dot{x}(t) = g\left(\rho_0\big(x(t)\big), \sum_{i=1}^{m} \rho_{\tau_i}\big(x(t-\tau_i)\big); \mu\right), \tag{6}$$

that is,

$$\dot{x}(t) = f\big(x(t), x(t-\tau_1), \ldots, x(t-\tau_m); \mu\big), \tag{7}$$

where $f : \mathbb{R}^n \times \ldots \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $f(0, 0, \ldots, 0; \mu) = 0$, which is the general form of DDEs with m discrete time delays. For $m = 1$ we have the form

$$\dot{x}(t) = f\big(x(t), x(t-\tau); \mu\big), \tag{8}$$

which we focus on in Section 4.

3. Stability and Bifurcations

According to the *Riesz Representation Theorem*, the linearisation of functional G with respect to x_t is defined by a *Stieltjes integral*, that is the variational system of RFDE (1) is given as

$$\dot{x}(t) = \int_{-r}^{0} \mathrm{d}_\vartheta \eta(\vartheta; \mu) x(t+\vartheta). \tag{9}$$

Note that it can also be obtained from the example (4) by considering $\rho_\vartheta(x) = x$ and taking the linear part of the function g.

Similarly to the case of linear ODEs, one may substitute the trial solution $x(t) = ke^{\lambda t}$ into *Eq.* (9) with a constant vector $k \in \mathbb{C}^n$ and characteristic exponent $\lambda \in \mathbb{C}$. It results in the characteristic equation

$$D(\lambda; \mu) = \det\left(\lambda \mathrm{I} - \int_{-r}^{0} e^{\lambda\vartheta} \mathrm{d}_\vartheta \eta(\vartheta; \mu)\right) = 0, \tag{10}$$

which has infinitely many solutions for the characteristic exponent λ.

The trivial solution (3) of the non-linear RFDE (1) is asymptotically stable (that is, stable in the Lyapunov sense, too) for the bifurcation parameter μ if all the infinitely many characteristic exponents are situated on the left-hand side of the imaginary axis. Hopf bifurcation takes place at the critical parameter value μ_{cr} if there exists a complex conjugate pair of pure imaginary characteristic exponents:

$$\lambda_{1,2}(\mu_{\text{cr}}) = \pm \mathrm{i}\omega. \tag{11}$$

In the parameter space of the RFDE, the corresponding stability boundaries are described by the so-called D-curves

$$R(\omega) = \operatorname{Re} D(\mathrm{i}\omega), \qquad S(\omega) = \operatorname{Im} D(\mathrm{i}\omega), \tag{12}$$

that are parameterised by the frequency $\omega \in \mathbb{R}^+$ referring to the imaginary part of the critical characteristic exponents (11). Since *Eq.* (10) has infinitely many solutions for λ, an infinite-dimensional version of the *Routh-Hurwitz* criterion is needed to decide on which side of the D-curves the steady state is stable or unstable. These kind of criteria can be determined by calculating complex integrals around the characteristic exponents; see [10, 11, 15] for detailed calculations.

We note, when not only one but two pairs of pure imaginary characteristic exponents (with two different frequencies) coexist at μ_{cr} then a co-dimension-two double Hopf bifurcation occurs as demonstrated by STÉPÁN & HALLER [16] for robot dynamics and by GREEN et al. [5] in laser systems. In the case, when a zero exponent and a pair of pure imaginary exponents coexist at μ_{cr} then a fold bifurcation occurs together with a Hopf bifurcation as investigated by SIEBER & KRAUSKOPF [13] in the case of a controlled inverted pendulum.

There is another condition for the existence of a Hopf bifurcation: the critical characteristic exponents $\lambda_{1,2}$ (11) have to cross the imaginary axis with a non-zero speed due to the variation of the bifurcation parameter μ:

$$\operatorname{Re}\left(\frac{\mathrm{d}\lambda_{1,2}(\mu_{\text{cr}})}{\mathrm{d}\mu}\right) = \operatorname{Re}\left(-\frac{\partial D(\lambda;\mu_{\text{cr}})}{\partial \mu}\left(\frac{\partial D(\lambda;\mu_{\text{cr}})}{\partial \lambda}\right)^{-1}\right) \neq 0, \tag{13}$$

where the first equality can be verified by implicit differentiation of the characteristic function (10).

The above conditions (11) and (13) can be checked using the variational system (9). Contrarily, the super- or subcritical nature of the Hopf bifurcation, *i.e.*, the stability and estimated amplitudes of the periodic solutions arising about the stable or unstable trivial solution (3) can be determined only by the investigation of the third degree power series of the original non-linear RFDE (1). In the subsequent section, the type of the Hopf bifurcation is determined. The algorithm will be presented for (8), *i.e.*, the case of a single discrete time delay.

4. Hopf Bifurcation in Case of one Discrete Delay

The analysis presented in this section is based on the examples in [14, 15]. However, we carry out the calculations for an arbitrary number of DDEs in a more general case. Furthermore, the overview below gives general forms for the linear algebraic equations resulting from the operator formalism.

Let us consider the non-linear system (8) with one discrete delay $\tau \in \mathbb{R}^+$ in the form

$$\dot{x}(t) = \Lambda x(t) + \mathrm{P}x(t-\tau) + \Phi(x(t), x(t-\tau)), \tag{14}$$

where $\Lambda, \mathrm{P} \in \mathbb{R}^{n\times n}$ are constant matrices and $\Phi\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an analytic function with the near-zero feature $\Phi(0,0) = 0$.

Since we assume that $\tau > 0$, we may introduce the dimensionless time $\tilde{t} = t/\tau$. Note that the characteristic exponents and the associated frequencies are also transformed as $\tilde{\lambda} = \tau\lambda$ and $\tilde{\omega} = \tau\omega$, respectively. To simplify notation, we remove tildes, so the rescaled form of equation (14) becomes

$$\dot{x}(t) = \tau\Lambda x(t) + \tau\mathrm{P}x(t-1) + \tau\Phi(x(t), x(t-1)). \tag{15}$$

Hereafter, we consider the time delay τ as the bifurcation parameter μ. This is a natural choice in applications where the mathematical models are extended by modelling delay effects. The calculations below can still be carried out in the same way if different bifurcation parameters are chosen.

The characteristic function of (15) assumes the form

$$D(\lambda;\tau) = \det(\lambda\mathrm{I} - \tau\Lambda - \tau\mathrm{P}\mathrm{e}^{-\lambda}). \tag{16}$$

We suppose, that the necessary conditions (11) and (13) are also fulfilled, that is, there exists a critical time delay τ_{cr} such that

$$\lambda_{1,2}(\tau_{\mathrm{cr}}) = \pm\mathrm{i}\omega, \qquad \mathrm{Re}\left(\frac{\mathrm{d}\lambda_{1,2}(\tau_{\mathrm{cr}})}{\mathrm{d}\tau}\right) \neq 0, \tag{17}$$

while all the other characteristic exponents λ_k, $k = 3, 4, \ldots$ are situated on the left-hand side of the imaginary axis when the time delay is in a finite neighbourhood of its critical value.

4.1. Operator Differential Equation

The dimensionless delay-differential equation (15) can be rewritten in the form of an operator-differential equation (OpDE). In the case $\tau = \tau_{\mathrm{cr}}$ we obtain

$$\dot{x}_t = \mathcal{A}x_t + \mathcal{F}(x_t), \tag{18}$$

where the linear and non-linear operators $\mathcal{A}, \mathcal{F}\colon \mathbb{X}_{\mathbb{R}^n} \to \mathbb{X}_{\mathbb{R}^n}$ are defined as

$$\mathcal{A}\phi(\vartheta) = \begin{cases} \phi'(\vartheta) & \text{if} \quad -1 \leq \vartheta < 0 \\ \mathrm{L}\phi(0) + \mathrm{R}\phi(-1) & \text{if} \quad \vartheta = 0, \end{cases} \tag{19}$$

$$\mathcal{F}(\phi)(\vartheta) = \begin{cases} 0 & \text{if} \quad -1 \leq \vartheta < 0 \\ F(\phi(0), \phi(-1)) & \text{if} \quad \vartheta = 0, \end{cases} \tag{20}$$

respectively. Here, dot still refers to differentiation with respect to the time t, while prime stands for differentiation with respect to ϑ. The $n \times n$ matrices L, R, and the near-zero non-linear function F are given as

$$\mathrm{L} = \tau_{\mathrm{cr}} \Lambda, \quad \mathrm{R} = \tau_{\mathrm{cr}} \mathrm{P}, \quad \text{and} \quad F = \tau_{\mathrm{cr}} \Phi. \tag{21}$$

Note that the consideration of first rows of the operators $\mathcal{A}$, $\mathcal{F}$ (19,20) on domains of $\mathbb{X}_{\mathbb{R}^n}$ which are restricted by their second rows, gives the same mathematical description as shown in [2].

It is possible to prove that the operator $\mathcal{A}$ has the same characteristic roots as the linear part of the delay-differential equation (15), because

$$\mathrm{Ker}(\lambda\mathcal{I} - \mathcal{A}) \neq \{0\} \quad \Leftrightarrow \quad \det(\lambda\mathrm{I} - \mathrm{L} - \mathrm{R}e^{-\lambda}) = 0, \tag{22}$$

and the corresponding two critical characteristic exponents (17) are also the same:

$$\lambda_{1,2}(\tau_{\mathrm{cr}}) = \pm\mathrm{i}\omega. \tag{23}$$

Although the OpDE (18) can be defined for any value of the bifurcation parameter τ, the calculations are presented for the critical value τ_{cr} only, since the subsequent Hopf bifurcation calculations use the system parameters at the critical point.

4.2. Centre-Manifold Reduction

We present the algorithm of Hopf bifurcation calculation for the general OpDE (18). In order to do this, let us determine the real eigenvectors $s_{1,2} \in \mathbb{X}_{\mathbb{R}^n}$ of the linear operator $\mathcal{A}$ associated with the critical eigenvalue $\lambda_1 = \mathrm{i}\omega$. These eigenvectors satisfy

$$\mathcal{A}s_1(\vartheta) = -\omega s_2(\vartheta), \quad \mathcal{A}s_2(\vartheta) = \omega s_1(\vartheta). \tag{24}$$

Substituting the definition (19), these equations form the $2n$-dimensional coupled linear first order boundary value problem

$$\begin{bmatrix} s_1'(\vartheta) \\ s_2'(\vartheta) \end{bmatrix} = \omega \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} s_1(\vartheta) \\ s_2(\vartheta) \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{L} & \omega\mathrm{I} \\ -\omega\mathrm{I} & \mathrm{L} \end{bmatrix} \begin{bmatrix} s_1(0) \\ s_2(0) \end{bmatrix} + \begin{bmatrix} \mathrm{R} & 0 \\ 0 & \mathrm{R} \end{bmatrix} \begin{bmatrix} s_1(-1) \\ s_2(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{25}$$

Its solution is

$$\begin{bmatrix} s_1(\vartheta) \\ s_2(\vartheta) \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \cos(\omega\vartheta) + \begin{bmatrix} -S_2 \\ S_1 \end{bmatrix} \sin(\omega\vartheta), \tag{26}$$

with constant vectors $S_{1,2} \in \mathbb{R}^n$ having two freely eligible scalar variables while satisfying the linear homogeneous equations

$$\begin{bmatrix} \mathrm{L} + \mathrm{R}\cos\omega & \omega\mathrm{I} + \mathrm{R}\sin\omega \\ -(\omega\mathrm{I} + \mathrm{R}\sin\omega) & \mathrm{L} + \mathrm{R}\cos\omega \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{27}$$

In order to project the system to the plane spanned by s_1 and s_2, and to its complementary space, we also need to determine the adjoint of the operator $\mathcal{A}$ (see [7] for details):

$$\mathcal{A}^*\psi(\sigma) = \begin{cases} -\psi'(\sigma) & \text{if} \quad 0 \le \sigma < 1 \\ \mathrm{L}^*\psi(0) + \mathrm{R}^*\psi(1) & \text{if} \quad \sigma = 0, \end{cases} \tag{28}$$

where * denotes either adjoint operator or transposed conjugate matrix.

The real eigenvectors $n_{1,2}$ of $\mathcal{A}^*$ associated with $\lambda_1^* = -\mathrm{i}\omega$ are determined by

$$\mathcal{A}^* n_1(\sigma) = \omega n_2(\sigma), \quad \mathcal{A}^* n_2(\sigma) = -\omega n_1(\sigma). \tag{29}$$

It results in the boundary value problem

$$\begin{bmatrix} n_1'(\sigma) \\ n_2'(\sigma) \end{bmatrix} = \omega \begin{bmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{bmatrix} \begin{bmatrix} n_1(\sigma) \\ n_2(\sigma) \end{bmatrix},$$

$$\begin{bmatrix} \mathrm{L}^* & -\omega\mathrm{I} \\ \omega\mathrm{I} & \mathrm{L}^* \end{bmatrix} \begin{bmatrix} n_1(0) \\ n_2(0) \end{bmatrix} + \begin{bmatrix} \mathrm{R}^* & 0 \\ 0 & \mathrm{R}^* \end{bmatrix} \begin{bmatrix} n_1(1) \\ n_2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{30}$$

when one uses definition (28). It has the solution

$$\begin{bmatrix} n_1(\sigma) \\ n_2(\sigma) \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \cos(\omega\sigma) + \begin{bmatrix} -N_2 \\ N_1 \end{bmatrix} \sin(\omega\sigma), \tag{31}$$

with constant vectors $N_{1,2} \in \mathbb{R}^n$ having two freely eligible scalar variables while satisfying

$$\begin{bmatrix} \mathrm{L}^* + \mathrm{R}^*\cos\omega & -(\omega\mathrm{I} + \mathrm{R}^*\sin\omega) \\ \omega\mathrm{I} + \mathrm{R}^*\sin\omega & \mathrm{L}^* + \mathrm{R}^*\cos\omega \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{32}$$

We define here the inner product (see [7])

$$\langle \psi, \phi \rangle = \psi^*(0)\phi(0) + \int_{-1}^{0} \psi^*(\xi + 1)\mathrm{R}\phi(\xi)d\xi, \tag{33}$$

which is used to calculate the orthonormality conditions

$$\langle n_1, s_1 \rangle = 1, \quad \langle n_1, s_2 \rangle = 0. \tag{34}$$

These determine two of the four freely eligible scalar values in vectors $S_{1,2}$, $N_{1,2}$. The application of (33) results in two linear non-homogeneous equations, which are arranged for the two free parameters in $N_{1,2}$:

$$\frac{1}{2} \begin{bmatrix} S_1^*\left(2\mathrm{I} + \mathrm{R}^*\left(\cos\omega + \frac{\sin\omega}{\omega}\right)\right) + S_2^*\mathrm{R}^*\sin\omega & -S_1^*\mathrm{R}^*\sin\omega + S_2^*\mathrm{R}^*\left(\cos\omega - \frac{\sin\omega}{\omega}\right) \\ -S_1^*\mathrm{R}^*\sin\omega + S_2^*\left(2\mathrm{I} + \mathrm{R}^*\left(\cos\omega + \frac{\sin\omega}{\omega}\right)\right) & -S_1^*\mathrm{R}^*\left(\cos\omega - \frac{\sin\omega}{\omega}\right) - S_2^*\mathrm{R}^*\sin\omega \end{bmatrix}$$

$$\times \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{35}$$

Note that there are still two free scalar parameters. For example, one may take 1 as the first component of S_1 and 0 as the first component of S_2; see [1].

With the help of the right and left eigenvectors $s_{1,2}$ and $n_{1,2}$ of operator $\mathcal{A}$, we are able to introduce the new state variables

$$\begin{cases} z_1 = \langle n_1, x_t \rangle, \\ z_2 = \langle n_2, x_t \rangle, \\ w = x_t - z_1 s_1 - z_2 s_2, \end{cases} \tag{36}$$

where $z_{1,2}\colon \mathbb{R} \to \mathbb{R}$ and $w\colon \mathbb{R} \to \mathbb{X}_{\mathbb{R}^n}$. Using the eigenvectors (26, 31) satisfying (24, 29) and the inner product definition (33), the OpDE (18) can be rewritten with the new variables (36):

$$\begin{aligned} \dot z_1 &= \langle n_1, \dot x_t \rangle = \langle n_1, \mathcal{A} x_t + \mathcal{F}(x_t) \rangle = \langle \mathcal{A}^* n_1, x_t \rangle + \langle n_1, \mathcal{F}(x_t) \rangle \\ &= \omega \langle n_2, x_t \rangle + n_1^*(0) \mathcal{F}(x_t)(0) = \omega z_2 + N_1^* \mathcal{F}(x_t)(0), \\ \dot z_2 &= -\omega z_1 + N_2^* \mathcal{F}(x_t)(0), \\ \dot w &= \dot x_t - \dot z_1 s_1 - \dot z_2 s_2 \\ &= \mathcal{A} x_t + \mathcal{F}(x_t) - \omega z_2 s_1 + \omega z_1 s_2 - N_1^* \mathcal{F}(x_t)(0) s_1 - N_2^* \mathcal{F}(x_t)(0) s_2, \end{aligned} \tag{37}$$

that is,

$$\begin{aligned} \begin{bmatrix} \dot z_1 \\ \dot z_2 \\ \dot w \end{bmatrix} &= \begin{bmatrix} 0 & \omega & \mathcal{O} \\ -\omega & 0 & \mathcal{O} \\ 0 & 0 & \mathcal{A} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ w \end{bmatrix} \\ &\quad + \begin{bmatrix} N_1^* \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) \\ N_2^* \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) \\ -\sum_{j=1,2} N_j^* \mathcal{F}(z_1 s_1 + z_2 s_2 + w)(0) s_j + \mathcal{F}(z_1 s_1 + z_2 s_2 + w) \end{bmatrix}. \end{aligned} \tag{38}$$

It shows the structure of OpDE (18) after projection to the plane spanned by s_1 and s_2, and to its complementary space.

Now, we need to expand the non-linearities in power series form, and to keep only those, which result in terms of degree up to three after the reduction to the centre-manifold. To this end, only those terms are calculated for $\dot z_{1,2}$ that have second and third order in $z_{1,2}$ and the terms $z_{1,2} w_i$, $(i = 1, \dots, n)$, while for $\dot w$, only the second order terms in $z_{1,2}$ are needed. This calculation is possible directly by the Taylor expansion of the analytic function $F\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ of (21) in the definition of (20) of the near-zero operator $\mathcal{F}$. The resulting truncated system of

OpDE (38) assumes the form:

$$\begin{bmatrix}\dot z_1\\ \dot z_2\\ \dot w\end{bmatrix}=\begin{bmatrix}0&\omega&\mathcal{O}\\ -\omega&0&\mathcal{O}\\ 0&0&\mathcal{A}\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ w\end{bmatrix}+\begin{bmatrix}\sum_{j,k\geq 0}^{j+k=2,3} f^{(1)}_{jk0}z_1^j z_2^k\\ \sum_{j,k\geq 0}^{j+k=2,3} f^{(2)}_{jk0}z_1^j z_2^k\\ \frac{1}{2}\sum_{j,k\geq 0}^{j+k=2}\big(f^{(3c)}_{jk0}\cos(\omega\vartheta)+f^{(3s)}_{jk0}\sin(\omega\vartheta)\big)z_1^j z_2^k\end{bmatrix}$$

$$+\begin{bmatrix}\sum_{i=1}^{n}\Big(\big(f^{(1l)}_{101,i}z_1+f^{(1l)}_{011,i}z_2\big)w_i(0)+\big(f^{(1r)}_{101,i}z_1+f^{(1r)}_{011,i}z_2\big)w_i(-1)\Big)\\ \sum_{i=1}^{n}\Big(\big(f^{(2l)}_{101,i}z_1+f^{(2l)}_{011,i}z_2\big)w_i(0)+\big(f^{(2r)}_{101,i}z_1+f^{(2r)}_{011,i}z_2\big)w_i(-1)\Big)\\ \frac{1}{2}\begin{cases}0 & \text{if}\quad -1\leq\vartheta<0\\ \sum_{j,k\geq 0}^{j+k=2} f^{(3)}_{jk0}z_1^j z_2^k & \text{if}\quad \vartheta=0\end{cases}\end{bmatrix}, \tag{39}$$

where the subscripts of the constant coefficients $f^{(1,2)}_{jkm}\in\mathbb{R}$, and the vector ones $f^{(3)}_{jkm}\in\mathbb{R}^n$ refer to the corresponding jth, kth and mth orders of z_1, z_2 and w, respectively. The terms with the coefficients $f^{(3s)}_{jk0}$, $f^{(3c)}_{jk0}$ come from the linear combinations of $s_1(\vartheta)$ and $s_2(\vartheta)$, while the terms with coefficients $f^{(3)}_{jk0}$ and the zero above them refer to the structure of the non-linear operator $\mathcal{F}$ (20).

The plane spanned by the eigenvectors s_1 and s_2 is tangent to the centre-manifold (CM) at the origin. This means, that the CM can be approximated locally as a truncated power series of w depending on the second order of the co-ordinates z_1 and z_2:

$$w(\vartheta)=\frac{1}{2}\left(h_{20}(\vartheta)z_1^2+2h_{11}(\vartheta)z_1z_2+h_{02}(\vartheta)z_2^2\right). \tag{40}$$

The unknown coefficients h_{20}, h_{11} and $h_{02}\in\mathbb{X}_{\mathbb{R}^n}$ can be determined by calculating the derivative of w in (40). On the one hand, it is expressed to the second order by the substitution of the linear part of first two equations of (39):

$$\dot w(\vartheta)=-\omega h_{11}(\vartheta)z_1^2+\omega(h_{20}(\vartheta)-h_{02}(\vartheta))z_1z_2+\omega h_{11}(\vartheta)z_2^2, \tag{41}$$

on the other hand, this derivative can also be expressed by the third equation of (39). The comparison of the coefficients of z_1^2, z_1z_2 and z_2^2 gives a linear boundary value problem for the unknown coefficients of the CM, that is, the differential equation

$$\begin{bmatrix}h'_{20}(\vartheta)\\ h'_{11}(\vartheta)\\ h'_{02}(\vartheta)\end{bmatrix}=\begin{bmatrix}0&-2\omega\mathrm{I}&0\\ \omega\mathrm{I}&0&-\omega\mathrm{I}\\ 0&2\omega\mathrm{I}&0\end{bmatrix}\begin{bmatrix}h_{20}(\vartheta)\\ h_{11}(\vartheta)\\ h_{02}(\vartheta)\end{bmatrix}$$
$$-\begin{bmatrix}f^{(3c)}_{200}\\ \frac{1}{2}f^{(3c)}_{110}\\ f^{(3c)}_{020}\end{bmatrix}\cos(\omega\vartheta)-\begin{bmatrix}f^{(3s)}_{200}\\ \frac{1}{2}f^{(3s)}_{110}\\ f^{(3s)}_{020}\end{bmatrix}\sin(\omega\vartheta), \tag{42}$$

with the boundary condition

$$\begin{bmatrix} \mathrm{L} & 2\omega\mathrm{I} & 0 \\ -\omega\mathrm{I} & \mathrm{L} & \omega\mathrm{I} \\ 0 & -2\omega\mathrm{I} & \mathrm{L} \end{bmatrix} \begin{bmatrix} h_{20}(0) \\ h_{11}(0) \\ h_{02}(0) \end{bmatrix} + \begin{bmatrix} \mathrm{R} & 0 & 0 \\ 0 & \mathrm{R} & 0 \\ 0 & 0 & \mathrm{R} \end{bmatrix} \begin{bmatrix} h_{20}(-1) \\ h_{11}(-1) \\ h_{02}(-1) \end{bmatrix} = -\begin{bmatrix} f_{200}^{(3c)} + f_{200}^{(3)} \\ \frac{1}{2}\left(f_{110}^{(3c)} + f_{110}^{(3)}\right) \\ f_{020}^{(3c)} + f_{020}^{(3)} \end{bmatrix}. \tag{43}$$

The general solution of (42) takes the form

$$\begin{aligned} \begin{bmatrix} h_{20}(\vartheta) \\ h_{11}(\vartheta) \\ h_{02}(\vartheta) \end{bmatrix} &= \begin{bmatrix} H_0 \\ 0 \\ H_0 \end{bmatrix} + \begin{bmatrix} H_1 \\ H_2 \\ -H_1 \end{bmatrix} \cos(2\omega\vartheta) + \begin{bmatrix} -H_2 \\ H_1 \\ H_2 \end{bmatrix} \sin(2\omega\vartheta) \\ &+ \frac{1}{3\omega}\left(\begin{bmatrix} f_{110}^{(3c)} + f_{200}^{(3s)} + 2 f_{020}^{(3s)} \\ -\frac{1}{2} f_{110}^{(3s)} - f_{200}^{(3c)} + f_{020}^{(3c)} \\ - f_{110}^{(3c)} + 2 f_{200}^{(3s)} + f_{020}^{(3s)} \end{bmatrix} \cos(\omega\vartheta) \right. \\ &\left. + \begin{bmatrix} f_{110}^{(3s)} - f_{200}^{(3c)} - 2 f_{020}^{(3c)} \\ \frac{1}{2} f_{110}^{(3c)} - f_{200}^{(3s)} + f_{020}^{(3s)} \\ - f_{110}^{(3s)} - 2 f_{200}^{(3c)} - f_{020}^{(3c)} \end{bmatrix} \sin(\omega\vartheta) \right), \end{aligned} \tag{44}$$

where the unknown constant vectors H_0, H_1 and $H_2 \in \mathbb{R}^n$ are determined by the linear non-homogeneous equations

$$\begin{aligned} &\begin{bmatrix} \mathrm{L}+\mathrm{R} & 0 & 0 \\ 0 & \mathrm{L}+\mathrm{R}\cos(2\omega) & 2\omega\mathrm{I}+\mathrm{R}\sin(2\omega) \\ 0 & -\left(2\omega\mathrm{I}+\mathrm{R}\sin(2\omega)\right) & \mathrm{L}+\mathrm{R}\cos(2\omega) \end{bmatrix} \begin{bmatrix} H_0 \\ H_1 \\ H_2 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} f_{200}^{(3)} + f_{020}^{(3)} \\ f_{200}^{(3)} - f_{020}^{(3)} \\ f_{110}^{(3)} \end{bmatrix} \\ &+ \frac{1}{6\omega} \begin{bmatrix} (\mathrm{L}+\mathrm{R}\cos\omega)\left(-3 f_{200}^{(3s)} - 3 f_{020}^{(3s)}\right) + (\omega\mathrm{I}+\mathrm{R}\sin\omega)\left(-3 f_{200}^{(3c)} - 3 f_{020}^{(3c)}\right) \\ (\mathrm{L}+\mathrm{R}\cos\omega)\left(-2 f_{110}^{(3c)} + f_{200}^{(3s)} - f_{020}^{(3s)}\right) + (\omega\mathrm{I}+\mathrm{R}\sin\omega)\left(2 f_{110}^{(3s)} + f_{200}^{(3c)} - f_{020}^{(3c)}\right) \\ (\mathrm{L}+\mathrm{R}\cos\omega)\left(f_{110}^{(3s)} + 2 f_{200}^{(3c)} - 2 f_{020}^{(3c)}\right) + (\omega\mathrm{I}+\mathrm{R}\sin\omega)\left(f_{110}^{(3c)} - 2 f_{200}^{(3s)} + 2 f_{020}^{(3s)}\right) \end{bmatrix}, \end{aligned} \tag{45}$$

arising from (43). Note that these equations are decoupled in variable H_0 and in variables $H_{1,2}$.

The above calculation based on (40)–(45) is called *centre-manifold reduction*, which is one of the key components of the Hopf bifurcation calculation.

4.3. *Poincaré Normal Form*

One may solve (45) and reconstruct the approximate equation of the CM by (44) and (40). Then calculating only the components $w(0)$ and $w(-1)$ of $w(\vartheta)$, and substituting them into the first two scalar equations of (39), we obtain the equations

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} \sum_{j,k\geq 0}^{j+k=2,3} a_{jk}^{(1)} z_1^j z_2^k \\ \sum_{j,k\geq 0}^{j+k=2,3} a_{jk}^{(2)} z_1^j z_2^k \end{bmatrix}, \tag{46}$$

which describe the flow restricted onto the two-dimensional CM. We note that the coefficients of the second order terms in the first two equation of (39) are not changed by the CM reduction, *i.e.*, $f_{jk0}^{(1,2)} = a_{jk}^{(1,2)}$ when $j + k = 2$. Using the coefficients of the Poincaré normal form (46), the so-called Poincaré-Lyapunov coefficient can be determined by the Bautin formula (see [15]):

$$\Delta = \frac{1}{8}\bigg(\frac{1}{\omega}\Big((a_{20}^{(1)} + a_{02}^{(1)})(-a_{11}^{(1)} + a_{20}^{(2)} - a_{02}^{(2)}) + (a_{20}^{(2)} + a_{02}^{(2)})(a_{20}^{(1)} - a_{02}^{(1)} + a_{11}^{(2)})\Big) \\ + \Big(3a_{30}^{(1)} + a_{12}^{(1)} + a_{21}^{(2)} + 3a_{03}^{(2)}\Big)\bigg). \tag{47}$$

It shows the type of the bifurcation and approximate amplitude of the periodic solution, so that the bifurcation is supercritical (subcritical) if $\Delta < 0$ ($\Delta > 0$), and the amplitude of the stable (unstable) oscillation is expressed by

$$A = \sqrt{-\frac{1}{\Delta}\mathrm{Re}\frac{\mathrm{d}\lambda_{1,2}(\tau_{\mathrm{cr}})}{\mathrm{d}\tau}(\tau - \tau_{\mathrm{cr}})}. \tag{48}$$

So the first Fourier term of the oscillation on the centre-manifold is

$$\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = A \begin{bmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}. \tag{49}$$

Note that it is valid with and without tildes, since it only includes the frequency and the time in the product form $\omega t = \tilde{\omega}\tilde{t}$. Since $x(t) = x_t(0)$ by definition and not too far from the critical bifurcation (delay) parameter $x_t(\vartheta) \approx z_1(t)s_1(\vartheta) + z_2(t)s_2(\vartheta)$, the formula (49) of the periodic solution yields

$$\begin{aligned} x(t) &\approx z_1(t)s_1(0) + z_2(t)s_2(0) \\ &= A\big(s_1(0)\cos(\omega t) - s_2(0)\sin(\omega t)\big) \\ &= A\big(S_1\cos(\omega t) - S_2\sin(\omega t)\big). \end{aligned} \tag{50}$$

The non-linear oscillations around the equilibrium (3) are well approximated with this harmonic oscillation when $\tau - \tau_{\mathrm{cr}}$ is sufficiently small.

5. Conclusion

We have given a general overview of the algorithm for the Hopf bifurcation calculation in time delayed systems with one discrete delay. The centre-manifold reduction has also been carried out in the infinite-dimensional phase space. The calculations resulted in closed-form linear algebraic equations which can be solved analytically in systems where the number of parameters is not too large. This formal structure will make such calculations simpler in a very wide range of applications in future.

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