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SPACE FILLINGS WITH MANY SYMMETRIES

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Abstract

After the discovery of the 219 (230) Euclidean three-dimensional space groups many interesting questions have been investigated, e.g. questions related to space filling polyhedra. The goal of this paper is to illustrate maximal tilings in E^3 in the sense of Delaney–Dress symbols, using just a few barycentric simplex orbits and many symmetries.

Keywords: space tiling, D-symbol, polyhedra.

1. Introduction

Assume that a group Γ acts from the right discretely on a d-dimensional, simply connected manifold X^d in such a way that one can find a Γ-*invariant cell decomposition*. That is, if we denote the set of cells by $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}^\gamma := \{A^\gamma : A \in \mathcal{T}\}$ holds for all $\gamma \in \Gamma$. The elements of $\mathcal{T}$ are the so-called *cells*. Every point of X^d belongs to at least one tile and no two tiles have an inner point in common. The points of X^d, belonging to exactly two tiles, constitute the $(d-1)$-hyperfaces, or *facets* of $\mathcal{T}$. By intersections we consequently define $(d-2)$-faces, $\ldots$, r-faces, …, 1-faces or *edges*, then 0-faces or *vertices*, as usual for compact (topological) d-politopes. The above pair $(\mathcal{T}, \Gamma)$ is called *equivariant tiling*. In our examinations the *symmetry group* Γ consists of isometries and contains at least d independent translations, so it is always *periodic*. In the following we restrict ourselves to $X^d = E^3$, although the theory also works on any X^d. Since about 1890 it has been well-known that in E^3 there are 230, resp. 219 possibilities for Γ, depending on whether one distinguishes the enantiomorphic group pairs or not.

Two tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ will be considered equivalent if they are topologically equivariant (homeomeric). It means that there exists a homeomorphism ψ that maps $\mathcal{T}$ onto $\mathcal{T}'$ preserving all incidences of tiles, faces, edges and vertices such that $\psi^{-1}\Gamma\psi = \Gamma'$.

Now we define the *formal barycentric subdivision* of $\mathcal{T}$ in the usual way: For every r-dimensional constituent of $\mathcal{T}$ $(r = 0, 1, 2, 3)$ we choose an interior point, called r-center of $\mathcal{T}$. Consider a fixed tile, one of its faces, an edge lying on it, finally an incident vertex. These four centers form the vertices of a three-dimensional simplex. Other sequence of r-centers leads to other simplex in the tile. Using the method for every tile we finally get the barycentric subdivision made up by simplices called *chambers*. The chamber system is denoted by $\mathcal{C}$. Every chamber has an *i-face* opposite to its i-vertex ($i \in I := \{0, 1, 2, 3\}$). It is obvious that for every chamber $C_1 \in \mathcal{C}$ there exists exactly one chamber C_2 such that their i-face is common. In this case we say that C_1 and C_2 are *i-adjacent* or *i-neighbors*. These adjacencies imply the so-called *adjacency operations* σ_i (for $i = 0, 1, 2, 3$):

$$\sigma_i : \mathcal{C} \to \mathcal{C}, \quad C \mapsto \sigma_i C$$

that maps every $C \in \mathcal{C}$ onto its i-neighbor.

The adjacency operations form a free Coxeter group:

$$\Sigma_I := \langle \sigma_i \big| 1 = \sigma_i \sigma_i = \sigma_i^2 : i = 0, 1, 2, 3 \rangle$$

that acts transitively from the left on $\mathcal{C}$, if Γ acts from the right, by our convention.

Note that the chamber system $\mathcal{C}$ can always be constructed in a way compatible with the action of Γ on $\mathcal{T}$, and suppose in the following that this is the case. Take a chamber $C \in \mathcal{C}$ and form its *orbit* by Γ:

$$C^{\Gamma} := \{C^{\gamma} : \gamma \in \Gamma\}.$$

Let $\mathcal{D} := \mathcal{C}/\Gamma$ be the set of different chamber orbits under Γ and let D_k be any orbit ($1 \leq k \leq n$, the number of orbits). Any $\gamma \in \Gamma$ maps i-neighbors onto i-neighbors, hence the operations σ_i commute with Γ on $\mathcal{C}$, for any i. Thus we can introduce the concept of *i-adjacencies of D_k's*: D_j and D_k are i-adjacent or i-neighbor iff for any $C_j \in D_j$ there exists $C_k \in D_k$ such that $C_k = \sigma_i C_j$ holds.

The set $\mathcal{D}$ and the mappings σ_i define a finite, connected, four-colored graph in which the nodes refer to the orbits and two nodes are linked by an i-colored edge ($i = 0, 1, 2, 3$) if the corresponding orbits are i-neighbors. Such a graph is called a *Delaney–Dress graph (diagram)* or shortly *D-graph*. Of course, $D = \sigma_i D$ is also possible, in this case we get an i-loop.

For short D_k will simply be denoted by k in the following.

Let us introduce a matrix function $\left(m_{ij}\right) : \mathcal{D} \to \mathbf{N}_{I \times I}$ in the following way. For any $D \in \mathcal{D}$ let

$$m_{ij}(D) := \min \left\{ m \big| \left(\sigma_j \sigma_i\right)^m C = C, \quad C \in D \in \mathcal{D} \right\}, (0 \leq i \leq j \leq 3).$$

It is easy to see that in a tiling this function has the properties 1–5:

1. $m_{ii}(D) = 1$;
2. $m_{ij}(D) = m_{ji}(D)$;
3. $m_{ij}(D) = m_{ij}(\sigma_i D) = m_{ij}(\sigma_j D)$;
4. $m_{ij}(D) = 2$, if $|i - j| > 1$;
5. $m_{ij}(D) > 2$, if $|i - j| = 1$ in the usual tilings.

A pair $(\mathcal{D}; m)$, consisting of a finite, connected, colored D-graph and the matrix function fulfilling the properties 1–5, is called a *Delaney–Dress symbol*, or shortly *D-symbol*.

Two D-symbols $(\mathcal{D}; m)$, $(\mathcal{D}'; m')$ are called *isomorphic* if there exists a bijection $\pi : \mathcal{D} \to \mathcal{D}'$ (called *D-isomorphism*) such that $\sigma_k (D^\pi) = (\sigma_k D)^\pi$ moreover, $m'_{ij} (D^\pi) = m_{ij}(D)$ hold for any $D \in \mathcal{D}$, $0 \le k \le 3$, $0 \le i \le j \le 3$.

Any surjective mapping $\psi : \mathcal{D} \to \mathcal{D}'$ is called *D-morphism* if it satisfies the following conditions: $|\mathcal{D}| \ge |\mathcal{D}'|$, $\sigma_k \left(D^\psi\right) = (\sigma_k D)^\psi$, and $m'_{ij} \left(D^\psi\right) = m_{ij}(D)$ hold for any $D \in \mathcal{D}$, $0 \le k \le 3$, $0 \le i \le j \le 3$.

A tiling $(\mathcal{T}, \Gamma)$ is a *maximal* one if $\Gamma = Aut\mathcal{T}$, viz. the symmetry group is the set of all automorphisms of $\mathcal{T}$ as incidence structure of faces.

A maximal tiling $(\mathcal{T}, \Gamma)$ has essential role because it can be considered as a representant of a family of tilings, where its D-symbol $(\mathcal{D}; m)$ is D-morphic image of a D-symbol $(\mathcal{D}'; m')$ of any member $(\mathcal{T}', \Gamma')$ of the family. Then any member of the family is called a *symmetry breaking* of the maximal representative tiling $(\mathcal{T}, \Gamma)$. In this case there is a homomorphism φ between the tilings $(\mathcal{T}', \Gamma')$ and $(\mathcal{T}, \Gamma)$ such that $\varphi : \mathcal{T}' \to \mathcal{T}$ and $\varphi^{-1}\Gamma'\varphi \subset \Gamma$ hold.

The following basic lemma provides the advantages of D-symbols in relation to classification problems:

Lemma 1 *Two tilings $(\mathcal{T}, \Gamma)$ and $(\mathcal{T}', \Gamma')$ are equivariantly equivalent (homeomeric, or lying in the same homeomorphism equivariance class), if and only if the corresponding D-symbols $(\mathcal{D}; m)$ and $(\mathcal{D}'; m')$ are isomorphic* [3]. □

Analogously as before, we can introduce other matrix functions r and v:

$$r : \mathcal{D} \to \mathbf{N}_{I\times I}, \qquad r_{ij}(D) := \min\left\{r : \left(\sigma_j\sigma_i\right)^r D = D\right\}$$

for any $D \in \mathcal{D}$, $(0 \le i \le j \le 3)$; and

$$v : \mathcal{D} \to \mathbf{N}_{I\times I}, \qquad v_{ij}(D) := m_{ij}(D)/r_{ij}(D),$$

where the above division is meant for the elements of matrices.

The theory of D-symbols has been elaborated for 2-dimensional tilings in details (see e.g. [3], the list of references in [1], [2]), however, beside results (e.g. [4], [6] and their references) there are a lot of open questions in higher dimensions.

2. Building Blocks of Space Fillings

The main goal of this paper is to present figures of simple constructions of space tilings in the sense that we shall have just 5 barycentric simplex orbits (with other words for the cardinality of the D-symbol $|\mathcal{D}| = 5$ holds) and we deal just with maximal tilings. Therefore we shall have very symmetric constructions of simple objects.

If somebody wants to make such tilings, the starting point is to determine the corresponding D-symbol. Firstly its D-graph is to be constructed as a connected, four-colored graph satisfying some conditions concerning the adjacency operations. It is important that the matrix function r inherits the properties 1–3 of m and $r_{ij}(D) = 1$ or 2, if $|i - j| > 1$. In [1] we have developed an algorithm which allows to derive D-graphs of arbitrary dimension and elements. In that publication we gave the complete enumeration of three-dimensional D-graphs of cardinality five as a result of a concrete computer program.

To make a D-symbol complete we need a matrix function m, where the corresponding tilings might be realizable in different homogeneous spaces (for more details see [6]). It would be difficult to find, just in a pure geometric manner, those values by which we get Euclidean tilings, although one can formulate at once necessary conditions using the two-dimensional subgraphs of the D-graph in question. Despite the lack of useful geometrical ideas the problem of determining those matrix functions that allow Euclidean tilings was solved by an algebraic topological approach by Olaf DELGADO–FRIEDRICHS in his doctoral thesis [2]. By the help of a sophisticated algorithm and its implementation he could list the Euclidean Delaney–Dress symbols up to cardinality ten. In Appendix A in [2] we can find nine possibilities for $|\mathcal{D}| = 5$, but we immediately exclude from the further investigation the case no. 222. 77, where – unlike in ordinary tilings – at some vertices of a tile only two faces meet.

In the other eight cases we first identified the corresponding D-graph by reading off the adjacency operations from the given involutive permutations. The D-graph itself contains information about the number of transitivity classes of tiles, faces, edges and vertices; e. g. omitting the σ_i operation in the D-graph the number k of the remaining connected subgraphs implies that we have k transitivity classes of i-dimensional centres. Moreover, if we consider those connected subgraphs that contain just i and $i+1$ colored edges, we know which orbits have the same $m_{i,i+1}$ values and which differ. In [2] the values of the function m are given in three comma-separated groups ordered for the simplex orbits by their smallest number. The given numbers characterize the following: the values m_{01} refer to the types of faces (e. g. 3 means triangle faces), m_{12} says how many faces of a tile meet at the vertices, while m_{23} informs us about how many tiles surround the edges. (The other values of m are fixed by the properties 1–5.)

Knowing these data, the construction of the simplices becomes possible. First in a topological sense, by adjacencies we can glue them together, forming the simplex orbits and building up the tiles from the fundamental domain of the corresponding space group [5]. One can easily find the necessary angles and distances and reconstruct the building blocks.

In the following we show the tiles of the eight maximal Euclidean tilings with five barycentric simplex orbits. For every case we also represent the fundamental domain $\mathcal{F}$ with the corresponding face identifications. Beside them the space group Γ will be given as well. That is, by the Poincaré algorithm the face identifications

Table 1. Euclidean D-symbols of elements five (extraction from [2])

D-graph by adjacency operations $\sigma_0, \dots, \sigma_3$	Values of $m_{i,i+1}$	Number of tiling	Space group type
σ_0=(1)(2)(3)(45)	333, 333, 644	45. 2	$Fm\bar{3}m$
σ_1=(1)(2)(34)(5)	333, 363, 333	45. 3	$F\bar{4}3m$
σ_2=(1)(23)(4)(5)	333, 433, 633	45. 7	$Fm\bar{3}m$
σ_3=(12)(3)(4)(5)	336, 433, 344	45. 12	$Fm\bar{3}m$
(1)(2)(3)(45), (1)(2)(34)(5),	333, 34, 36	54. 2	$Fd\bar{3}m$
(1)(23)(45), (12)(3)(4)(5)	333, 44, 34	54. 4	$Im\bar{3}m$
(1)(2)(34)(5), (1)(23)(45),	34, 34, 36	222. 2	$P6_3/mmc$
(12)(3)(4)(5), (1)(25)(34)	44, 34, 34	222. 52	$I4/mmm$

$\mathcal{F}$ for generators with the defining relations as edge-equivalence classes allow us to determine the symmetry group as a factor group of some free group (for more details see [5]).

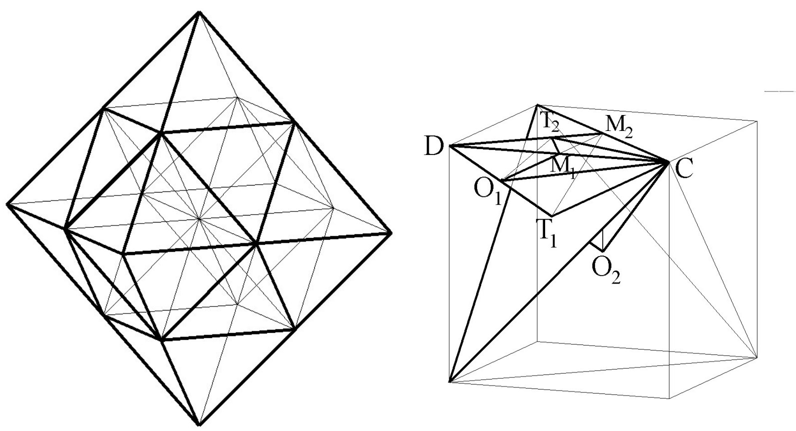

Fig. 1. a. A tiling of regular octahedra and tetrahedra by $\Gamma = Fm\bar{3}m$.
b. The fundamental domain is O_2M_2CD. The face pairings are here just reflections in the faces of this tetrahedron (reflection simplex).

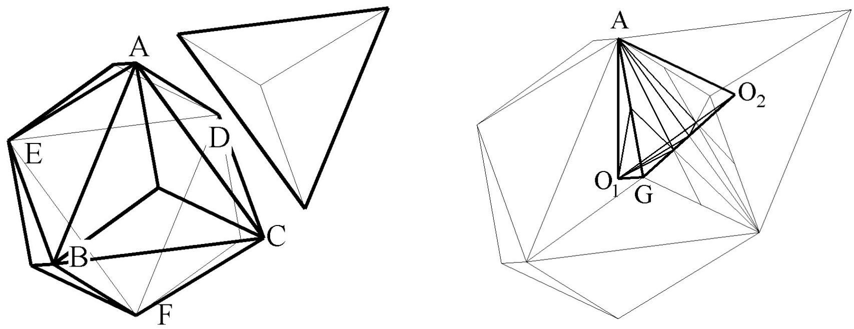

Fig. 2. a. A tiling of two types of blocks. For the first one take a regular octahedron $ABCDEF$ and let build quarters of regular tetrahedra on four of its faces (top: front and back, down: left and right), while the second block is a regular tetrahedron. The symmetry group is $\Gamma = F\bar{4}3m$.
b. The corresponding fundamental domain is a reflection simplex O_1O_2AG, again.

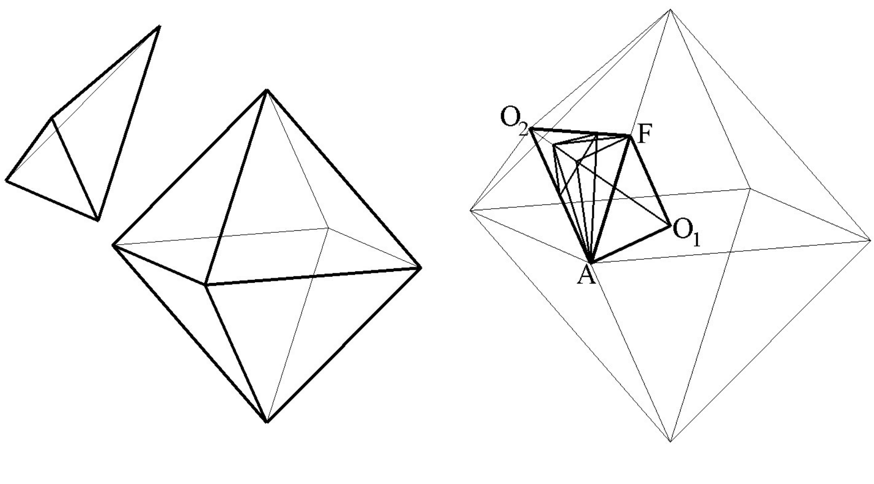

Fig. 3. a. A tiling of regular octahedra and quarters of regular tetrahedra by $\Gamma = Fm\bar{3}m$.
b. The corresponding fundamental domain is O_1O_2FA, a reflection simplex.

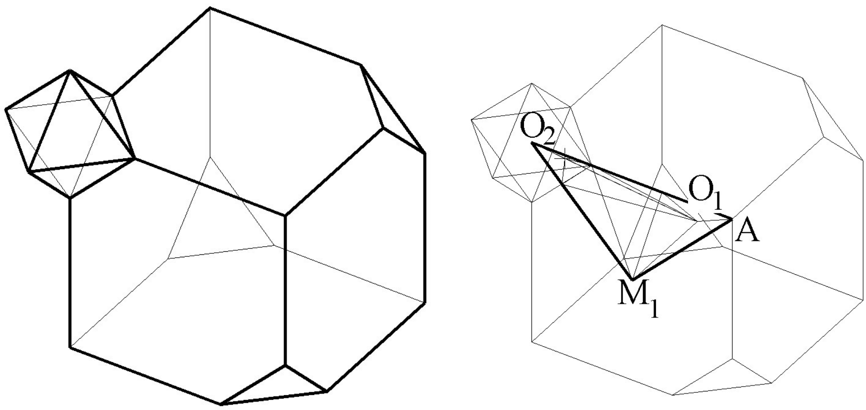

Fig. 4. a. A tiling of truncated cubes and regular octahedra by $\Gamma = Fm\bar{3}m$.
b. The reflection simplex $O_1O_2M_1A$ serves as fundamental domain.

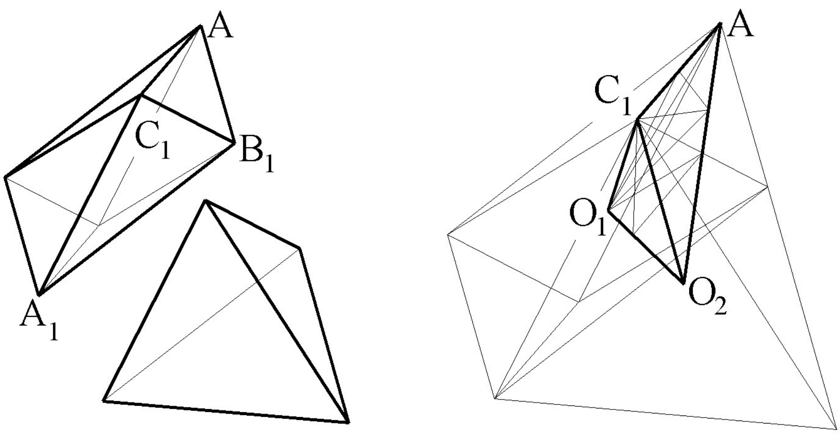

Fig. 5. a. The tiling consists of two types of tiles. For the first one take a regular triangle $(A_1B_1C_1)$, rotate it with π and lift it till the triangle faces of the deformed octahedra (e. g. AB_1C_1) would halve any regular tetrahedron built on the regular triangles. The second block is a regular tetrahedron. The corresponding group is $Fd\bar{3}m$.
b. The corresponding fundamental domain is $O_1O_2C_1A$. The generators are a twofold rotation $r : O_1C_1A \to O_1AC_1$ and reflections in the other three faces.

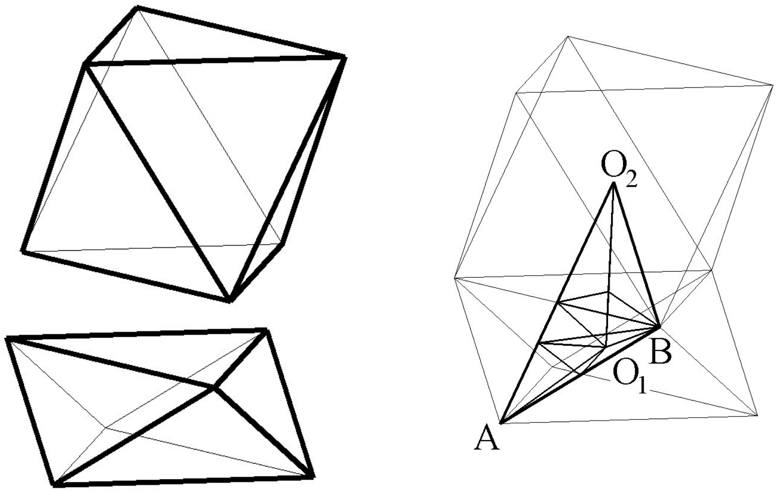

Fig. 6. a. A similar tiling to the previous one, except that we have regular octahedra instead of tetrahedra. $\Gamma = Im\overline{3}m$.
b. The corresponding fundamental domain is O_1O_2AB, with the following face identifications: a twofold rotation $r : O_1AB \to O_1BA$ and three reflections in the remaining faces.

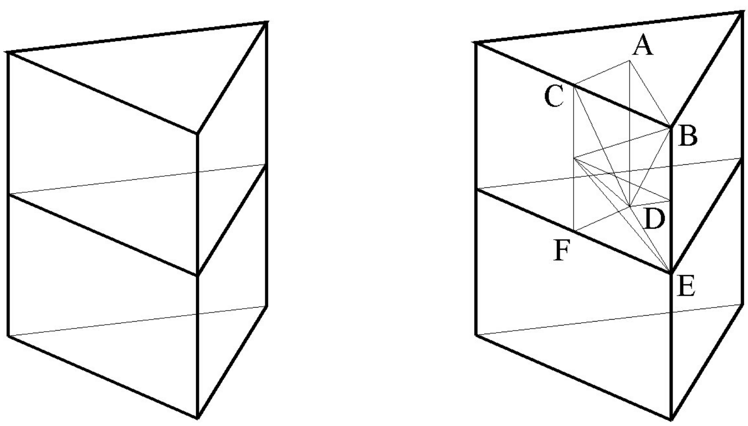

Fig. 7. a. A tiling with regular trigonal prisms. The rectangle faces should be considered as two congruent complanar faces. The corresponding group is $P6_3/mmc$.
b. The fundamental domain is a trigonal prism $ABCDEF$, the generators are plane reflections in its faces.

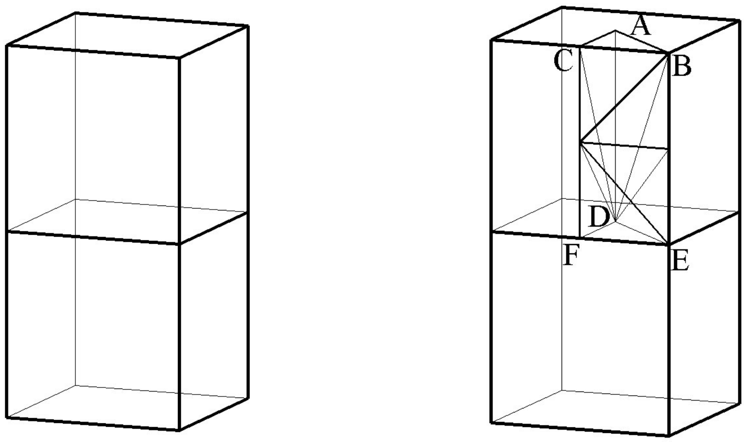

Fig. 8. a. A tiling with square based prisms, similarly as before. $\Gamma = I4/mmm$.
b. The fundamental domain is a trigonal prism $ABCDEF$, the generators are plane reflections in its faces.

3. Closing Remark

Since the maximal Euclidean Delaney–Dress symbols are known for $|\mathcal{D}| \leq 10$ [2], the construction of space filling polyhedra of all pairs $(\mathcal{T}, \Gamma)$ with up to ten barycentric simplex orbits would also be possible. By symmetry breakings other nice, less symmetrical constructions could be formed, as well.

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