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ON THE AFFINE MAPS OF $\mathbb{E}^n$

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Abstract

According to well-known methods of standard calculus approximation of smooth functions f : $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ are usually done by Taylor polynomials of first degree $AX + B$, where A is the Jacobi matrix of f at a given point $X_0 \in \mathbb{R}^n$ and B denotes the constant column matrix $f(X_0) - AX_0 \in \mathbb{R}^n$. However, the number $n^2 + n$ of the real input parameters characterizing the map $AX + B$ (which is considered here as an affine map of $\mathbb{E}^n$ in a given coordinate-system) can be reduced considerably by using a suitably chosen new coordinate-system. The paper answers also the question how to find the minimal translation part of any affine map.

Keywords: affine transformation, change of coordinates, linear algebra, minimal translation.

1. Introduction

In order to give an affine map $\mathcal{A} : \mathbb{E}^n \longrightarrow \mathbb{E}^n$ of the Euclidean space $\mathbb{E}^n$ it is enough to give the image of a non-degenerated $(n + 1)$-tuple of points:

$$(Q_0 = \mathcal{A}(P_0),\ Q_1 = \mathcal{A}(P_1), \dots, Q_n = \mathcal{A}(P_n)).$$

Then the image $Q = \mathcal{A}(P)$ of any point $P \in \mathbb{E}^n$ will be defined on the basis of the equality

$$\overrightarrow{Q_0 Q} = \sum_{j=1}^{n} x^j\, \overrightarrow{Q_0 Q_j},$$

where $x^1, \dots, x^n \in \mathbb{R}$ are the unique coefficients in the decomposition $\overrightarrow{P_0 P} = \sum_{j=1}^{n} x^j\, \overrightarrow{P_0 P_j}$.

Notice that a linear map (endomorphism) $\alpha :\ V^n \longrightarrow V^n$ has been induced on the related Euclidean vectorspace V^n as the images of the linearly independent base vectors $\mathbf{u}_j = \overrightarrow{P_0P_j}$ have been defined by

$$\mathbf{v}_j = \overrightarrow{Q_0Q_j} = \sum_{i=1}^{n} a_{ij}\mathbf{u}_i \quad \text{for } j = 1, \dots n,$$

and so for any $\mathbf{u} = \sum_{j=1}^{n} x^j \mathbf{u}_j \in V^n$

$$\alpha(\mathbf{u}) = \mathbf{v} = \sum_{j=1}^{n} x^j \alpha(\mathbf{u}_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x^j \mathbf{u}_i$$

should hold, as well.

The matrix $A = (a_{ij}),\ a_{ij} \in \mathbb{R}$ for $i, j = 1, \dots, n$ represents the linear part α of the affine map $\mathcal{A}$ with respect to the given base $\mathbf{B} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$, and so the equality of vectors

$$\mathbf{B}Y = \overrightarrow{P_0Q} = \overrightarrow{Q_0Q} + \overrightarrow{P_0Q_0} = \mathbf{B}(AX + Y_0)$$

can easily be translated to the well-known representation

$$\overline{\mathcal{A}}_{\{P_0,\mathbf{B}\}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

of the affine map $\mathcal{A}$, with respect to the given affine coordinate-system $\{P_0, \mathbf{u}_1, \dots, \mathbf{u}_n\}$, by the equation

$$Y = AX + Y_0,$$

where X, Y and Y_0 denote the column matrices containing the coordinates of the vectors $\overrightarrow{P_0P}$, $\overrightarrow{P_0Q}$ and $\overrightarrow{P_0Q_0}$, respectively.

Remark. Throughout this paper also the degenerated case where the vectors $\alpha(\mathbf{u}_j) = \mathbf{v}_j$ for $j = 1, \dots, n$ are linearly dependent, i.e. $\det(A) = 0$, will be allowed, as well.

2. How to Describe the Given Affine Map $\mathcal{A}$ of $\mathbb{E}^n$ in a New Affine Coordinate-System $\{P_0', \mathbf{B}' = (\mathbf{u}'_1, \dots, \mathbf{u}'_n\}$?

It has been shown that after having fixed the coordinate-system $\{P_0, \mathbf{B} = (\mathbf{u}_1, \dots, \mathbf{u}_n)\}$ the affine map $\mathcal{A}$ can be completely described by $n^2 + n$ real input parameters. In fact, we have to give n^2 independent entries of the matrix A and also n independent entries of the matrix Y_0. This statement remains true if the base $\mathbf{B} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is supposed to be orthonormal. From now on, without the loss of generality, we will restrict ourselves to the use of orthonormal bases. Our aim is to find an orientation preserving isometry which carries the first coordinate-system $\{P_0, \mathbf{B}\}$ into a new orthonormal one $\{P_0', \mathbf{B}'\}$ which may become well adapted to the considered affine map $\mathcal{A}$.

Let us define first the transition from the original coordinate-system to the second one by the following relations:

$$\mathbf{B}' = \mathbf{B}R, \text{ where } R \in SO(n)$$

and

$$\overrightarrow{P_0 P_0'} = \mathbf{B}X_0, \overrightarrow{P_0' P_0} = \mathbf{B}' X_0',$$

respectively. Thus

$$\begin{aligned}
\mathbf{B}'X' = \overrightarrow{P_0' P} = \overrightarrow{P_0' P_0} + \overrightarrow{P_0 P} = \mathbf{B}(-X_0 + X),\\
\mathbf{B}'A'X' = \overrightarrow{Q_0' Q} = \alpha(\overrightarrow{P_0' P}) = \mathbf{B}(-AX_0 + AX),\\
\mathbf{B}'Y_0' = \overrightarrow{P_0' Q_0'} = \overrightarrow{P_0' P_0} + \overrightarrow{P_0 Q_0} + \alpha(\overrightarrow{P_0 P_0'}) = \mathbf{B}(-X_0 + Y_0 + AX_0),\\
\mathbf{B}'Y' = \overrightarrow{P_0' Q} = \overrightarrow{P_0' P_0} + \overrightarrow{P_0 Q} = \mathbf{B}(-X_0 + Y)
\end{aligned}$$

hold and so

$$R(Y_0' + A'X') = Y_0 - X_0 + AX$$

holds as well. Using that $A' = R^T AR$ and $X' = R^T(X - X_0)$ the relation

$$Y_0' = R^T((A - I)X_0 + Y_0)$$

is obtained.

3. How to Choose the New Origin P_0' in Order to Get Minimal Translation?

We are looking for the coordinates $(x_0^1, \ldots, x_0^n)^T$ of the vector $\overrightarrow{P_0 P_0'} = \mathbf{B}X_0$ by the condition that the norm of the vector

$$\begin{aligned}
\mathbf{B}'Y_0' = \mathbf{B}((A - I)X_0 + Y_0)\\
= x_0^1(\mathbf{v}_1 - \mathbf{u}_1) + x_0^2(\mathbf{v}_2 - \mathbf{u}_2) + \cdots + x_0^n(\mathbf{v}_n - \mathbf{u}_n) + \overrightarrow{P_0 Q_0}
\end{aligned}$$

should be minimal.

Case 1 $\det(A - I) \neq 0$ i.e. $\dim(\ker(\alpha - \mathrm{id})) = 0$.
As the vectors $\mathbf{v}_j - \mathbf{u}_j$ $(j = 1, 2, \ldots, n)$ are linearly independent the equation

$$x_0^1(\mathbf{v}_1 - \mathbf{u}_1) + x_0^2(\mathbf{v}_2 - \mathbf{u}_2) + \cdots + x_0^n(\mathbf{v}_n - \mathbf{u}_n) + \overrightarrow{P_0 Q_0} = 0$$

has a unique solution and so $Y_0' = 0$, i.e. $\overrightarrow{P_0'Q_0'} = 0$.

Case 2 rank of $(A - I) = k < n$, i.e. $\dim(\ker(\alpha - \mathrm{id})) = n - k > 0$.

Let $(\overrightarrow{P_0 Q_0})_{pr}$ be the orthogonal projection of $\overrightarrow{P_0 Q_0}$ on the k dimensional subspace spanned by the vectors $\mathbf{v}_j - \mathbf{u}_j$ $(j = 1, 2, \ldots, n)$. However, the set of solutions of the equation

$$x_0^1(\mathbf{v}_1 - \mathbf{u}_1) + x_0^2(\mathbf{v}_2 - \mathbf{u}_2) + \cdots + x_0^n(\mathbf{v}_n - \mathbf{u}_n) + (\overrightarrow{P_0 Q_0})_{pr} = 0$$

is now $n - k$ dimensional (and so there is a certain freedom in the choice of the new origin P_0') for the shortest translation we get always the same vector $\overrightarrow{P_0'Q_0'} = \overrightarrow{P_0 Q_0} - (\overrightarrow{P_0 Q_0})_{pr}$.

Remark. If $\overrightarrow{P_0'Q_0'} = 0$ (i.e. if $\overrightarrow{P_0 Q_0} \in \mathrm{Im}\,(\alpha - \mathrm{id})$) the above defined points P_0' form the set of fix points of the affine map $\mathcal{A}$, which is an $n - k$ dimensional flat of $\mathbb{E}^n$. In the opposite case if the vector $\overrightarrow{P_0 Q_0}$ does not belong to the subspace spanned by the vectors $\mathbf{v}_1 - \mathbf{u}_1, \ldots, \mathbf{v}_n - \mathbf{u}_n$ the affine map $\mathcal{A}$ has no fix point.

4. How to Choose a Well Adapted New Base $\mathbf{B}' = (\mathbf{u}'_1, \ldots, \mathbf{u}'_n\}$?

As a second step the vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ should be turned to the unit eigenvectors $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n$ corresponding to the real eigenvalues

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

of the Gram matrix $G = A^T A$. (The matrix $R \in SO(n)$ representing this rotation has $\frac{n(n-1)}{2}$ independent entries.)

Then the linear map $\alpha : V^n \longrightarrow V^n$ turns to be simply a composition of a rotation represented by an orthogonal matrix $\overline{R} \in SO(n)$ and n axial dilatations represented by a diagonal matrix D whose diagonal elements are

$$\sqrt{\lambda_1}, \sqrt{\lambda_2}, \ldots, \sqrt{\lambda_{n-1}}, \varepsilon\sqrt{\lambda_n},$$

where $\varepsilon = \operatorname{sgn}\det(A)$. In other words the affine map $\mathcal{A}$ keeps the orthogonality of the vectors $\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n$ and their dilatations are described by the non-negative eigenvalues of the Gram matrix G as it can be proved in the following straightforward way:

$$\begin{aligned}\langle \alpha(\mathbf{u}'_i), \alpha(\mathbf{u}'_j)\rangle &= \langle \mathbf{B}'A'E_i, \mathbf{B}'A'E_j\rangle = E_i^T(R^TAR)^T(R^TAR)E_j \\ &= E_i^T(R^TGR)E_j = \langle \mathbf{u}'_i, \lambda_j\mathbf{u}'_j\rangle = \lambda_j\delta_{ij},\end{aligned}$$

where E_i, E_j are the column matrices containing the coordinates of $\mathbf{u}'_i, \mathbf{u}'_j$ with respect to the base $\mathbf{B}'$ and δ_{ij} denotes the symbol of Kronecker for $i, j = 1, \ldots, n$.

5. Final Remarks, Applications

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a map (vector field) which is differentiable at $X_0 \in \mathbb{R}^n$. Then the approaching map $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$g(X) = AX + Y_0,$$

where A is the Jacobi matrix and $Y_0 = f(X_0) - AX_0$, represents an affine map $\mathcal{A}$ of $\mathbb{E}^n$ with respect to a standard coordinate-system. According to the above described method it may always be convenient to carry the original coordinate-system into a well adapted new one. (So the number of characterizing parameters will be diminished considerably.) This situation is illustrated for example in thermodynamics where the translation part in the equation $V = V_0(1 + \beta t)$ could be eliminated by transforming it into the well-known Gay-Lussac equation $V = \frac{V_0}{T_0} T$.

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