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LOCALIZATION THEOREMS IN TOPOLOGY: A BRIEF SUMMARY

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Abstract

We present some applications of localization theorems in algebraic topology. The example of the Grassmannian of 2-planes in 4-space is analyzed in detail.

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1. Introduction

This short note has been written to serve as an introduction for the uninitiated to the applications of localization theorems in topology. We will not attempt to give a complete treatment and all the proofs will be omitted. The goal here is to exhibit the power, and the combinatorial and algebraic complexity of these methods. A small amount of knowledge of algebraic topology and differential geometry will be assumed. The basic textbooks on the subject are [8, 6].

2. A Simple Problem of Classical Geometry. A Reformulation

Our toy example will be computing certain quantities in enumerative geometry. Consider for example, imposing conditions on lines in 3-dimensional complex space. One could ask:

1. How many lines go through two points?
2. How many lines go through a point and intersect 2 given lines?
3. How many lines intersect 4 given lines?

Naturally, all the given data in the questions are in generic position.

The answer to the first question is clear. The answer to the second is also easy; it is a little clearer if one poses the dual question: *How many lines in 3-space intersect two given lines and are contained in a given plane?* The answer is clearly 1, since the two given lines intersect the given plane in two points, and our line has to go through these.

Now the 3rd question is a little more difficult, although any geometer worth its salt will give you the answer quickly: 2. These numbers are the ones that we will try to compute in several ways below.

The first idea is that we projectivize our picture: we replace 3-space by 4-space, and lines by planes going through the origin, etc. Thus we will consider complex 2-planes in complex 4-dimensional space. This will not change the enumerative data and results. Then our enumerative numbers above maybe represented as intersection numbers on a smooth complex manifold as follows. Consider the space of all possible planes in 4-space; this is a compact 4-dimensional manifold $\mathrm{Gr}(2, 4)$ called the Grassmannian. Denote by $C_2(R)$ the set of planes contained in a 3-dimensional subspace R, and by $C_1(P)$ the set of planes intersecting a fixed plane P along a line. The indices here stand for the complex codimensions of these subsets in $\mathrm{Gr}(2, 4)$. Then forming the intersection

$$C_2(R) \cap C_1(P_1) \cap C_1(P_2)$$

we obtain a finite set of points whose number is exactly the answer to question 2. Since this number does not depend on the particular 3-subspace R and on the planes, we could informally write $C_2C_1^2 = 1$, meaning that the number of elements of the intersection is one. Such a number is called an *intersection number* of the manifold. Note that the 3rd question on our list reduces to computing $\cap C_1^4$.

The next thing is to introduce the notion of a *vector bundle* over a manifold M: this is a smoothly varying family E of vector spaces parameterized by M. This means, in particular, that E is also a manifold and that there is a mapping $\pi : E \to M$ such that $E_P = \pi^{-1}(P)$ is a vector space. A section of the vector bundle E is a mapping $s : M \to E$ such that $\pi(s(P)) = P$ for every $P \in M$. An example of a vector bundle over M is the tangent bundle TM; a section of this bundle is called a vector field. Note that for simplicity, we mean here the complex tangent bundle and holomorphic sections.

It turns out that vector bundles and their sections create a convenient language to ask general questions about manifolds. Returning to our example, there is a natural vector bundle E over $\mathrm{Gr}(2, 4)$ such that for each point $P \in \mathrm{Gr}(2, 4)$, E_P is exactly the 2-plane in $\mathbb{C}^4$ represented by P. The only section that this, so-called *tautological*, vector bundle E has is the one that is zero everywhere, however its dual E^*, i.e. the family of dual vector spaces, has some interesting sections. Pick a 3-dimensional subspace R and a nonzero linear functional $\mu : \mathbb{C}^4 \to \mathbb{C}$ which vanishes on R. Then for each $P \in \mathrm{Gr}(2, 4)$ the functional μ restricts to a linear functional $s_\mu(P) : E_P \to \mathbb{C}$, thus μ provides a section s_μ of E^*. Now we leave as an exercise to the reader to check that $C_2(R)$ is exactly the set of those planes where s_μ vanishes, while if $L = R_1 \cap R_2$, then $C_1(L)$ is the set of those $P \in \mathrm{Gr}(2, 4)$ for which $s_{\mu_1}(P)$ and $s_{\mu_2}(P)$ are collinear.

3. The First Result: Euler's Theorem and the Euler Characteristic

With these preparations we can embark on our excursion into localization techniques of computation of intersection numbers of complex manifolds.

Let X be a compact complex manifold of complex dimension n, on which a circle T acts compatibly with the complex structure. The simplest and most basic example of this setup is the rotation of the sphere by the angle θ:

$$(z_1, z_2) \longrightarrow (e^{2\pi i\theta} z_1, z_2).$$

Here we used the representation of the sphere as the complex projective line; z_1 and z_2 are the complex projective coordinates, thus $z_1 z_2 \neq 0$, and (z_1, z_2) is identified with $(\epsilon z_1, \epsilon z_2)$. Note that projective spaces are special cases of Grassmannians.

Returning to the general case, denote by V the generating vector field on X for this action. Denote by F the set $\{x \in X |\, V(x) = 0\}$ of fixed points of the action. Pick one of the fixed points $p \in F$, and assume that it is isolated. Choose complex coordinates $z_1, \dots, z_n$ centered at this point. Then the vector field V vanishes at the origin, and has the form

$$V = \sum_j A_{ij}(z) z_i \frac{\partial}{\partial z_j} + \text{ complex conjugate},$$

where $A_{ij}(z)$ is a holomorphic function near the origin.

Now, if the matrix $A_{ij} = A_{ij}(0)$ is nondegenerate, then we call p a *nondegenerate fixed point* of V. The first theorem in our hierarchy of localization theorems is the theorem of Euler:

Theorem 1 *Assume F is finite, and that each fixed point of the action is nondegenerate. Then the Euler characteristic $\chi(X)$ of X is equal to the number of fixed points F.*

Recall the definition of the Euler characteristic of a manifold. Assume that $\mathfrak{T}$ is a triangulation of X with $\#\mathfrak{T}_i$ faces of dimension i; thus, for example, the number of vertices will be $\#\mathfrak{T}_0$. Then the Euler characteristic is given by

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \#\mathfrak{T}_i,$$

and this number does not depend on the triangulation $\mathfrak{T}$. Thus it provides a topological invariant of X.

Consider the case $n = 1$: here it is easy to construct explicit triangulations, and one has the sphere, with $\chi = 2$, the torus with $\chi = 0$, and, in general, the sphere with g handles with $\chi = 2 - 2g$. Projecting the sphere on the complex plane, we can write down (the holomorphic) part of an appropriate vector field as $z\frac{\partial}{\partial z}$. It is clearly nondegenerate and has appropriately 2 fixed points: one at zero and one at infinity.

Note that one could also consider the vector field $z^2 \frac{\partial}{\partial z}$. This has only one fixed point, which is degenerate, however. One can extract the same number, 2 that is, from this vector field as well, but that requires a little more work.

This listing of the Euler characteristics of 1-dimensional complex manifolds also points at a limitation of the above theorem. It, essentially, shows that surfaces with more than one handle do not have any circle actions, since there cannot be a negative number of fixed points. This problem may be remedied as well (cf. [2]).

4. Bott's Residue Formula

Now we generalize Euler's Theorem and find more general topological invariants of manifolds. To define these, recall that the conjugation-invariant functions on the set of n-by-n matrices which are polynomial in the entries are in one-to-one correspondence with symmetric polynomials in the eigenvalues, which, in turn, form a polynomial algebra of the elementary symmetric polynomials: $c_1, c_2, \dots, c_n$. Denote the homogeneous degree k invariant polynomials on the n-by-n matrices by $P_n^{[k]}$. Then for a compact complex manifold X, there is a *characteristic* linear mapping

$$\chi^X : P_n^{[n]} \longrightarrow \mathbb{C},$$

whose values are again topological invariants.. These values are called the *characteristic numbers* of the manifold X. As an example, consider c_n thought of as an element of $P_n^{[n]}$. Clearly, c_n, being the product of the eigenvalues, represents the determinant of a matrix, thus we can write $c_n(A) = \det(A)$. The corresponding characteristic number turns out to be the Euler characteristic of X. We will not give the precise meaning of the other invariants, except for a special case to be detailed below.

Now we are ready to formulate the Residue Theorem of Raoul BOTT [4, 5]:

Theorem 2 (Bott) *Assume that V is a holomorphic vector field on a compact complex manifold X with only nondegenerate fixed points. Denote the matrix of the local form of the vector field around a fixed point p by $A(P)$ as above. Then for any $\phi \in P_n^{[n]}$ one has*

$$\chi^X(\phi) = \sum_{p\in F} \frac{\phi(A(p))}{\det(A(p))}. \tag{1}$$

This theorem clearly includes Euler's theorem as a special case: $\phi = c_n$. It implies, in particular, that no matter how one picks the vector field with nondegenerate fixed points, not only the number of these fixed points, but also the sum on the right hand side of (1) remains unchanged.

The Residue Theorem has several important generalizations. First, often the action of T is lifted to a vector bundle E on X. For example, clearly, a linear action of circle T on the complex vector space $\mathbb{C}^4$ induces an action on the Grassmannian $\mathrm{Gr}(2, 4)$ and on the tautological bundle as well, since the points of the tautological bundle can be thought of lying in $\mathbb{C}^4$.

Assuming the dim $E = r$, now there is a characteristic map $\chi^{X,E}$ from the space of degree n invariant polynomials on r-by-r matrices:

$$\chi^{X,E} : P_r^{[n]} \longrightarrow \mathbb{C}.$$

The values of this map are topological invariants of the bundle E. We will explain the exact geometric meaning of these values for the case of Grassmannians below.

Then the formula reads

Theorem 3 (Bott) *Assume that, in addition to the setup of the previous theorem, the action of the circle lifts to a holomorphic vector bundle E over X. At a fixed point p, one obtains then a matrix $B(p)$ representing the infinitesimal action of the vector field on the vector space E_p. Then for an invariant polynomial $\phi \in P_r^{[n]}$, one has*

$$\chi^{X,E}(\phi) = \sum_{p \in F} \frac{\phi(B(p))}{\det(A(p))}.$$

Our basic example is that of the complex Grassmannian $\mathrm{Gr}(r, n)$ of r-planes in $\mathbb{C}^n$. This is a complex manifold of dimension $r(n-r)$, endowed with a tautological vector bundle E of rank r. In this case the geometric meaning of the characteristic map of the bundle E that we mentioned above may be given in a particularly elegant form.

Theorem 4 *Let $F = (V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n)$ be a flag of subspaces of $\mathbb{C}^n$ with $\dim V_j = j$. Associate to c_j, the jth elementary symmetric polynomial in r variables, the compact subset $C_j(F) \subset \mathrm{Gr}(r, n)$ of complex codimension j defined by*

$$C_j(F) = \{S \in \mathrm{Gr}(r, N) |\ \dim(S \cap V_{n-r+j-1}) \geq j\}.$$

Then if $\sum_{j=1}^{r} j\alpha_j = n$, then

$$\chi^{E^*}\left(\prod c_j^{\alpha_j}\right) = C_j^{\alpha_j}.$$

A little explanation for the last formula. Since $\sum_{j=1}^{r} j\alpha_j = n$, we can expect that the intersection on the right hand side is of dimension zero, i.e. it consists of a finite set of points. This may be achieved by choosing different flags for every element C_j. (That would make $\sum \alpha_j$ different flags.) On the right hand side, we identified the intersection with the number of points in it.

Note that our definition of the subset C_j here is consistent with the definitions in the $n = 4$, $r = 2$ case given earlier. Combined with Theorem 3, this result then allows us to compute our numbers. Let us see how that goes.

We start with the simpler case of the projective space: $r = 1$. Since $P_1^{[n]}$ is one-dimensional, spanned by the monomial c_1^n, we only need to compute a single quantity here. According to Theorem 4 this quantity will count the common lines lying in n generic hyperplanes in an $n+1$-dimensional vector space. The answer is, naturally, 1. The localization formula gives something much more complex here. For simplicity consider the case $n = 2$. Thus our space is $\mathbb{P}^2$, the space of lines in

$\mathbb{C}^3$ going through the origin. As usual, our space is endowed with a tautological line bundle L, and canonical injection

$$i_p : L_p \to \mathbb{C}^3 \text{ for every } p \in \mathbb{P}^2.$$

Choosing three integers a, b and c, we can define an action of the circle on $\mathbb{C}^3$ by

$$w \cdot (z_1, z_2, z_3) = (w^a z_1, w^b z_2, w^c z_3),$$

where we thought of the circle as being embedded in the complex plane as the set of numbers of unit length, and w is an element of this unit circle. To apply Theorem 3, first we need to determine the fixed points of this action. These are, clearly, the coordinate lines in $\mathbb{C}^3$; there is 3 of them: $p_1 = (\lambda, 0, 0)$, $p_2 = (0, \lambda, 0)$, $p_3 = (0, 0, \lambda)$, where $\lambda \in \mathbb{C}$. Next we find the eigenvalues of the corresponding matrices. The B matrices are of rank 1 here, so they are simply numbers:

$$B(p_1) = a, \quad B(p_2) = b, \quad B(p_3) = c.$$

The matrices $A(p_i)$ are of rank 2, and they are a bit trickier to find. The computation is based on the isomorphism

$$T_p\mathbb{P}^2 \cong \mathrm{Hom}(\mathbb{C}^3 / \ker(i_p^*), L_p^*),$$

where i_p^* is the dual of the linear mapping i_p defined above. The reader is encouraged to check this isomorphism. Armed with it, it is easy to determine the eigenvalues: for p_1, for example, these are $a - b$ and $a - c$. The resulting formula for the intersection number coming from Theorem 3 is

$$\frac{a^2}{(a-b)(a-c)} + \frac{b^2}{(b-a)(c-a)} + \frac{c^2}{(c-a)(c-b)},$$

which does not look like it wants to be equal to 1, but it is. In fact, there is an elegant way to see this: consider the differential form

$$\frac{z^2\, dz}{(z-a)(z-b)(z-c)};$$

This has 4 residues: the ones at a, b, c give the above contributions, while the residue at ∞ is equal to -1. Now applying the residue theorem in the complex plane we obtain the answer.

The first interesting case is that of $\mathrm{Gr}(2, 4)$. Recall that the 3rd question about the number of lines in 3-space intersecting 4 given generic lines reduces to computing the intersection number $\# \cap C_1^4$. Again we consider the action

$$w \cdot (z_1, z_2, z_3, z_4) = (w^a z_1, w^b z_2, w^c z_3, w^d z_4).$$

In this case, there are 6 fixed points corresponding to the 6 coordinate 2-planes in $\mathbb{C}^4$. At the first of these, at $p = (\lambda, \mu, 0, 0)$, $\lambda, \mu \in \mathbb{C}$, the eigenvalues of the infinitesimal action on E_p^* are a and b, while the eigenvalues on the tangent space at this point are computed similarly to the case of $\mathbb{P}^2$ above. The localization formula gives the following result:

$$\#C_1^4 = \sum_{\sigma \in \binom{4}{2}} \sigma \cdot \frac{(a+b)^4}{(a-c)(a-d)(b-c)(b-d)}, \tag{2}$$

where the summation is over the 6 different rational fractions which may be obtained from the one on the right by permuting the symbols (a, b, c, d). Hardened by the previous example, we are not surprised to find that the sum on the right hand side is equal to 2, independently from the values of a, b, c, d. It would be interesting to construct a residue form on a 2-dimensional manifold, which would reproduce the fact that this sum is equal to 2, using a global residue theorem, just as in the case of projective case.

We can conclude that the localization formula gives a clear, albeit computationally somewhat cumbersome procedure for computing enumerative quantities.

5. Localization via Reduction

In this section we give a completely different way of computing enumerative quantities via localization. Here the group will not act on the space of all objects that we want to consider. Rather, our space will emerge as a quotient by a group acting on a larger space. We start with the formula of DUISTERMAAT–HECKMAN, which somehow interpolates between these two points of view.

Unfortunately, it was not possible to keep the discussion on the same elementary level as in the previous section. Thus our goal was to give a flavor of the resulting formulas and give some references.

First we need to introduce the more flexible language of differential forms instead of cycles. Cycles $C_r(E)$ of a vector bundle E over a manifold M are replaced by Poincaré dual differential forms $c_r(E)$. These differential forms are closed, i.e. $dc_r = 0$. In this language, the intersection numbers appear as integrals of differential forms:

$$\prod C_i^{\alpha_i} = \int_M \prod c_i^{\alpha_i}.$$

Recall that a symplectic manifold is one endowed with a nondegenerate 2-form, and that any projective complex manifold is symplectic in a natural way. If a group acts on such a manifold with a generating vector field V, then there is a so-called *moment map* μ, satisfying the property $d\mu = V \cap \omega$, where ω is the symplectic form.

Theorem 5 ([7], Duistermaat–Heckman) *Let (M, ω) be a symplectic manifold of real dimension $2n$, and let $\mu : M \to \mathbb{R}$ be a* moment map *of the symplectic action of a circle on M, i.e. assume that $d\mu = V \cap \omega$. Assume that $\omega = c_1(L)$ for some line bundle L. Then*

$$\begin{aligned} -\langle \mu + \omega, V\rangle &\rightsquigarrow -(\mu + \omega) \\ -\langle \mu(p), V\rangle &\rightsquigarrow -\mu(p) \\ \langle \beta_i^{(p)}, V\rangle &\rightsquigarrow \beta_i^{(p)}, \end{aligned}$$

where the $\beta_i^{(p)}$ are the weights of the action on the tangent bundle at p.

Note that the denominator in the formula is essentially the same thing as the denominator of the Bott residue formula.

The Bott residue formula for the volume, $\int c_1^n$, may be obtained by letting V approach 0 here. For example, for $M = \mathbb{P}^1$ we obtain that

$$\int e^{zu}\, d\,\mathrm{vol} = \frac{e^u - e^{-u}}{2u},$$

where we think of $\mathbb{P}^1$ as the 2-sphere embedded into 3-space the standard way, and the circle acts via rotations around the z-axis. It is easy to see that the z coordinate is a moment map; u is a parameter here. Setting u to 0, we recover the answer from the Bott residue formula for the volume of the sphere. What is interesting here though, is that we can interpret the LHS as the Laplace transform of the push-forward measure: if we denote the symplectic quotient by N_ξ for $\xi \in \mathfrak{t}^*$, then we can write the LHS as

$$\mathrm{vol}(T) \int_{\mathfrak{t}^*} e^{\langle \xi, u\rangle}\, \mathrm{vol}(N_\xi)\, d\xi.$$

The basic formula of the Laplace transform $L\{f\}(s) = \int_{\mathbb{R}} e^{-st} f(t)\, dt$ that is relevant for us is that $L\{H(t-a)\}(s) = e^{-as}/s$, where $H(x) = (x + |x|)/2$. For our example, this means that the inverse Laplace transform of $\sinh(u)/u$ is one half of the indicator function of the interval $[-1, 1]$. In this case the volume of T is to be interpreted as 2, because the square root of the identity acts trivially.

This works in the noncompact case as well, as long as the image of the moment map is bounded from below. Say the standard action on $\mathbb{C}$ gives contribution e^u/u, whose inverse Laplace transform is the indicator function of $[-1, \infty)$. In the multidimensional case, where the μ maps to the dual of the Lie algebra $\mathfrak{t}$ of the torus, one obtains a function on the cone of those $V \in \mathfrak{t}$ for which the function $\langle \mu, V\rangle$ is bounded from below, and the inverse Laplace transform can be taken similarly.

The next result, which we will not use in this paper, is the BERLINE–VERGNE/ATIYAH–BOTT localization theorem [1, 3]. This is a direct generalization of the DH formula. It uses equivariant cohomology, which is a new ring structure on the cohomology of the manifold with coefficients in the polynomials on the Lie algebra of the torus. In this formulation we do not assume isolated fixed points.

Theorem 6 ([3, 1]) *For an arbitrary equivariant class $\alpha \in H^*(M) \otimes \mathbb{C}[u_i]$, the integral*

$$\int_M \alpha = \sum_{C \subset F} \int_C \frac{\alpha}{E(N_C)},$$

where the sum is over the connected components of the fixed point set, N_C is the (equivariant) normal bundle of C in M and E is the equivariant normal class.

Now we generalize the idea mentioned above, which says that the DH formula can be used to compute the volume of the reduced space. Assume that a 1-dimensional torus acts on a symplectic manifold X with moment map μ, and denote the subquotient $\mu^{-1}(0)/T$ by M. The Kirwan map is a map from the equivariant cohomology of X to the ordinary cohomology of X, but I like to think about it in terms of K-theory: as long as the action of T on $\mu^{-1}(0)$ is free, every equivariant vector bundle V on X reduces to an ordinary vector bundle V_μ on the symplectic quotient M. If the action is not free then one needs to take an appropriate twist or power of V to make sure that the bundle descends.

Theorem 7 ([10, 9]) *For each invariant polynomial ϕ of total degree* $\dim M$, *we have*

$$\int_M \phi(V_\mu) = n_0 \operatorname*{Res}_{u=0} \sum_{C \subset F^+} \int_C \frac{\phi(V)\, du}{E(N_C)},$$

where F^+ is the set of fixed points with a positive value of the moment map, and n_0 is the size of the subgroup of the torus which acts trivially.

If the torus is replaced by the group SU(2), the formula remains the same except for the multiplication by the Weyl factor:

$$\int_M \phi(V_\mu) = n_0 \operatorname*{Res}_{u=0} -2u^2 \sum_{C \subset F^+} \int_C \frac{\phi(V)\, du}{E(N_C)}.$$

These formulas also work in the noncompact case, when the moment map is proper.

Let us revisit our example of the Grassmannian of 2-planes in 4-space. We may obtain it by symplectic reduction from $W = \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^4)$ via the group $U(2)$. Thinking of the elements of W as 2-by-4 matrices S, we see that the moment map is S^*S if we identify the dual of the Lie algebra $u(2)$ with Hermitian matrices via the product $H \mapsto -i\,\mathrm{tr}(H\cdot)$. Note that the moment map of the torus action only is the projection of S^*S onto its diagonal. This is a case of a noncompact reduction. We carry out the computation in a second, but first we do the same with a compact reduction.

For this we take the 7-dimensional projective space $\mathbb{P}W$ and act on it by SU(2). Here the moment map is $S^*S/\mathrm{tr}(S^*S)$ and we fix its value to be the identity matrix (this induces the zero functional on the Lie algebra of SU(2)). The fixed points set consists of two 3-dimensional projective spaces, but only one of them has positive value of the moment map. The equivariant Euler class of the normal bundle is easy to identify: $(c+2u)^4$, but the classes of the lifts of the characteristic classes c_1, c_2 are a bit harder. The answer is $c^2+2cu \mapsto c_2$ and $2(c+u) \mapsto c_1$, which can be seen from identifying the appropriate bundles. Note that the class $c+u$ does not descend to an integral class; this is because of the nontrivial action on the fiber of the tautological bundle of the central element of the group. In our case, $n_0 = 2$, and indeed we obtain

$$-4u^2 \int_{\mathbb{P}^3} \frac{(c^2+2cu)^2}{(c+2u)^4} = \frac{1}{u}, \qquad -4u^2 \int_{\mathbb{P}^3} \frac{(2c+2u)^4}{(c+2u)^4} = \frac{2}{u}.$$

The noncompact version is much more transparent, however. Here, there is only one fixed point: the origin. A 2-dimensional torus acts now, with weights a and b. Here the identification is much easier: the Euler class of the normal bundle is a^4b^4, $a+b \mapsto c_1$, $ab \mapsto c_2$ and the Weyl factor is $-(a-b)^2/2$. Again we have

$$\operatorname*{Res}_{a=0} \operatorname*{Res}_{b=0} \frac{-(a-b)^2(ab)^2}{2a^4b^4} = 1, \qquad \operatorname*{Res}_{a=0} \operatorname*{Res}_{b=0} \frac{-(a-b)^2(a+b)^4}{2a^4b^4} = 2. \tag{3}$$

Note that this computation is essentially equivalent to the formulas of Sean MARTIN [11].

Thus finally we obtained two rather different formulas, (2) and (3) which express 2, the number of lines intersecting 4 given lines in complex 3-space. Neither of these two formulas are easy to compute, but they do give a definite path to the result and that is what counts!

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