

SKEW LINES IN HYPERBOLIC SPACE¹

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Dedicated to the memory of Imre Vermes

Abstract

In this paper we will show and prove a model-independent construction of the normal transversal of two skew lines in the hyperbolic 3-space.

Keywords: hyperbolic space, skew-lines.

1. Introduction, Historical Remarks

Interestingly enough, in the standard literature of hyperbolic geometry there is no trace of how to construct a common perpendicular line intersecting two skew ones. Since it is well-known that such a line exists, the problem arises whether a model-independent or absolute construction can be found. In this paper we give three constructions continuing the sort of famous analogous constructions by J. BOLYAI [1], D. HILBERT [5], R. CARSLAW [2] and others of the hyperbolic plane. In *Fig. 2* and *Fig. 3* we can see Bolyai's and Hilbert's constructions of a line parallel to a given line through a given point and perpendicular to two ultraparallel lines, respectively.

As I have been informed at the conference 'Geometrie Tagung Vorau', where I lectured on this topic, Mr. Wilhelm Fuhs (TU Wien) has obtained similar results independently. We intended to publish a joint paper, but it was not possible this time. I thank for his interest in studying my manuscript.

2. Construction in Poincaré's Halfspace Model

The normal transversal has a nice interpretation in the halfspace model, that can be seen in *Fig. 1*.

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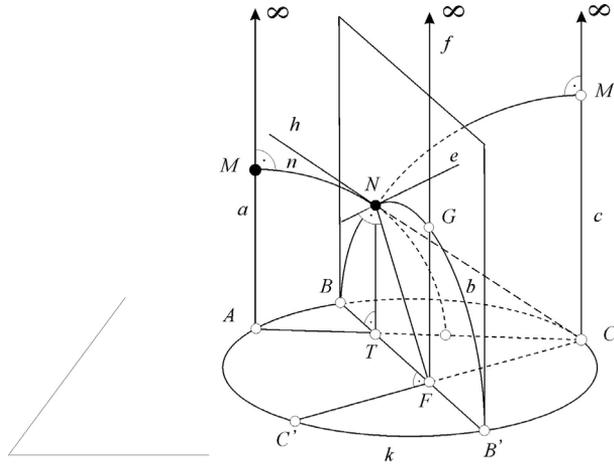


Fig. 1. The common perpendicular to two skew lines in the halfspace model

The original data are: The halfspace model embedded in the Euclidean 3-space, the skew lines a and b with ‘ideal ends’ A, ∞ and B, B' , respectively. We can assume that the representations of these lines in the model are a line and a halfcircle, respectively.

The steps of the construction, using the axiom of Euclidean space-constructions are:

1. In the boundary plane of the model we draw the circle k determined by the points A, B, B' . (This circle exists since a and b are skew lines.)
2. Draw the bisector of the Euclidean segment BB' and determine the intersections of this line with the circle k . The intersection point lying on the other arc as A will be denoted by C .
3. The intersection point of the Euclidean segments AC and BB' is the point M which is the orthogonal projection of a point N of the hyperbolic line b .
4. In the model, the plane determined by the hyperbolic line a and the point N represents the halfcircle with center A through the point N . This is the normal transversal n of the two skew lines a and b . (The intersection of a and n is the point M in Fig. 1.)

Proof. The proof is based on the properties of the embedding Euclidean 3-space and the model. Let the line e be tangent to the halfcircle – representing a hyperbolic line – b at the point N . This line is orthogonal to the Euclidean segments FN and FC , meaning that it is orthogonal to the Euclidean segment NC , too. Since this latter line is tangent the halfcircle representing the hyperbolic line n , the lines b and n are perpendicular to each other in the hyperbolic 3-space. On the other hand the line n is also perpendicular to the line a by definition, so it is the sought one. \square

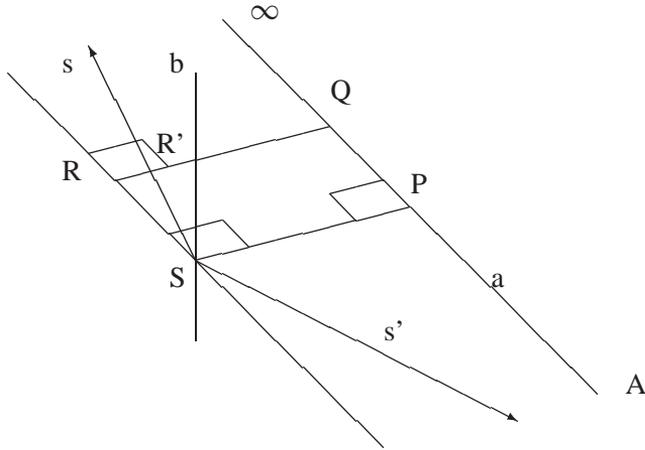


Fig. 2. The construction of the parallels through S to a . SP is orthogonal to a , SR is orthogonal to SP , RQ is orthogonal to SR . Then the point R' for which $|SR'| = |PQ|$ belongs to the ray s .

3. A Model-Independent Construction

First of all we clear up the axiom of the hyperbolic 3-space constructions. We will work using the following ones:

I: A plane is constructed if we can construct three non-collinear (nonideal!) points of it.

II: If we consider a constructed plane in the space then in this plane we can execute the known (model independent) hyperbolic constructions.

First we give a model independent construction using only the axioms above.

1. First we choose an arbitrary point S on the line b and therefore the plane determined by the point S and line a is considered to be constructed. Using J.Bolyai's construction for parallels to a line through a point in a plane, (see section 34 in [1]) we construct a ray s in the plane (S, a) through $S(\in b)$ with ideal point ∞ of line a . The plane (b, ∞) in question now is spanned by b and s . (See Fig. 2.)
2. Secondly, in the plane (b, ∞) we construct a line f which is perpendicular to the given line b and parallel to the ray s . (This is also a well-known construction given by Carlsaw (see Fig. 3 and p. 76 in [2]). Let the ideal points of the line f be ∞ and F , respectively. Let the intersection of the lines f and b be denoted by G . If we draw a line through G in the plane (b, A) , orthogonally to b , then we get its ideal points C and C' .
3. Finally, we construct (by a consequence of D. Hilbert's construction see p. 151 in [5] and Fig. 3) the lines parallel to the given rays GC, SA and GC', SA , respectively. (The construction of (b, A) is similar to the construction of (b, ∞) and may use the point S , too.) Choosing that one, which intersects the line b , we get the intersection point N . (The mutual position of the two

the bisector of the planes (a, B) , (a, B') and β is the bisector of the planes (b, A) , (b, A') , respectively.

Proof of the hyperbolic case. The hyperbolic planes determined by the line a and ideal points B and B' , respectively are intersecting (distinct) ones, therefore they determine two other planes which are the bisectors of the two angles of them. In the model all of the planes through the line a are represented by Euclidean halfplanes, so are these planes, too. Since the arcs BC and CB' of k equal, the angles BAC and CAB' equal, too. This means that one of the mentioned bisectors is the hyperbolic plane represented by the Euclidean halfplane through a and C . Thus the point N is the intersection point of this bisector with the original line b . (Since the other bisector avoids the Euclidean domains of angle BAB' it does not intersect the line b .)

□

Proof of the Euclidean case. The planes (a, B) , (a, B') and (b, A) , (b, A') coincide, respectively, furthermore (a, B) and (b, A) are parallel to each other. The planes α and β are perpendicular to the above planes containing the lines a and b , respectively. Thus their intersection is the normal transversal of a and b , too.

□

Finally, we present a nice absolute proof of this construction given by CSIKÓS [4]. The proof is based on the following absolute lemma:

Lemma 1 *Let us denote two lines of the space by a and b and move the points P and Q uniformly along these lines, respectively. Then the function defined by the distance of the points is convex.*

Proof. It is clear that we have to check that the inequality

$$|P(0)Q(0)| + |P(1)Q(1)| \geq 2|P(\frac{1}{2})Q(\frac{1}{2})|$$

holds if $|P(t)Q(t)|$ denotes the distance of the points P and Q at time t . Reflecting the point $P(0)$ to the point $Q(\frac{1}{2})$, the assertion immediately follows from the triangle inequality and the absolute version of the theorem on the center line of a triangle. (See Fig. 4):

$$\begin{aligned} |P(\frac{1}{2})Q(\frac{1}{2})| &\leq \frac{1}{2}|P(1)P'(0)| \leq \frac{1}{2}(|P(1)Q(1)| + |Q(1)P'(0)|) \\ &= \frac{1}{2}(|P(1)Q(1)| + |P(0)Q(0)|). \end{aligned}$$

□

Absolute proof for the construction of Theorem 1. First we remark that the only possibility for the normal transversal is the intersection of the two bisectors. In fact, the orthogonalities of the pairs of lines a,n and b,n imply (by reflection) that the line n belongs to the planes α and β , respectively. The only thing we

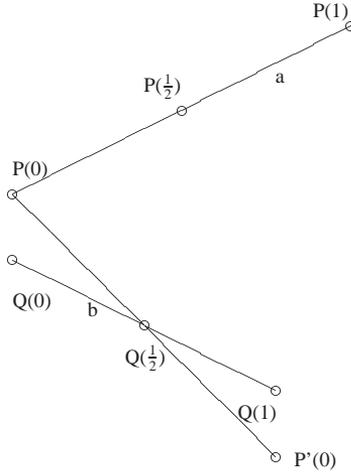


Fig. 4. The convexity of the distance function

have to prove is that the normal transversal exists. Reflect the line a to b the image will be a' . If we consider a point $P(t)$, its reflected image $P'(t)$ and its orthogonal projection $T(t)$, respectively, we get that $|P(t)P'(t)| = 2|P(t)T(t)|$ holds for every t . If the movement of the point $P(t)$ is uniform then the points $P'(t)$ and $T(t)$ move uniformly, too. Thus, by the above lemma, the function $t \rightarrow |P(t)P'(t)|$ is (strictly) convex, continuous and positive, meaning that it has a unique minimum value. Denote the corresponding parameter by t_0 . Since this position gives the unique minimum of the function $t \rightarrow |P(t)T(t)|$, too, we get a segment orthogonal to the line b , which intersects the line a and b and has minimal length from among the segments with the first two properties. Consequently, it is also orthogonal to the line a , as it can be easily seen. By projecting the point $T(t_0)$ orthogonally onto the line a , then this new point $T'(t_0)$ onto the line b ($T''(t_0)$). In this way we get $|T'(t_0)T''(t_0)| < |P(t_0)T(t_0)|$, which is a contradiction. \square

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