PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 47, NO. 1, PP. 41–49 (2003)

ON Nil GEOMETRY

Emil MOLNÁR

Department of Geometry
Institute of Mathematics
Budapest University of Technology and Economics
H–1521 Budapest, Hungary

Received: March 5, 2003

Dedicated to the Memory of Professor Imre Vermes

Abstract

Nil geometry is a homogeneous 3-space derived from the Heisenberg matrix group in formula (1), where the matrix multiplication provides the non-commutative addition of translations. The Lie theory, combined with projective geometry [1], makes possible to illustrate some phenomena, e. g. the discrete lattices and the geodesics in **Nil**. I think the method, aided by computer, gives new possibilities in this field [3] in the future.

Keywords: **Nil** space, lattice, geodesics, balls.

1. The Nil Space Modelled in $\mathbf{E}^3 \subset \mathcal{P}^3$

In studying magnetic fields, Werner HEISENBERG found his famous *real matrix group* $\mathbf{L}(\mathbf{R})$ whose left (row-column) multiplication by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

provided a new addition of points (translations)

$$(x, y, z) * (a, b, c) = (a + x, b + y, c + xb + z). \tag{2}$$

Our *Fig. 1* (in a *Cartesian coordinate system* of *the usual Euclidean 3-space* $\mathbf{E}^3$) shows that

$$(1, 0, 0) * (1, 2, 1) = (2, 2, 3), \quad \boxed{1} \tag{3}$$

$$(1, 2, 1) * (1, 0, 0) = (2, 2, 1), \quad \boxed{2}$$

i.e. *the translations are not commutative*, in general.

The matrices $\mathbf{K}(z) \lhd \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ & 1 & 0 \\ & & 1 \end{pmatrix} \mapsto (0, 0, z), \tag{4}$$

however, constitute the *cyclic centre*, i.e. each of them is commuting with all elements of **L**. The elements of **K** are called *fibre translations*, as well, and they can be visualized by straight lines, growing out from the points of the $(x, y, 0)$ plane. Any *fibre line* is an orbit of a point $(x, y, 0) \mapsto (x, y, z)$ under the fibre translations $\mathbf{K}(z)$, where $z \in \mathbf{R}$ is varied.

In the following we consider **L** as *projective collineation group* (see [1], but here) with *right actions* in homogeneous coordinates as follows

$$(1, a, b, c) \begin{pmatrix} 1 & x & y & z \\ & 1 & 0 & 0 \\ & & 1 & x \\ & & & 1 \end{pmatrix} = (1, x + a, y + b, z + bx + c). \tag{5}$$

The points of **Nil** will be visualized in $\mathbf{E}^3$ and embedded into the projective space $\mathcal{P}^3$, where the ideal points $(0, u, v, w)$, with direction vector (u, v, w), will be taken under the collineations in (5), as well.

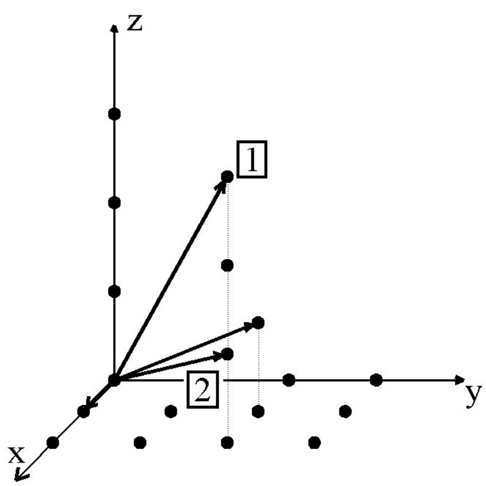

Fig. 1. The group $\mathbf{L}(\mathbf{Z})$ is not commutative

Any plane $\underline{\mathbf{u}} \sim (u_0, u_1, u_2, u_3)^T$, with linear equation for its points (row matrices) $\mathbf{x} \sim (x^0, x^1, x^2, x^3) \sim (1, x, y, z)$ ($\sim$ means a freedom up to a non-zero

R factor), i.e.

$$0 = \mathbf{x}\underline{\mathbf{u}} = (x^0, x^1, x^2, x^3) \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} =$$

$$x^0u_0 + x^1u_1 + x^2u_2 + x^3u_3 \sim 1u_0 + xu_1 + yu_2 + zu_3 \tag{6}$$

is described by a linear form $\underline{\mathbf{u}}$ (column matrix, upper T means transposition), again up to a non-zero **R** factor. The collineation in (5) for points induces the corresponding collineation for planes by *inverse matrix* (with *left action*) as follows

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} \mapsto \begin{pmatrix} 1 & -x & -y & xy-z \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}. \tag{7}$$

Namely, this is the criterion, that any incident point and plane will be mapped under the collineation onto incident point and plane.

In particular, the *horizontal plane pencil* $\underline{\mathbf{u}}(p) \sim (p, 0, 0, 1)^T$, along the fibre $(1, 0, 0, z)$ over the origin $(1, 0, 0, 0)$ has the equation for the variable $(\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3) \sim (1, \overline{x}, \overline{y}, \overline{z})$:

$$0 = (\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3) \begin{pmatrix} p \\ 0 \\ 0 \\ 1 \end{pmatrix} = \overline{x}^0 p + \overline{x}^3 1 \sim p + \frac{\overline{x}^3}{\overline{x}^0} = p + \overline{z}, \tag{8}$$

with any fixed $p \in \mathbf{R}$, i.e. we have the intersection point $(1, 0, 0, -p)$ with the fibre.

This plane pencil will be mapped by (7) onto the *sloped plane pencil* (along the fibre over $(1, x, y, z)$)

$$\begin{pmatrix} 1 & -x & -y & xy-z \\ & 1 & 0 & 0 \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} p \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p + xy - z \\ 0 \\ -x \\ 1 \end{pmatrix},$$

i.e. with equation for $(\overline{x}^0, \overline{x}^1, \overline{x}^2, \overline{x}^3) \sim (1, \overline{x}, \overline{y}, \overline{z})$

$$0 = \overline{x}^0(p + xy - z) + \overline{x}^2(-x) + \overline{x}^3 \cdot 1 \sim$$

$$p + xy - z + \frac{\overline{x}^2}{\overline{x}^0}(-x) + \frac{\overline{x}^3}{\overline{x}^0} \cdot 1 = (p + xy - z) + \overline{y}(-x) + \overline{z} \cdot 1. \tag{9}$$

Now we can extend the translation group **L** defined by formulas (5) and (7) to a larger group **G** of collineations, preserving the fibering, that will be the (orientation preserving) *isometry group of* **Nil**. We indicate how to introduce the rotation about the fibre over the origin about angle ω by the usual matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{10}$$

leaving invariant the *infinitezimal arc-length-square*

$$(ds)^2 = (d\overline{x})^2 + (d\overline{y})^2 + (d\overline{z})^2 \tag{11}$$

as a positive definite quadratic differential form at the origin. By the *Lie theory* this will be extended to the rotation about the fibre over any point $(1, x, y, 0)$ by conjugacy (see (5) and (7)):

$$\begin{pmatrix} 1 & -x & -y & xy-z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\omega & \sin\omega & 0 \\ 0 & -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = \tag{12}$$

$$\begin{pmatrix} 1 & x(1-\cos\omega)+y\sin\omega & -x\sin\omega+y(1-\cos\omega) & -x^2\sin\omega+xy(1-\cos\omega) \\ 0 & \cos\omega & \sin\omega & x\sin\omega \\ 0 & -\sin\omega & \cos\omega & -x(1-\cos\omega) \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Moreover, we have the 'pull-back transform'

$$(0, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z) \begin{pmatrix} 1 & -x & -y & xy-z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0, \mathrm{d}\overline{x}, \mathrm{d}\overline{y}, \mathrm{d}\overline{z}) \tag{13}$$

for the basis differential forms at $(1, x, y, z)$ and at the origin, respectively. From this we obtain the *infinitezimal arc-length-square* by (11) at any point of **Nil** as follows

$$\begin{gathered} (\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (-x\mathrm{d}y + \mathrm{d}z)^2 = \\ (\mathrm{d}x)^2 + (1+x^2)(\mathrm{d}y)^2 - 2x(\mathrm{d}y)(\mathrm{d}z) + (\mathrm{d}z)^2 =: (\mathrm{d}s)^2. \end{gathered} \tag{14}$$

Hence we get the symmetric metric tensor field g on **Nil** by components, furthermore its inverse:

$$g_{ij} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+x^2 & -x \\ 0 & -x & 1 \end{pmatrix}, \quad g^{jk} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & 1+x^2 \end{pmatrix}. \tag{15}$$

Thus **Nil** is a *homogeneous Riemann space* where the arc-length of any piecewise smooth curve can be computed by integration as usual for surface curves in the classical differential geometry.

2. The Discrete Translation Group L(Z)

If we substitute *integers*, their set is denoted by **Z**, into the formulas (1–2) or (5) for x, y, z, then we get *discrete group actions* whose set will be denoted by **L(Z)**, as integer lattice of **Nil**.

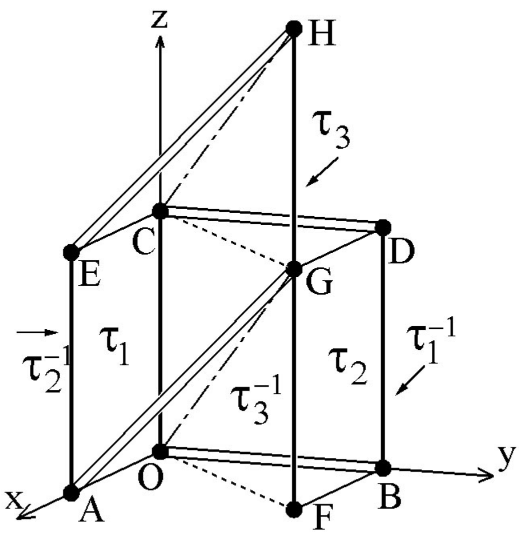

Fig. 2. A fundamental domain $\tilde{\mathcal{F}}$ for **L**(**Z**), representing the **Nil** space form **Nil**/**L**(**Z**)

As a surprising phenomenon, we illustrate the action of **L**(**Z**) on **Nil** in *Fig.2* by a *fundamental domain* $\mathcal{F} = OABCDEFGH$. We remark that the Euclidean integer lattice may have a cube as fundamental domain, whose opposite side faces are mapped under the *three generating translations* [2]. Now (5) provides us the face pairing generators as follows

$$\begin{aligned}
\underline{\tau_1} : OBDC =: \tau_1^{-1} &\mapsto \tau_1 := AGHE, && \text{i.e.} \\
(1, 0, b, c) &\mapsto (1, 1, b, c + b) && 0 \le b \le 1, 0 \le c \le 1; \\
\underline{\tau_2} : OAEC =: \tau_2^{-1} &\mapsto \tau_2 := BFGD; \\
\underline{\tau_3} : OAGFB =: \tau_3^{-1} &\mapsto \tau_3 := CEHGD.
\end{aligned} \tag{16}$$

Here the bent faces τ_3^{-1} and τ_3 are remarkable. Of course, e.g. the inverse translation $\underline{\tau_3}^{-1} : \tau_3 \mapsto \tau_3^{-1}$ has also been defined.

These generators induce three **L**(**Z**) equivalence classes of edges, each class provides a so-called defining relation for the generators:

$$
\begin{aligned}
&= \{OB, AG, EH, CD\} : \underline{\tau_1\tau_3\tau_1}^{-1}\underline{\tau_3}^{-1} = \mathbf{1} \ (\textit{identity map});\\
&- \{OA, BF, DG, CE\} : \underline{\tau_2\tau_3\tau_2}^{-1}\underline{\tau_3}^{-1} = \mathbf{1};\\
&\blacksquare \{OC, AE, FG, GH, BD\} : \underline{\tau_1\tau_2\tau_3\tau_1}^{-1}\tau_2^{-1} = \mathbf{1},
\end{aligned}
\tag{17}
$$

as indicated in *Fig. 2*. Now we only remark that any relation above can be read off a standard procedure (Poincaré algorithm, see [2]): The image edge domains belonging to any edge class amount a complete tubular neighbourhood of each edge in the class.

The vertices of $\mathcal{F}$ also fall into one equivalence class, and the image corner domains amount a ball-like neighbourhood of each vertex in the class. All these arguments imply that the fundamental domain $\tilde{\mathcal{F}}$, with face pairing identifications (˜), represents a compact **Nil** manifold or **Nil** space form, denoted by **Nil**/**L**(**Z**).

The last relation of (17) provides $\underline{\tau_3} = \underline{\tau_2}^{-1}\underline{\tau_1}^{-1}\underline{\tau_2\tau_1}$ as a *commutator*, generating the centre **K**(**Z**) (as in (4)) of **L**(**Z**). Substituting $\underline{\tau_3}$ into the first two relations of (17), we get a *minimal presentation*:

$$
\mathbf{L}(\mathbf{Z}) = \tag{18}
$$
$$
(\underline{\tau_1}, \underline{\tau_2} - \mathbf{1} = \underline{\tau_2\tau_2\tau_1}^{-1}\underline{\tau_2}^{-1}\underline{\tau_1\tau_1\tau_2}^{-1}\underline{\tau_1}^{-1} = \underline{\tau_1}^{-1}\underline{\tau_2\tau_1\tau_2}^{-1}\underline{\tau_1}^{-1}\underline{\tau_2}^{-1}\underline{\tau_1\tau_2}).
$$

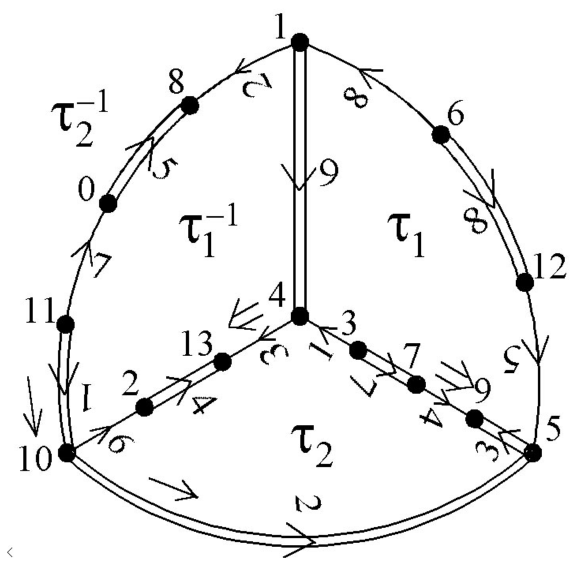

Fig. 3. The minimally presenting fundamental tetrahedron $\tilde{\mathcal{T}}$ for **Nil**/**L**(**Z**)

This minimal presentation has a geometrically realizing fundamental domain $\tilde{\mathcal{T}}$, a topological tetrahedron with face pairing generators $\underline{\tau_1} : \tau_1^{-1} \mapsto \tau_1$, $\underline{\tau_2} : \tau_2^{-1} \mapsto \tau_2$ as above (*Fig. 3*).

This *Schlegel diagram* has a coordinate realization, analogously to *Fig.2*, with great freedom, but this will be a *computer graphic problem* to solve later on. We have to produce the vertices of $\tilde{\mathcal{T}}$ with an appropriate starting vertex, first e. g. with the origin O, then its images as *Fig. 3* dictates:

$$O,\ 1 := O^{\underline{\tau_1}},\ 2 := O^{\underline{\tau_2}},\ 3 := 1^{\underline{\tau_1}},\ 4 := 1^{\underline{\tau_2}},\ 5 := 2^{\underline{\tau_1}},\ 6 := 3^{\underline{\tau_2}^{-1}},\ 7 := 4^{\underline{\tau_1}},$$
$$8 := 4^{\underline{\tau_1}^{-1}},\ 9 := 5^{\underline{\tau_2}},\ 10 := 5^{\underline{\tau_2}^{-1}},\ 11 := 6^{\underline{\tau_1}^{-1}},\ 12 := 7^{\underline{\tau_2}^{-1}},\ 13 := 8^{\underline{\tau_2}}. \qquad (19)$$

Then we form the edges. An appropriate centre, e. g. the *barycentre* of the above vertices of the face $\underline{\tau}_1^{-1}$, enables us to form the star-like face τ_1^{-1}, indeed. The $\underline{\tau_1}$ image of the former centre also provides the star-like face τ_1. Similarly, we can construct the faces τ_2^{-1} and τ_2 and the polyhedron $\tilde{\mathcal{T}}$ by computer. A simplicial subdivision of $\tilde{\mathcal{T}}$ can be produced by the barycentre of all vertices in (19) as a formal centre for $\tilde{\mathcal{T}}$.

This new polyhedron type shows how to apply our method in the group theory, and many new problems arise.

3. Nil Geodesics

We are interested in determining the *geodesic curves* in our **Nil** geometry. As it is well-known, these curves are generally defined as *having locally minimal* (stationary) *arc length* between their any two (near enough) points.

Then it holds a second order differential equation (system)

$$\ddot{y}^k + \dot{y}^i \dot{y}^j \Gamma^k_{ij} = 0, \qquad (20)$$

where $y^1(t) =: x(t)$, $y^2(t) =: y(t)$, $y^3(t) =: z(t)$ are the coordinate components of the parametrized geodesic curves, upper point means the derivation $\frac{\mathrm{d}}{\mathrm{d}t}$ by the parameter t, as usual. The Einstein–Schouten index conventions will be applied for recalling the general theory. Namely, the Levi-Civita connection by

$$\Gamma^k_{ij} = \frac{1}{2}\left(\frac{\partial g_{jl}}{\partial y^i} + \frac{\partial g_{li}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^l}\right) g^{lk} \qquad (21)$$

can be expressed by (14) and (15) from the metric tensor field, by an easy but lengthy computation. Finally we obtain the system to solve

$$\begin{aligned}
&\text{(i)} && \ddot{x} + \dot{y}\dot{y}(-x) + \dot{y}\dot{z} = 0 && \text{with} \quad x(0) = y(0) = z(0) = 0,\\
&\text{(ii)} && \ddot{y} + \dot{x}\dot{y}(x) + \dot{x}\dot{z}(-1) = 0, && \dot{x}(0) = c\cos\alpha, \quad \dot{y}(0) = c\sin\alpha,\\
&\text{(iii)} && \ddot{z} + \dot{x}\dot{y}(x^2 - 1) + \dot{x}\dot{z}(-x) = 0, && \dot{z}(0) = w,
\end{aligned} \qquad (22)$$

as initial values. For simplicity we have chosen the origin as starting point, by the homogeneity of **Nil** this can be assumed, because of (5) we can transform a curve into an another starting point. From $(-x)$(ii) + (iii) we get the consequence

$$-\ddot{y}x + \ddot{z} - \dot{x}\dot{y} = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}t}(\dot{z} - x\dot{y}) = 0,$$

hence

$$\text{(iv)} \quad \dot{z} = w + x\dot{y} \Leftrightarrow z = w \cdot t + \int_0^t x(\tau)\dot{y}(\tau)d\tau. \tag{23}$$

Substituting this into (22) (i) and (ii), respectively, we get

$$\text{(v)} \quad \ddot{x} + w\dot{y} = 0, \qquad \text{(vi)} \quad \ddot{y} - w\dot{x} = 0. \tag{24}$$

Then by (v) $\dot{x}$ + (vi)$\dot{y}$ we get

$$(\dot{x})^2 + (\dot{y})^2 = c^2 \quad \text{constant},$$

and

$$\text{(v}'\text{)} \quad \dot{x} + wy = c \cdot \cos\alpha, \qquad \text{(vi}'\text{)} \quad \dot{y} - wx = c \cdot \sin\alpha. \tag{25}$$

Finally, by easy steps, we get the $w \neq 0$ solution for $(x(t), y(t), z(t))$ as follows

$$\begin{aligned} x(t) &= \frac{c}{w}\big[\sin(wt+\alpha) - \sin\alpha\big], \quad y(t) = -\frac{c}{w}\big[\cos(wt+\alpha) - \cos\alpha\big], \\ z(t) &= wt + \frac{c^2}{2w}t - \frac{c^2}{4w^2}\big(\sin(2wt+2\alpha) - \sin 2\alpha\big) \\ &\quad + \frac{c^2}{2w^2}\big[\sin(wt+2\alpha) - \sin 2\alpha - \sin(wt)\big]. \end{aligned} \tag{26}$$

Here we can introduce the arc length parameter

$$s = \sqrt{c^2 + w^2} \cdot t,$$

moreover,

$$w = \sin\theta, \quad c = \cos\theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \tag{27}$$

i.e. unit velocity can be assumed.

We remark that there is no more simple relation among the distance s, and the coordinates (x, y, z), as it has been in the Euclidean space.

In other form we obtain the solution

$$\begin{aligned} & w \neq 0, \\ x(t) &= \frac{2c}{w}\sin\frac{wt}{2}\cos\Big(\frac{wt}{2} + \alpha\Big), \quad y(t) = \frac{2c}{w}\sin\frac{wt}{2}\sin\Big(\frac{wt}{2} + \alpha\Big), \\ z(t) &= wt \cdot \bigg\{1 + \frac{c^2}{2w^2}\bigg[\Big(1 - \frac{\sin(2wt+2\alpha) - \sin 2\alpha}{2wt}\Big) \\ &\quad + \Big(1 - \frac{\sin(wt)}{wt}\Big) - \Big(1 - \frac{\sin(wt+2\alpha) - \sin 2\alpha}{2wt}\Big)\bigg]\bigg\} = \\ &= wt \cdot \bigg\{1 + \frac{c^2}{2w^2}\bigg[\Big(1 - \frac{\sin(wt)}{wt}\Big) + \frac{1 - \cos(wt)}{wt} \cdot \sin(wt + 2\alpha)\bigg]\bigg\} \end{aligned} \tag{28}$$

as a helix-like geodesic curve.

$$
\begin{aligned}
&c = 0 \quad \text{leads to} \quad (x, y, z) = (0, 0, wt) \quad \text{as solution;} \\
&w = 0 \quad \text{leads to} \quad x = c \cdot \cos\alpha \cdot t, \quad y = c \cdot \sin\alpha \cdot t, \\
&z = \frac{1}{2}c^2 \cos\alpha \sin\alpha \cdot t^2
\end{aligned} \tag{29}
$$

as a parabola on the hyperbolic paraboloid surface

$$2 \cdot Z - XY = 0. \tag{30}$$

Again, a nice computer visualization problem arises: *Determine the sphere of radius r in the* **Nil** *geometry!*

Connecting the Sections **2** and **3** of this paper, it is natural *to ask for the densest lattice-like ball packing of the* **Nil** *space.* Gauss had already solved this problem in the Euclidean space $\mathbf{E}^3$. The face-centred cubic lattice serves the density $\frac{\pi}{\sqrt{18}} \approx 0,7404805$.

Now the general concept of lattice in **Nil** should be defined first. Then an optimal ball packing should be constructed, where the ball centres form a point lattice in **Nil** and no two balls intersect each other.

The Euclidean analogies can help!?

Acknowledgement

I thank my colleague Attila Bölcskei for preparing the manuscript and for designing the figures.

References

[1] Ledneczki, P. – Molnár, E., Projective Geometry in Engineering, *Periodica Polytechnica Ser. Mechanical Engineering* **39** No. 1 (1995), pp. 43–60.

[2] Molnár, E., Some Old and New Aspects on the Crystallographic Groups, *Periodica Polytechnica Ser. Mechanical Engineering* **36** Nos. 3–4 (1992), pp. 191–218.

[3] Molnár, E., The Projective Interpretation of the Eight 3-Dimensional Homogeneous Geometries, *Beiträge zur Algebra und Geometrie (Contributions to Algebra and Geometry)* **38** No. 2, (1997), pp. 261–288.