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ON THE SECOND-ORDER REED–MULLER CODE[1]

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Abstract

In this paper we shall give a recursion and a new explicite formula for some functions connected with the weight distribution of the second-order Reed–Muller code. We define some new subcodes of it and determine their information rates, respectively.

Keywords: Reed–Muller code, weight distribution.

1. Definitions and Lemmas

If $\underline{u} = (u_1, \cdots, u_m)$ and $\underline{v} = (v_1, \cdots, v_n)$ are two vectors then denote by $| \underline{u} | \underline{v} |$ the vector $(u_1, \cdots, u_m, v_1, \cdots, v_n))$ of length $n + m$. We shall use Theorem 2 of Ch.13.§3 in [2] which says the following:

Theorem 1

$$R(2, n+1) = \{| \underline{u} | \underline{u} + \underline{v} | \ \ \textit{where}\ \underline{u} \in R(2, n),\ \underline{v} \in R(1, n)\},$$

where $R(1, n)$ denotes the first-order Reed–Muller code of length 2^n.

Let $EG(n, 2)$ be the Euclidean geometry of dimension n over $GF(2)$ and let H be a subset of $EG(n, 2)$. Denote by $[H]$ the incidence vector of the subset H so $[H]$ is a $(0 - 1)$ vector of dimension 2^n indexed by the elements of $EG(n, 2)$, for which:

$$[H]_\alpha = \begin{cases} 1 & \text{if } \alpha \in H, \\ 0 & \text{if } \alpha \notin H. \end{cases}$$

We shall say that $H \subset EG(n, 2)$ is a codeword of $R(2, n)$ if and only if $[H] \in R(2, n)$.

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Definition 1 (of the codes $R_k(2,n)$) Let $V_1 < V_2 < \cdots < V_n$ be a nested sequence of subspaces of the geometry $EG(n,2)$, for which dim $V_k = k$. We define the code $R_k(2,n)$ ($k = 1,\dots,n$) as the set of all codewords $H \subset EG(n,2)$ of $R(2,n)$ which satisfy the condition:

$$\mid H \cap V_k \mid = 2^{k-1}.$$

It is clear that this definition depends only on the dimension k, because a regular linear transformation of the space $EG(n,2)$ induces a bijection of the code $R(2,n)$ onto itself.

Remark 1 *First of all the results of this paper enlarge the aspect of the very important code $R(2,n)$ though the necessity of examination of the codes defined above arose immediately in the theme 'Geometry of numbers'. In the works* [5] *and* [6] *the author defined some new N-dimensional point-lattices with a 'lot of $O(2^{\frac{1}{2}\{\log^2 N+\log N\}})$ minima'. These constructions are based on the method of Barnes and Wall* (see [4]) *and the setting up of the second-order Reed–Muller code.* Denote by $A^{n,k}_{2^{k-1}}$ the number of codewords of the code $R_k(2,n)$ and let $A^k_{2^{k-1}}$ be the number of codewords of weight 2^{k-1} in $R(2,k)$. By Theorem 1 we can determine the connection of the numbers $A^{n,k}_{2^{k-1}}$ and $A^k_{2^{k-1}}$, so we now prove Lemma 1:

Lemma 1

$$A^{n,k}_{2^{k-1}} = A^k_{2^{k-1}} 2^{\binom{n+1}{2}-\binom{k+1}{2}}.$$

Proof. Let k be a fix number for which $1 \le k \le n$. It is clear that the equality $A^{k,k}_{2^{k-1}}=A^k_{2^{k-1}}$ holds. Regard now the code $R_k(2,k+1)$. From Theorem 1 we get that a codeword in $R_k(2,k+1)$ has the form $\mid \underline{u} \mid \underline{u}+\underline{v} \mid$, where $\underline{u} \in R_k(2,k)$, and $\underline{v} \in R(1,k)$. But the number of codewords of $R(1,k)$ is equal to $2^{1+\binom{k+1}{1}}$ so we have the equality:

$$A^{k+1,k}_{2^{k-1}} = A^k_{2^{k-1}} 2^{1+\binom{k}{1}}.$$

Similarly, the codewords in $R_k(2,k+2)$ have the form

$$\mid \mid \underline{u} \mid \underline{u}+\underline{v} \mid \mid \mid \underline{u} \mid \underline{u}+\underline{v} \mid + \underline{w} \mid,$$

where $\mid \underline{u} \mid \underline{u}+\underline{v} \mid \in R_k(2,k+1)$, and $\underline{w}$ is an arbitrary element of $R(1,k+1)$. For this reason we get that:

$$A^{k+2,k}_{2^{k-1}} = A^k_{2^{k-1}} 2^{1+\binom{k}{1}} 2^{1+\binom{k+1}{1}}.$$

Since we can continue these conversions in this way, we proved the statement of our lemma:

$$A^{n,k}_{2^{k-1}} = A^k_{2^{k-1}} 2^{\sum_{j=k}^{n-1}\left(1+\binom{j}{1}\right)} = A^k_{2^{k-1}} 2^{\binom{n+1}{2}-\binom{k+1}{2}}.$$

Remark 2 *It is obvious that the condition of the definition can be replaced by*

$$|H \cap V_k| = i,$$

where i is a possible weight of a codeword in $R(2,k)$. Thus we have $i = 2^{k-1}$ or $i = 2^{k-1} \pm 2^{k-1-\delta}$ where $0 \le \delta \le [\frac{k}{2}]$. If A_i^k is the number of codewords of weight i in $R(2,k)$ then the number of codewords of the new code $R_{k,i}(2,n)$ is equal to

$$A_i^{n,k} = 2^{\binom{n+1}{2}-\binom{k+1}{2}} A_i^k.$$

From Theorem 8 of Ch.15.§2 in [2] we know the number $A_{2^{k-1}}^k$. This formula is the following:

$$A_{2^{k-1}}^k = 2^{1+\binom{k}{1}+\binom{k}{2}} - \sum_{\delta=1}^{[\frac{k}{2}]} 2^{\delta(\delta+1)} \frac{(2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+1}-1)}{(4^\delta-1)(4^{\delta-1}-1)\cdots(4-1)} - 2.$$

The expression is rather complicated, but it can be simplified by a deeper investigation of the generator function $g_k(x)$:

$$g_k(x) = \sum_{\delta=1}^{[\frac{k}{2}]} 2^{\delta(\delta-1)} \frac{(2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+1}-1)}{(4^\delta-1)(4^{\delta-1}-1)\cdots(4-1)} x^\delta.$$

So with this notation we get that

$$A_{2^{k-1}}^k = 2^{1+\binom{k+1}{2}} - g_k(4) - 2.$$

Lemma 2

$$g_k(1) = 2^{\binom{k}{2}} - 1.$$

Proof. If $k = 2, 3$ or 4, the equality holds trivially. At the same time the right hand side satisfies the following recurrence relation:

$$T_k = 2^{k-1} T_{k-1} + (2^{k-1} - 1),$$

where T_k is equal to $2^{\binom{k}{2}} - 1$. We prove that this relation holds for the left hand side, too. Now, let T_k be the following sum:

$$T_k = \sum_{\delta=1}^{[\frac{k}{2}]} 2^{\delta(\delta-1)} \frac{(2^k-1)\cdots(2^{k-2\delta+1}-1)}{(4^\delta-1)\cdots(4-1)}.$$

Then

$$
\begin{aligned}
&T_k - (2^{k-1} - 1)\\
&= \frac{(2^k - 1)(2^{k-1} - 1)}{4 - 1} - (2^{k-1} - 1) + \sum_{\delta=2}^{[\frac{k}{2}]} 2^{\delta(\delta-1)} \frac{(2^k - 1)\cdots(2^{k-2\delta+1} - 1)}{(4^\delta - 1)\cdots(4 - 1)}\\
&= 2^{k-1}\frac{(2^{k-1} - 1)(2^{k-2} - 1)}{4 - 1} + 2^2\frac{(2^k - 1)(2^{k-1} - 1)(2^{k-2} - 1)(2^{k-3} - 1)}{(4^2 - 1)(4 - 1)}\\
&\quad - (2^{k-1} - 2^2)\frac{(2^{k-1} - 1)(2^{k-2} - 1)}{4 - 1} + \sum_{\delta=3}^{[\frac{k}{2}]} 2^{\delta(\delta-1)} \frac{(2^k - 1)\cdots(2^{k-2\delta+1} - 1)}{(4^\delta - 1)\cdots(4 - 1)}\\
&= 2^{k-1}\left[\frac{(2^{k-1} - 1)(2^{k-2} - 1)}{4 - 1} + 2^2\frac{(2^{k-1} - 1)(2^{k-2} - 1)(2^{k-3} - 1)(2^{k-4} - 1)}{(4^2 - 1)(4 - 1)}\right]\\
&\quad - (2^{k-1} - 2^4)\frac{(2^{k-1} - 1)(2^{k-2} - 1)(2^{k-3} - 1)(2^{k-4} - 1)}{(4^2 - 1)(4 - 1)}\\
&\quad + \sum_{\delta=3}^{[\frac{k}{2}]} 2^{\delta(\delta-1)} \frac{(2^k - 1)\cdots(2^{k-2\delta+1} - 1)}{(4^\delta - 1)\cdots(4 - 1)} = \cdots = 2^{k-1}T_{k-1}.
\end{aligned}
$$

So we have the same recurrence relation for the two sides, therefore Lemma 2 is proved.

2. Recurrence Relation for the Numbers $A^{n,k}_{2^{k-1}}$

First we introduce the 4-ary Gaussian binomial coefficients $\begin{bmatrix} s \\ \delta \end{bmatrix}$:

$$
\begin{bmatrix} s \\ 0 \end{bmatrix} = 1,
$$

$$
\begin{bmatrix} s \\ \delta \end{bmatrix} = \frac{(4^s - 1)(4^{s-1} - 1)\cdots(4^{s-\delta+1} - 1)}{(4^\delta - 1)(4^{\delta-1} - 1)\cdots(4 - 1)}, \quad \delta = 1, 2, \ldots.
$$

(Here s is a real number.) Denote by $[\delta]$ the following product:

$$
[\delta] = (4^\delta - 1)(4^{\delta-1} - 1)\cdots(4 - 1) \text{ for } [\delta] = 1, 2, \ldots.
$$

The basic properties of these coefficients can be seen for example in [2]. With these notations we can write the expression of $g_k(x)$ in the form:

$$
g_k(x) = \sum_{\delta=1}^{[\frac{k}{2}]} 2^{\delta(\delta-1)}[\delta]\begin{bmatrix} \frac{k}{2} \\ \delta \end{bmatrix}\begin{bmatrix} \frac{k-1}{2} \\ \delta \end{bmatrix} x^\delta.
$$

In this section we shall prove the following theorem:

Theorem 2 *The recurrence relations*

i: $g_{k+1}(4) = 2^{\binom{k+2}{2}} - 2^{k+1} - (2^{k+1}-1)g_k(4)$;

ii: $2^k A^{n,k}_{2^{k-1}} = (2^k - 1)\left[2^{\binom{n+1}{2}+1} - A^{n,k-1}_{2^{k-2}}\right]$

are valid for each $k \geq 4$.

Proof. We define $T_{k,\delta} = 0$ for $\delta < 1$ and $\delta > [\frac{k}{2}]$, moreover in the case of $1 \leq \delta \leq [\frac{k}{2}]$ let $T_{k,\delta}$ be given by the following expression:

$$T_{k,\delta} = 2^{\delta(\delta-1)}[\delta]\begin{bmatrix}\frac{k}{2}\\ \delta\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta\end{bmatrix}.$$

Now assume that $k \geq 4$ and $1 \leq \delta \leq [\frac{k}{2}] - 1$. Then we have the formulas:

$$T_{k,\delta} = 2^{\delta(\delta-1)}[\delta]\begin{bmatrix}\frac{k}{2}\\ \delta\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta\end{bmatrix},$$
$$T_{k,\delta+1} = 2^{\delta(\delta+1)}[\delta+1]\begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta+1\end{bmatrix},$$
$$T_{k+1,\delta+1} = 2^{\delta(\delta+1)}[\delta+1]\begin{bmatrix}\frac{k+1}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}.$$

At this time we know that

$$(2^{2k-2\delta-1} - 2^{k-1})T_{k,\delta} + T_{k,\delta+1} =$$
$$= 2^{\delta(\delta+1)}[\delta+1]\left(\frac{2^{2k-2\delta-1} - 2^{k-1}}{2^{2\delta}(4^{\delta+1}-1)}\begin{bmatrix}\frac{k}{2}\\ \delta\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta\end{bmatrix} + \begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta+1\end{bmatrix}\right).$$

But

$$\frac{2^{2k-2\delta-1} - 2^{k-1}}{2^{2\delta}(4^{\delta+1}-1)} = 2^{k-2\delta-1}\frac{2^{k-2\delta}-1}{4^{\delta+1}-1} = 2^{k-2\delta-1}\frac{4^{\frac{k}{2}-\delta}-1}{4^{\delta+1}-1},$$

so we have the equality:

$$(2^{2k-2\delta-1} - 2^{k-1})T_{k,\delta} + T_{k,\delta+1}$$
$$= 2^{\delta(\delta+1)}[\delta+1]\left(2^{k-2\delta-1}\begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta\end{bmatrix} + \begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k-1}{2}\\ \delta+1\end{bmatrix}\right)$$
$$= 2^{\delta(\delta+1)}[\delta+1]\begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\left(2^{k-2\delta-1}\begin{bmatrix}\frac{k-1}{2}\\ \delta\end{bmatrix} + \begin{bmatrix}\frac{k-1}{2}\\ \delta+1\end{bmatrix}\right)$$
$$= 2^{\delta(\delta+1)}[\delta+1]\begin{bmatrix}\frac{k}{2}\\ \delta+1\end{bmatrix}\begin{bmatrix}\frac{k+1}{2}\\ \delta+1\end{bmatrix} = T_{k+1,\delta+1}.$$

From this relation we get that

$$\begin{aligned} g_{k+1}(x) &= \sum_{\delta=1}^{[\frac{k+1}{2}]} 2^{\delta(\delta-1)}[\delta]\begin{bmatrix}\frac{k+1}{2}\\ \delta\end{bmatrix}\begin{bmatrix}\frac{k}{2}\\ \delta\end{bmatrix}x^\delta = \sum_{\delta=1}^{[\frac{k+1}{2}]} T_{k+1,\delta}x^\delta \\ &= T_{k+1,1} + \sum_{\delta=2}^{[\frac{k+1}{2}]} T_{k+1,\delta}x^\delta = T_{k+1,1} + \sum_{\delta=1}^{[\frac{k+1}{2}]-1} T_{k+1,\delta+1}x^{\delta+1} \\ &= T_{k+1,1} + \sum_{\delta=1}^{[\frac{k+1}{2}]-1} (2^{2k-2\delta-1} - 2^{k-1})T_{k,\delta} + T_{k,\delta+1}x^{\delta+1} \\ &= T_{k+1,1}x + 2^{2k-1}x\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]-1} T_{k,\delta}\cdot\left(\frac{x}{4}\right)^\delta \\ &\quad - 2^{k-1}x\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]-1} T_{k,\delta}\cdot x^\delta + \sum_{\delta=2}^{\left[\frac{k+1}{2}\right]} T_{k,\delta}x^\delta. \end{aligned}$$

If the number k is even then

$$\left[\frac{k+1}{2}\right] - 1 = \frac{k}{2} - 1 = \left[\frac{k}{2}\right] - 1,$$

so we get that

$$\begin{aligned} g_{k+1}(x) &= T_{k+1,1}x + 2^{2k-1}x\left(g_k\left(\frac{x}{4}\right) - T_{k,\left[\frac{k}{2}\right]}\cdot\left(\frac{x}{4}\right)^{\left[\frac{k}{2}\right]}\right) \\ &\quad - 2^{k-1}x\left(g_k(x) - T_{k,\left[\frac{k}{2}\right]}\cdot x^{\left[\frac{k}{2}\right]}\right) + g_k(x) - T_{k,1}x \\ &= 2^{2k-1}xg_k\left(\frac{x}{4}\right) - (2^{k-1}x - 1)g_k(x) + x[T_{k+1,1} - T_{k,1}] \\ &= 2^{2k-1}xg_k\left(\frac{x}{4}\right) - (2^{k-1}x - 1)g_k(x) + x(2^k - 1)2^{k-1}. \end{aligned}$$

Finally, if the number k is odd, we get immediately that

$$\begin{aligned} g_{k+1}(x) &= T_{k+1,1}x + 2^{2k-1}xg_k\left(\frac{x}{4}\right) - 2^{k-1}xg_k(x) + g_k(x) - T_{k,1}x \\ &= 2^{2k-1}xg_k\left(\frac{x}{4}\right) - (2^{k-1}x - 1)xg_k(x) + x[T_{k+1,1} - T_{k,1}] \\ &= 2^{2k-1}xg_k\left(\frac{x}{4}\right) - (2^{k-1}x - 1)g_k(x) + x(2^k - 1)2^{k-1}. \end{aligned}$$

Here we used the precise values of the numbers $T_{k+1,1}$ and $T_{k,1}$ which are $\frac{(2^{k+1}-1)(2^k-1)}{4-1}$ and $\frac{(2^k-1)(2^{k-1}-1)}{4-1}$, respectively. Substitute now the value 4 into this equation. Then we get the formula:

$$g_{k+1}(4) = 2^{2k+1}g_k(1) - (2^{k+1}-1)g_k(4) + (2^k-1)2^{k+1}$$

and the first statement of this theorem can be seen from Lemma 2:

$$\begin{aligned} g_{k+1}(4) &= 2^{2k+1+\frac{k(k-1)}{2}} - 2^{2k+1} - (2^{k+1}-1)g_k(4) + (2^k-1)2^{k+1} \\ &= 2^{\frac{k^2}{2}+\frac{3k}{2}+1} - 2^{k+1} - (2^{k+1}-1)g_k(4) = 2^{k+1}(2^{\frac{k^2}{2}+\frac{k}{2}}-1) \\ &\quad - (2^{k+1}-1)g_k(4) = 2^{\binom{k+2}{2}} - 2^{k+1} - (2^{k+1}-1)g_k(4). \end{aligned}$$

Now apply the original formula of $A^k_{2^{k-1}}$. Then we have the equality:

$$\begin{aligned} A^k_{2^{k-1}} &= 2^{1+k+\binom{k}{2}} - 2g_k(4) - 2 \\ &= 2^{\binom{k+1}{2}+1} - 2^{\binom{k+1}{2}+1} + 2^{k+1} + (2^{k+1}-2)g_{k-1}(4) - 2 \\ &= (2^{k+1}-2)(g_{k-1}(4)+1) = -2^k\Big[2^{\binom{k}{2}+1} - 2g_{k-1}(4) - 2\Big] \\ &\quad + 2^{\binom{k+1}{2}+1} + \Big[2^{\binom{k}{2}+1} - 2g_{k-1}(4) - 2\Big] - 2^{\binom{k}{2}+1} \\ &= 2^{\binom{k+1}{2}+1} - 2^{\binom{k}{2}+1} - (2^k-1)A^{k-1}_{2^{k-2}} = (2^k-1)\Big[2^{\binom{k}{2}+1} - A^{k-1}_{2^{k-2}}\Big]. \end{aligned}$$

By virtue of Lemma 2 this means that the following recursive formulas hold:

$$\begin{aligned} A^k_{2^{k-1}} &= (2^k-1)\left[2^{\binom{k}{2}+1} - A^{k-1}_{2^{k-2}}\right], \\ 2^k A^{n,k}_{2^{k-1}} &= (2^k-1)\left[2^{\binom{n+1}{2}+1} - A^{n,k-1}_{2^{k-2}}\right]. \end{aligned}$$

So we proved the statement ii, of Theorem 2, too.

Remark 3 *Since we know the values of $A^k_{2^{k-1}}$ in the case of $k = 1, 2, 3$ and 4 (these are $2, 6, 70$ and 870, respectively) the other values of the function $A^n_{2^{n-1}} = A_{2^{n-1}}$ can be computed easily by the formula of Theorem 2.*

3. Explicit Formula for the Numbers $A^k_{2^{k-1}}$

In this section we prove the following statement:

Theorem 3

$$A^{n,k}_{2^{k-1}} = \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} (2^k-1)\cdots(2^{k-2\delta+2}-1)2^{\binom{n+1}{2}-\binom{k+1}{2}+\binom{k-2\delta+1}{2}+1},$$

$$A_{2^{n-1}} = A^{n,n}_{2^{n-1}} = \sum_{\delta=1}^{\left[\frac{n+1}{2}\right]} 2^{\binom{n-2\delta+1}{2}+1}(2^n-1)\cdots(2^{n-2\delta+2}-1).$$

Proof. Apply the recursion formulas for the function $A^{n,k}_{2^{k-1}}$! Then we get the undermentioned formula:

$$\begin{aligned} A^{n,k}_{2^{k-1}} = 2^{\binom{n+1}{2}+1}\Bigg\{ & \left(\frac{2^k-1}{2^k} - \frac{2^k-1}{2^k}\frac{2^{k-1}-1}{2^{k-1}}\right) \\ & + \left(\frac{(2^k-1)(2^{k-1}-1)(2^{k-2}-1)}{2^k 2^{k-1} 2^{k-2}}\right. \\ & \left. - \frac{(2^k-1)(2^{k-1}-1)(2^{k-2}-1)(2^{k-3}-1)}{2^k 2^{k-1} 2^{k-2} 2^{k-3}}\right) + \cdots \\ & + \left(\frac{(2^k-1)\cdots(2^{k-2l+2}-1)}{2^k\cdots 2^{k-2l+2}} - \frac{(2^k-1)\cdots(2^{k-2l+1}-1)}{2^k\cdots 2^{k-2l+1}}\right)\Bigg\} \\ & + \frac{(2^k-1)\cdots(2^{k-2l+1}-1)}{2^k\cdots 2^{k-2l+1}} A^{n,k-2l}_{2^{k-2l-1}}, \end{aligned}$$

where $l = 1, \ldots, \left[\frac{k}{2}\right]$. So if $k = 2l$ then

$$\begin{aligned} A^{n,2l}_{2^{2l-1}} = 2^{\binom{n+1}{2}+1} & \left\{ \sum_{\delta=1}^{l} \frac{(2^{2l}-1)\cdots(2^{2l-2\delta+2}-1)}{2^{2l}\cdots 2^{2l-2\delta+1}} \right\} \\ & + \frac{(2^{2l}-1)\cdots(2-1)}{2^{2l}\cdots 2} A^{n,0}_{2^{-1}}, \end{aligned}$$

and if $k = 2l+1$ then

$$\begin{aligned} A^{n,2l+1}_{2^{2l}} = 2^{\binom{n+1}{2}+1} & \left\{ \sum_{\delta=1}^{l} \frac{(2^{2l+1}-1)\cdots(2^{2l-2\delta+3}-1)}{2^{2l+1}\cdots 2^{2l-2\delta+2}} \right\} \\ & + \frac{(2^{2l+1}-1)\cdots(2^2-1)}{2^{2l+1}\cdots 2^2} A^{n,1}_{2^0}, \end{aligned}$$

where we used the equalities:

$$A^{n,1}_{2^0} = 2^{\binom{n+1}{2}-\binom{2}{2}} \cdot A^2_{2^0} = 2^{\binom{n+1}{2}} \quad \text{and} \quad A^{n,0}_{2^{-1}} = 0.$$

Therefore we have got the formulas:

$$A^{n,2l}_{2^{2l-1}} = 2^{\binom{n+1}{2}+1}\left\{\sum_{\delta=1}^{l} \frac{(2^{2l}-1)\cdots(2^{2l-2\delta+2}-1)}{2^{\binom{2l+1}{2}-\binom{2l-2\delta+1}{2}}}\right\}$$

and

$$A^{n,2l+1}_{2^{2l}} = 2^{\binom{n+1}{2}+1}\left\{\sum_{\delta=1}^{l+1} \frac{(2^{2l+1}-1)\cdots(2^{2l+1-2\delta+2}-1)}{2^{\binom{2l+2}{2}-\binom{2l-2\delta+2}{2}}}\right\},$$

so

$$A^{n,k}_{2^{k-1}} = \sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} (2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+2}-1)2^{\binom{n+1}{2}-\binom{k+1}{2}+\binom{k-2\delta+1}{2}+1}.$$

In the case of $k = n$ we get the simple explicit formula for the numbers $A_{2^{n-1}}$, too.

4. The Information Rates of the New Codes

Since the code $R_k(2, n)$ is not linear, the information rate R_k is defined by the quotient:

$$R_k = \frac{\log_2 A^{n,k}_{2^{k-1}}}{2^n}.$$

This is equal to

$$\frac{1}{2^n}\log_2\left(\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} (2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+2}-1)2^{\binom{n+1}{2}-\binom{k+1}{2}+\binom{k-2\delta+1}{2}+1}\right).$$

We shall prove that this number is asymptotically equal to $\frac{\binom{n+1}{2}}{2^n}$. More precisely we verify the statement:

Theorem 4 *For* $1 \leq k \leq n$ *the following inequalities hold:*

$$\frac{\binom{n+1}{2}-1}{2^n} \leq R_k \leq \frac{\binom{n+1}{2}+1}{2^n}.$$

Proof. Since the upper bound is the information rate of the second-order Reed–Muller code the second inequality trivially holds. On the other hand the value

$$\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]} (2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+2}-1)2^{\binom{n+1}{2}-\binom{k+1}{2}+\binom{k-2\delta+1}{2}+1}$$

can be written in the following form:

$$2^{\binom{n+1}{2}+1}\cdot\left[\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]}(2^k-1)(2^{k-1}-1)\cdots(2^{k-2\delta+2}-1)2^{-\binom{k+1}{2}+\binom{k-2\delta+1}{2}}\right]$$
$$=2^{\binom{n+1}{2}+1}\cdot\left[\sum_{\delta=1}^{\left[\frac{k+1}{2}\right]}\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k-1}}\right)\cdots\left(1-\frac{1}{2^{k-2\delta+2}}\right)\frac{1}{2^{k-2\delta+1}}\right].$$

Denote by L_k the sum in the bracket. It is easy to verify the following recursive formula for this number:

$$L_k=\left(1-\frac{1}{2^k}\right)\left[\frac{1}{2^{k-1}}+\left(1-\frac{1}{2^{k-1}}\right)L_{k-2}\right].$$

If k is odd then $L_1=\frac{1}{2}$ and it can be seen by induction with respect to k that $L_k\geq\frac{1}{2}$, because

$$L_k\geq\left(1-\frac{1}{2^k}\right)\left[\frac{1}{2^{k-1}}+\left(1-\frac{1}{2^{k-1}}\right)\frac{1}{2}\right]=\left(1-\frac{1}{2^k}\right)\left[\frac{1}{2^{k-1}}+\frac{1}{2}-\frac{1}{2^k}\right]$$
$$=\left(1-\frac{1}{2^k}\right)\left(\frac{1}{2}+\frac{1}{2^k}\right)=\frac{1}{2}+\frac{1}{2^{k+1}}-\frac{1}{2^{2k}}\geq\frac{1}{2}.$$

If k is even, a similar calculation shows that L_k is greater than or equal to $\frac{3}{8}$. This means that

$$R_k\geq\frac{\log_2 2^{1+\binom{n+1}{2}}\cdot\frac{3}{8}}{2^n}\geq\frac{\log_2 2^{\binom{n+1}{2}-1}}{2^n}.$$

So we have proved this theorem, too.

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