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# SOME NOTES ON MODELLING AND INTRINSIC GEOMETRY OF $K$-PATCHES

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*Dedicated to the memory of professor Julius Strommer*

## Abstract

The paper deals with the construction of $k$-patches in the $d$-dimensional projectively extended Euclidean space $\mathbf{P}E^d$ on the basis of $k$ consecutively applied one-parameter transformations of the space to a fixed point. A special 3-patch (solid) modelling is described and some notes on solid intrinsic geometric properties are given.

*Keywords:* projective geometry, $k$-patch, geometric transformations, modelling.

## 1. Introduction

Let $\mathbf{P}E^d$ be a $d$-dimensional projectively extended Euclidean space with the well-defined inherited Euclidean metric and the fixed Cartesian homogeneous rectangular normed reference system. With respect to [1] we can establish the following designations.

Any point $\mathbf{A}$ in the space $\mathbf{P}E^d$ can be represented by a $(d+1)$-tuple of real numbers, at least one of which is different from zero

$$\mathbf{A}\left(\mathbf{a}\left(a^0, a^1, \dots, a^d\right)^t\right)$$

denoted as point homogeneous coordinates. These refer to an orthonormed basis $(\mathbf{e}_0; \mathbf{e}_i)$ with $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}$ and $\langle \mathbf{e}_0, \mathbf{e}_\alpha \rangle = 0$ for any $\alpha = 0, 1, \dots, d$ and $i, j = 1, 2, \dots, d$.

In the case of a real point $\mathbf{A}$, the coordinate $a^0 \neq 0$, and $(d+1)$-tuples $\mathbf{a}\left(a^0, a^1, \dots, a^d\right)^t$ and $\mathbf{a}\lambda\left(a^0\lambda, a^1\lambda, \dots, a^d\lambda\right)^t$ for any real number $\lambda \neq 0$ represent the same point. Normalized form of the homogeneous coordinates of the real point $\mathbf{A}$ is a $(d+1)$-tuple

$$\mathbf{a}\left(1, \frac{a^1}{a^0}, \dots, \frac{a^d}{a^0}\right)^t, \qquad \text{where} \qquad \lambda = (a^0)^{-1} \neq 0.$$

For any point at infinity $\mathbf{V}^{\infty}$ determined by a proportional direction vector represented as a $(d+1)$-tuple $\mathbf{v}\left(v^0, v^1, \ldots, v^d\right)^t$ the coordinate $v^0 = 0$ and $\mathbf{v}\left(0, v^1, \ldots, v^d\right)^t$ is equivalent to $\mathbf{v}\lambda\left(0, v^1\lambda, \ldots, v^d\lambda\right)^t$ for any real number $\lambda \neq 0$.

A geometric transformation $\Phi$ of the space $\mathbf{P}E^d$ is a linear mapping of the space

$$\Phi : \mathbf{P}E^d \to \mathbf{P}E^d,$$

$$\mathbf{X}(\mathbf{x}) \to \mathbf{Y}(\mathbf{y}), \mathbf{x} \to \mathbf{y} \sim \Phi\mathbf{x}$$

represented by a real regular square matrix $\mathbf{T}$ of rank $d+1$, $\det(\mathbf{T}) \neq 0$

$$\mathbf{T} = T_\alpha^\beta = \begin{pmatrix} T_0^0 & T_1^0 & \ldots & T_d^0 \\ T_0^1 & T_1^1 & \ldots & T_d^1 \\ \div & \div & \div & \div \\ T_0^d & T_1^d & \ldots & T_d^d \end{pmatrix},$$

$$x^\alpha \mapsto y^\beta = T_\alpha^\beta x^\alpha \qquad \text{for} \qquad \alpha, \beta = 0, 1, \ldots, d$$

up to a nonzero real factor $\tau$, thus $\mathbf{T}$ and $\tau \cdot \mathbf{T}$ describe the same transformation $\Phi$, and for

$$\mathbf{x} = (\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_d)\left(x^0, x^1, \ldots, x^d\right)^t = \mathbf{e}_\alpha x^\alpha$$

and

$$\mathbf{y} = (\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_d)\left(y^0, y^1, \ldots, y^d\right)^t = \mathbf{e}_\beta y^\beta,$$

$$\mathbf{y} = \mathbf{T}\mathbf{x} = \mathbf{T}\mathbf{e}_\alpha x^\alpha = \mathbf{e}_\beta T_\alpha^\beta x^\alpha \sim \mathbf{e}_\beta y^\beta.$$

A composition of $n > 1$ geometric transformations ${}^1\mathbf{T}, {}^2\mathbf{T}, \ldots, {}^n\mathbf{T}$ in a predefined order – a concatenated transformation $\mathbf{T}$ can be determined as a composition of $n$ linear mappings and expressed as follows

$$\mathbf{x} \to \mathbf{y} = {}^1\mathbf{T}\mathbf{x} \to \mathbf{z} = {}^2\mathbf{T}\mathbf{y} = {}^2\mathbf{T}\,{}^1\mathbf{T}\mathbf{x} \to \cdots \to \mathbf{w} =$$

$$= {}^n\mathbf{T}\mathbf{v} = {}^n\mathbf{T} \ldots {}^2\mathbf{T}\,{}^1\mathbf{T}\mathbf{x} = \mathbf{T}\mathbf{x};$$

$$x^\alpha \to y^\beta = {}^1T_\alpha^\beta x^\alpha \to z^\gamma = {}^2T_\beta^\gamma y^\beta = {}^2T_\beta^\gamma \quad {}^1T_\alpha^\beta x^\alpha \to \cdots \to w^\rho = {}^nT_\sigma^\rho v^\sigma =$$

$${}^nT_\sigma^\rho \ldots {}^2T_\beta^\gamma \quad {}^1T_\alpha^\beta x^\alpha = T_\alpha^\rho x^\alpha.$$

Geometric transformations applied consecutively in the predefined order can serve for construction of $k$-patches from a given point in the space $\mathbf{P}E^d$.

Geometric transformations dependent on a parameter $u \in R$ can be determined as a one-parameter set of transformations defined on the domain $I \subset R$ and it is represented in the matrix form

$$\mathbf{T}(u) : x^\alpha \to y^\beta = T_\alpha^\beta(u) x^\alpha,$$

where the function $\mathbf{T}(u)$ that can be derived from the matrix $T_\alpha^\beta$ is a continuously differentiable function on the interval $I$ with the values in the set of real regular square matrices of rank $d+1$.

Entries of any matrix function $T_{\alpha}^{\beta}(u)$, for $\alpha, \beta = 0, 1, \ldots, d$ are real functions of one real variable, all defined and differentiable on the same interval $I$, while $\det\left(T_{\alpha}^{\beta}(u)\right) \neq 0$ for any value $u \in I$.

The same rules will be applied for a composition of $n > 1$ one-parameter sets of geometric transformations in a predefined order.

## 2. Definitions

**Definition 1** *A $k$-patch $S\left(\Omega^k\right)$ in a $d$-dimensional projectively extended Euclidean space $\mathbf{P}E^d$ is a non-empty subset, that is a continuously differentiable mapping of the region $\Omega^k \subset R^k$*

$$\varphi : \Omega^k \to S\left(\Omega^k\right) \subset \mathbf{P}E^d,$$

$$\left(u^1, u^2, \ldots, u^k\right) \mapsto \left(s^0, s^1, \ldots, s^d\right),$$

$$u^i \mapsto s^{\alpha}\left(u^j\right), \qquad \textit{for} \qquad i, j = 1, 2, \ldots, k \qquad \textit{and} \qquad \alpha = 0, 1, \ldots, d.$$

*A $k$-patch $S\left(\Omega^k\right)$ is denoted as a curve $S(\Omega)$ for $k = 1$, a surface $S\left(\Omega^2\right)$ for $k = 2$, a solid $S\left(\Omega^3\right)$ for $k = 3$ and an animation $S\left(\Omega^4\right)$ (of the space-time) for $k = 4$.*

**Definition 2** *Generating principle of a $k$-patch $S\left(\Omega^k\right)$ in a $d$-dimensional projectively extended Euclidean space $\mathbf{P}E^d$ will be a finite sequence of $k$ one-parameter sets of geometric transformations*

$$\left\{{}^1\mathbf{T}\left(u^1\right), {}^2\mathbf{T}\left(u^2\right), \ldots, {}^k\mathbf{T}\left(u^k\right)\right\}$$

*defined for $\left(u^1, u^2, \ldots, u^k\right) \in \Omega^k \subset R^k$, such that applying these transformations consecutively to an arbitrarily chosen fixed real point $\mathbf{X} \in \mathbf{P}E^d$, the related $k$-patch $S\left(\Omega^k\right)$ can be constructed.*

**Definition 3** *Creative representation of a $k$-patch $S\left(\Omega^k\right)$ is an ordered pair*

$$\left(\mathbf{X}, \left\{{}^1\mathbf{T}\left(u^1\right), {}^2\mathbf{T}\left(u^2\right), \ldots, {}^k\mathbf{T}\left(u^k\right)\right\}\right)$$

*of a fixed real point $\mathbf{X} \in \mathbf{P}E^d$ and a generating principle*

$$\left\{{}^1\mathbf{T}\left(u^1\right), {}^2\mathbf{T}\left(u^2\right), \ldots, {}^k\mathbf{T}\left(u^k\right)\right\}$$

*defined above.*

Further information concerning the form and structure of the geometric figure creative representation is available in [2].

## 3. Modelling of 3-patches

Let the real point $\mathbf{A}\left(\mathbf{a}\left(a^0, a^1, \dots, a^d\right)^t\right)$ be a fixed point in the space $\mathbf{P}E^d$ and the sequence of the three sets of one-parameter geometric transformations $\left\{{}^1\mathbf{T}\left(u^1\right), {}^2\mathbf{T}\left(u^2\right), {}^3\mathbf{T}\left(u^3\right)\right\}$ determined for real parameters $\left(u^1, u^2, u^3\right) \in \Omega^3 \subset R^3$ let be defined as a generating principle of a 3-patch $S\left(\Omega^3\right)$:

$$\left(\mathbf{A}, \left\{{}^1\mathbf{T}\left(u^1\right), {}^2\mathbf{T}\left(u^2\right), {}^3\mathbf{T}\left(u^3\right)\right\}\right) : \mathbf{a} \to$$

$$\to {}^3\mathbf{T}\left(u^3\right) {}^2\mathbf{T}\left(u^2\right) {}^1\mathbf{T}\left(u^1\right) \mathbf{a} := S\left(\Omega^3\right)$$

represented by homogeneous coordinates

$$s^\delta\left(u^1, u^2, u^3\right) = {}^3\mathbf{T}^\delta_\gamma\left(u^3\right) {}^2\mathbf{T}^\gamma_\beta\left(u^2\right) {}^1\mathbf{T}^\beta_\alpha\left(u^1\right) a^\alpha.$$

Choosing constant values of parameters $\left(u^1, u^2\right)$, $\left(u^1, u^3\right)$, $\left(u^2, u^3\right)$, where $u^1 = a$, $u^2 = b$, $u^3 = c$, $(a, b, c) \in \Omega^3$ respectively, isoparametric curve segments of the 3-patch (solid), forming three systems of curve segments, can be determined. For the constant value of only one of the parameters we speak about three systems of isoparametric surfaces. Any point $\mathbf{P}$ of the solid is determined by its curvilinear coordinates $(a, b, c) \in \Omega^3$, while homogeneous coordinates of the point can be calculated from

$$\mathbf{P} = \mathbf{s}(a, b, c) = {}^3\mathbf{T}(c) {}^2\mathbf{T}(b) {}^1\mathbf{T}(a)\mathbf{a}.$$

Boundary elements of the solid (facets, edges, vertices) naturally correspond to the boundary elements of the region $\Omega^3$, but compression of parts of $\Omega^3$ is also possible.

## 4. Intrinsic Geometric Properties of a Solid

Let the 3-patch (solid) $S\left(\Omega^3\right) \subset \mathbf{P}E^d$ be represented by

$$\mathbf{s}\left(u^i\right) = \left(s^0\left(u^1, u^2, u^3\right), s^1\left(u^1, u^2, u^3\right), \dots, s^d\left(u^1, u^2, u^3\right)\right)^t,$$

where the components $s^\alpha\left(u^i\right)$ are homogeneous coordinate functions in three real variables differentiable on $\Omega^3$. Let $\mathbf{P}$ be a point of the solid represented by homogeneous coordinates determined from the curvilinear coordinates $(a, b, c) \in \Omega^3$.

Partial derivatives of the function, with respect to all three variables, define in this point (*Fig. 1*):

tangent vectors to the isoparametric curve segments:

$$\frac{\partial \mathbf{s}}{\partial u^1}(a, b, c) = \mathbf{s}_1(a, b, c) = {}^3\mathbf{T}(c) {}^2\mathbf{T}(b) {}^1\mathbf{T}'(a)\mathbf{a},$$

$$\frac{\partial \mathbf{s}}{\partial u^2}(a, b, c) = \mathbf{s}_2(a, b, c) = {}^3\mathbf{T}(c) {}^2\mathbf{T}'(b) {}^1\mathbf{T}(a)\mathbf{a},$$

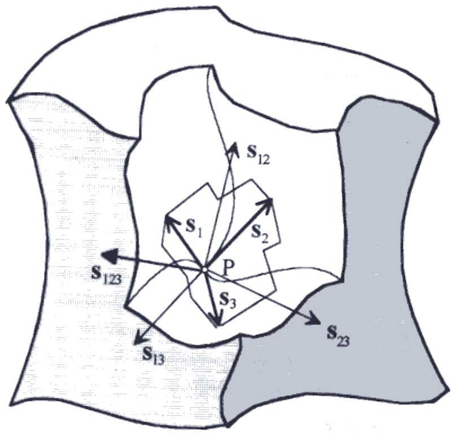


*Fig. 1.*

$$\frac{\partial \mathbf{s}}{\partial u^3}(a,b,c) = \mathbf{s}_3(a,b,c) =^3 \mathbf{T}'(c)^2\mathbf{T}(b)^1\mathbf{T}(a)\mathbf{a},$$

twist vectors to the isoparametric surface patches:

$$\frac{\partial^2 \mathbf{s}}{\partial u^1 \partial u^2}(a,b,c) = \mathbf{s}_{12}(a,b,c) =^3 \mathbf{T}(c)^2\mathbf{T}'(b)^1\mathbf{T}'(a)\mathbf{a},$$

$$\frac{\partial^2 \mathbf{s}}{\partial u^1 \partial u^3}(a,b,c) = \mathbf{s}_{13}(a,b,c) =^3 \mathbf{T}'(c)^2\mathbf{T}(b)^1\mathbf{T}'(a)\mathbf{a},$$

$$\frac{\partial^2 \mathbf{s}}{\partial u^2 \partial u^3}(a,b,c) = \mathbf{s}_{23}(a,b,c) =^3 \mathbf{T}'(c)^2\mathbf{T}'(b)^1\mathbf{T}(a)\mathbf{a},$$

density vector of distribution:

$$\frac{\partial^3 \mathbf{s}}{\partial u^1 \partial u^2 \partial u^3}(a,b,c) = \mathbf{s}_{123}(a,b,c) =^3 \mathbf{T}'(c)^2\mathbf{T}'(b)^1\mathbf{T}'(a)\mathbf{a}.$$

The given vectors $\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3$ form a tangent trihedron in the regular point of the solid. Trihedron facets are tangent planes to the isoparametric surface patches, edges are tangent lines to the isoparametric curve segments meeting in the solid point, that is the trihedron vertex. The tangent trihedron determines intrinsic geometric properties of the solid. They can be calculated by means of the coefficients of the first fundamental form of the solid.

In the regular point **P** of the solid there is a nonzero 3-minor determinant of the Jacobi matrix

$$\left(\frac{\partial \mathbf{s}^\alpha}{\partial u^i}\right), \qquad i = 1,2,3; \qquad \alpha = 0,1,\ldots,d$$

which together with the standard scalar product in $\mathbf{P}E^d$

$$\delta_{\alpha\beta} = \left\langle \mathbf{e}_\alpha, \mathbf{e}_\beta \right\rangle = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

induce the infinitesimal metric on the solid.

We can define

$$g_{ij} = \frac{\partial \mathbf{s}^\alpha}{\partial u^i} \delta_{\alpha\beta} \frac{\partial \mathbf{s}^\beta}{\partial u^j} =< \mathbf{s}_i, \mathbf{s}_j >= \mathbf{s}_i \mathbf{s}_j$$

in the sense above.

The arc length (or line element) quadrate is called the solid first fundamental form $\varphi_1\left(u^1, u^2, u^3\right)$ and it is determined as

$$(\mathrm{d}s)^2 = \mathrm{d}u^i g_{ij} \mathrm{d}u^j = (\mathbf{s}_1 \mathrm{d}u^1 + \mathbf{s}_2 \mathrm{d}u^2 + \mathbf{s}_3 \mathrm{d}u^3)^2 =$$

$$= \left({}^3\mathbf{T}(c)^2\mathbf{T}(b)^1\mathbf{T}'(a)\mathbf{a}\mathrm{d}u^1 +^3 \mathbf{T}(c)^2\mathbf{T}'(b)^1\mathbf{T}(a)\mathbf{a}\mathrm{d}u^2 + \right.$$

$$\left. + \,{}^3\mathbf{T}'(c)^2\mathbf{T}(b)^1\mathbf{T}(a)\mathbf{a}\mathrm{d}u^3\right)^2 =$$

$$= \mathbf{s}_1^2\left(\mathrm{d}u^1\right)^2 + 2\mathbf{s}_1\mathbf{s}_2\mathrm{d}u^1\mathrm{d}u^2 + \mathbf{s}_2^2\left(\mathrm{d}u^2\right)^2 + \mathbf{s}_1^2\left(\mathrm{d}u^1\right)^2 + 2\mathbf{s}_1\mathbf{s}_3\mathrm{d}u^1\mathrm{d}u^3 + \mathbf{s}_3^2\left(\mathrm{d}u^3\right)^2 +$$

$$+\mathbf{s}_2^2\left(\mathrm{d}u^2\right)^2 + 2\mathbf{s}_2\mathbf{s}_3\mathrm{d}u^2\mathrm{d}u^3 + \mathbf{s}_3^2\left(\mathrm{d}u^3\right)^2 - \left(\mathbf{s}_1^2\left(\mathrm{d}u^1\right)^2 + \mathbf{s}_2^2\left(\mathrm{d}u^2\right)^2 + \mathbf{s}_3^2\left(\mathrm{d}u^3\right)^2\right).$$

The first fundamental form $\varphi_1\left(u^1, u^2, u^3\right)$ of the solid is determined as the sum of the first fundamental forms $\varphi_1\left(u^1, u^2\right)$, $\varphi_1\left(u^1, u^3\right)$, $\varphi_1\left(u^2, u^3\right)$ of the isoparametric surface patches of the solid subtracted by the sum $\varphi$ of the first fundamental forms of the isoparametric curve segments of the solid, that are squares of the total differentials of the isoparametric curve segment point functions. With respect to this relation some of the intrinsic geometric properties of the solid can be related to the intrinsic geometric properties of the isoparametric subsets – curve segments and surface patches.

Discriminant of the first fundamental form $\varphi_1\left(u^1, u^2, u^3\right)$ of the solid is defined in a solid regular point as the value of the determinant

$$D = \left|g_{ij}\right| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} = \begin{vmatrix} \mathbf{s}_1\mathbf{s}_1 & \mathbf{s}_1\mathbf{s}_2 & \mathbf{s}_1\mathbf{s}_3 \\ \mathbf{s}_2\mathbf{s}_1 & \mathbf{s}_2\mathbf{s}_2 & \mathbf{s}_2\mathbf{s}_3 \\ \mathbf{s}_3\mathbf{s}_1 & \mathbf{s}_3\mathbf{s}_2 & \mathbf{s}_3\mathbf{s}_3 \end{vmatrix}$$

that is always positive.

Volume of the solid can be calculated from the discriminant $D$, as the triple integral over the region $\Omega^3$

$$\iiint_\Omega \sqrt{D\left(u^1, u^2, u^3\right)}\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3.$$

## 5. Example Calculations

We shall demonstrate how to use the machinery, introduced in previous sections.

### 5.1. Conical Solid

Revolving the point $\mathbf{A}(1, a, 0, 0)^t$ about the coordinate axis $z$, set of transformations ${}^1\mathbf{T}(u)$, up to the angle $2\pi$ a circle located in the ground projection plane $(x, y)$ can be created, from which by the set of scalings ${}^2\mathbf{T}(v)$ to its centre $\mathbf{O}$ with the coefficient 1 we can create a disc. This is the basic figure subdued to the set of scalings ${}^3\mathbf{T}(w)$ with the centre in the point $\mathbf{V}(1, b, c, d)^t$, $d \neq 0$, conical solid vertex, and a prescribed coefficient $k \neq 0$ (*Fig. 2*).

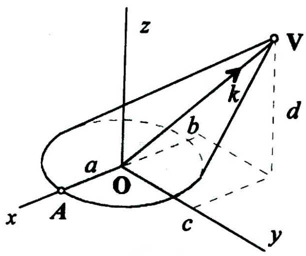


*Fig. 2.*

For the different values of $k$ we can create different forms of the conical solid.

If $k = 1$ a cone can be created, for $k > 1$ a doubled conical solid appears and for $k < 1$ there can be modelled a truncated conical solid in-between the basic disc and vertex $\mathbf{V}$ (for $k > 0$) or the opposite one (for $k < 0$), as illustrated in *Fig. 3*.

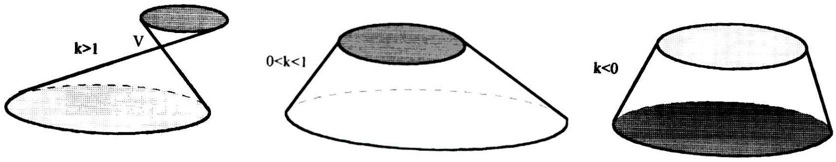


*Fig. 3.*

Matrices of the above transformations and their derivatives are in the following forms

$$^1\mathbf{T}(u)=\begin{pmatrix}1&0&0&0\\0&\cos 2\pi u&-\sin 2\pi u&0\\0&\sin 2\pi u&\cos 2\pi u&0\\0&0&0&1\end{pmatrix},$$

$$^1\mathbf{T}'(u)=\begin{pmatrix}0&0&0&0\\0&-2\pi\ \sin 2\pi u&-2\pi\ \cos 2\pi u&0\\0&2\pi\ \cos 2\pi u&-2\pi\ \sin 2\pi u&0\\0&0&0&0\end{pmatrix},$$

$$^2\mathbf{T}(v)=\begin{pmatrix}1&0&0&0\\0&1-v&0&0\\0&0&1-v&0\\0&0&0&1-v\end{pmatrix},\qquad ^2\mathbf{T}'(v)=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix},$$

$$^3\mathbf{T}(w)=\begin{pmatrix}1&0&0&0\\bwk&1-wk&0&0\\cwk&0&1-wk&0\\dwk&0&0&1-wk\end{pmatrix},$$

$$^3\mathbf{T}'(w)=\begin{pmatrix}0&0&0&0\\bk&-k&0&0\\ck&0&-k&0\\dk&0&0&-k\end{pmatrix}.$$

The conical solid point function for $(u,v,w)\in[0,1]^3$ can be calculated as

$$\mathbf{s}(u,v,w)={}^3\,\mathbf{T}(w)^2\mathbf{T}(v)^1\mathbf{T}(u)\mathbf{A}=$$

$$=(1,a(1-v)(1-wk)\cos 2\pi u+bwk,a(1-v)(1-wk)\sin 2\pi u+cwk,dwk)^t.$$

Solid first fundamental form will be

$$\Phi_1(u,v,w)=4\pi^2a^2(1-v)^2(1-wk)^2\mathrm{d}u^2+a^2(1-wk)^2\mathrm{d}v^2+$$
$$+k^2\left(b^2+c^2+d^2+a^2(1-v)^2-2a(1-v)(b\cos 2\pi u+c\sin 2\pi u)\right)\mathrm{d}w^2+$$
$$+2\pi ak(1-v)(1-wk)(c\cos 2\pi u-b\sin 2\pi u)\mathrm{d}u\mathrm{d}w+$$
$$+ak(1-wk)\left(a(1-v)-b\cos 2\pi u-c\sin 2\pi u\right)\mathrm{d}v\mathrm{d}w$$

by straightforward computations.

Then we can calculate all the data indicated in section 4.

For the volume we get formulae

$$\sqrt{D}=2\pi a^2dk(1-v)(1-wk)^2,$$

$$V=\int_0^1\int_0^1\int_0^1\sqrt{D\,(u,v,w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w=\pi a^2dk\left(1-k+\frac{k^2}{3}\right).$$

## 5.2. Toroidal Solid

Revolving the point $\mathbf{A} = (1, a, 0, 0)^t$ about the coordinate axis $z$ up to the angle $2\pi$, set of transformations ${}^1\mathbf{T}(u)$ from the previous example, a circle located in the ground projection plane can be created. From this, by the set of scalings ${}^2\mathbf{T}(v)$ (from the previous example) to its centre $\mathbf{O}$ with the scale coefficient 1 we can create a disc. That will be translated in the direction of the axis $x$ by an arbitrary distance $b > a$ (*Fig. 4*) in the simple geometric transformation $\mathbf{T}(b)$. This disk will be subdued to the set of revolutions ${}^3\mathbf{T}(w)$ about the coordinate axis $y$ up to the angle $2\pi$, while the toroidal solid illustrated in the *Fig. 5* can be obtained.

$$\mathbf{T}(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ b & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$${}^3\mathbf{T}(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi w & 0 & -\sin 2\pi w \\ 0 & 0 & 1 & 0 \\ 0 & \sin 2\pi w & 0 & \cos 2\pi w \end{pmatrix},$$

$${}^3\mathbf{T}'(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\pi \sin 2\pi w & 0 & -2\pi \cos 2\pi w \\ 0 & 0 & 0 & 0 \\ 0 & 2\pi \cos 2\pi w & 0 & -2\pi \sin 2\pi w \end{pmatrix}.$$

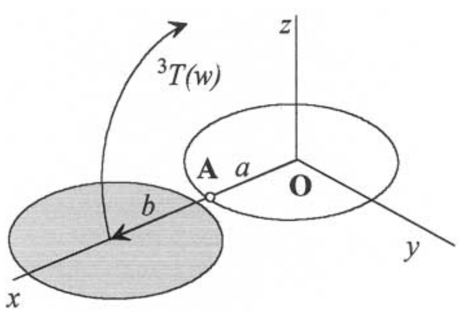


*Fig. 4.*

The toroidal solid point function

$$\mathbf{s}(u, v, w) = {}^3\mathbf{T}(w)\mathbf{T}(b){}^2\mathbf{T}(v){}^1\mathbf{T}(u)\mathbf{A} =$$

$$= (1, (b + a(1-v)\cos 2\pi u)\cos 2\pi w, a(1-v)\sin 2\pi u, (b + a(1-v)\cos 2\pi u)\sin 2\pi w)^t$$

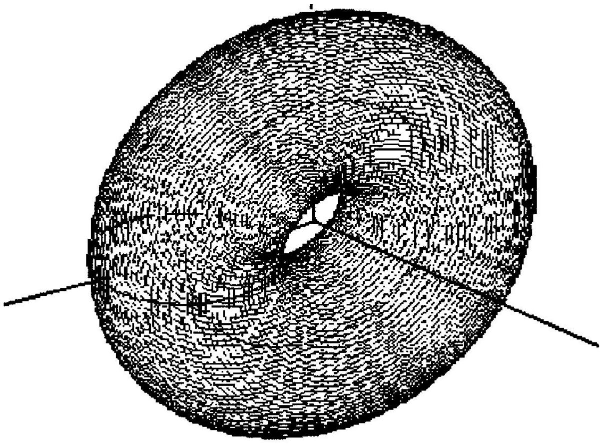

*Fig. 5.*

for $(u, v, w) \in [0, 1]^3$.

Toroidal solid first fundamental form can be expressed as a result

$$\Phi_1(u, v, w) = 4\pi^2 a^2 (1-v)^2 \mathrm{d}u^2 + a^2 \mathrm{d}v^2 + 4\pi^2 \left(b + a(1-v)\cos 2\pi u\right)^2 \mathrm{d}w^2.$$

To get this formula, tangent vectors to the solid isoparametric curves can be calculated from the above vector functions

$$\mathbf{s}_1(u, v, w) = {}^3\,\mathbf{T}(w)\mathbf{T}(b)^2\mathbf{T}(v)^1\mathbf{T}'(u)\mathbf{A} =$$

$$= 2\pi\ (0, -a(1-v)\sin 2\pi u \cos 2\pi w, a(1-v)\cos 2\pi u,$$
$$-a(1-v)\sin 2\pi u \sin 2\pi w)^t\ ,$$

$$\mathbf{s}_2(u, v, w) = {}^3\,\mathbf{T}(w)\mathbf{T}(b)^2\mathbf{T}'(v)^1\mathbf{T}(u)\mathbf{A} =$$

$$= (0, -a\cos 2\pi u \cos 2\pi w, -a\sin 2\pi u, -a\cos 2\pi u \sin 2\pi w)^t\ ,$$

$$\mathbf{s}_3(u, v, w) = {}^3\,\mathrm{T}'(w)\mathbf{T}(u)^2\mathrm{T}(v)^1\mathrm{T}(u)\mathbf{A} =$$

$$= 2\pi\ (0, -\left(b + a(1-v)\cos 2\pi u\right)\sin 2\pi w, 0, \left(b + a(1-v)\cos 2\pi u\right)\cos 2\pi w)^t\ .$$

For the volume of the toroidal solid we receive formulae

$$\sqrt{D(u, v, w)} = 4\pi^2 a^2 (1-v)\left(b + a(1-v)\cos 2\pi u\right),$$

$$V = \int_0^1\int_0^1\int_0^1 \sqrt{D(u, v, w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w = 2\pi^2 a^2 b.$$

### 5.3. *Helical Solid*

Revolving the fixed point $\mathbf{A} = (1, a, 0, 0)^t$ about the axis $z$ (set of transformations ${}^1\mathbf{T}(u)$ from the previous examples) up to the angle $2\pi$ a circle located in the ground projection plane can be created. From this we can create a disc by a set of scalings ${}^2\mathbf{T}(v)$ (from the previous examples) to its centre $\mathbf{O}$ with the scale coefficient 1. The disc will be translated in the direction of the axis $x$ by an arbitrary distance $b > a$, applying the transformation $\mathbf{T}(b)$ from the previous example. A helical solid can be created from the disc by subduing it to the helical movement about the axis in the arbitrary line. Let the axis of the helical movement ${}^3\mathbf{T}_z(w)$ be the coordinate axis $z$ up to the angle $2\pi$ and the pitch be $c \neq 0$, as illustrated in the *Fig. 6*.

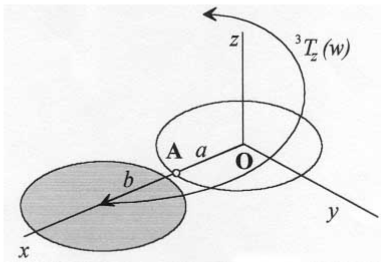


*Fig. 6.*

$$
{}^3\mathbf{T}_z(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi w & -\sin 2\pi w & 0 \\ 0 & \sin 2\pi w & \cos 2\pi w & 0 \\ 2\pi c w & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^3\mathbf{T}'_z(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\pi \sin 2\pi w & -2\pi \cos 2\pi w & 0 \\ 0 & 2\pi \cos 2\pi w & -2\pi \sin 2\pi w & 0 \\ 2\pi c & 0 & 0 & 0 \end{pmatrix}.
$$

The helical solid point function for $(u, v, w) \in [0, 1]^3$ is in the form

$$
\mathbf{s}(u, v, w) = {}^3\mathbf{T}_z(w)\mathbf{T}(b){}^2\mathbf{T}(v){}^1\mathbf{T}(u)\mathbf{A} =
$$

$$
= (1, b\cos 2\pi w + a(1-v)\cos 2\pi(u+w), b\sin 2\pi w +
$$
$$
+ a(1-v)\sin 2\pi(u+w), 2\pi c w)^t .
$$

Tangent vectors to the solid isoparametric curve segments can be determined from the vector functions

$$
\mathbf{s}_1(u, v, w) = 2\pi\ (0, -a(1-v)\sin 2\pi(u+w), a(1-v)\cos 2\pi(u+w), 0)^t ,
$$

$$
\mathbf{s}_2(u, v, w) = (0, -a\cos 2\pi(u+w), -a\sin 2\pi(u+w), 0)^t ,
$$

$$\mathbf{s}_3(u,v,w) = 2\pi\ (0, -b\sin 2\pi w - a(1-v)\sin 2\pi(u+w),$$
$$b\cos 2\pi w + a(1-v)\cos 2\pi(u+w), c)^t\ .$$

Helical solid first fundamental form:

$$\Phi_1(u,v,w) = 4\pi^2a^2(1-v)^2\mathrm{d}u^2 + a^2\mathrm{d}v^2 + 4\pi^2\left(b^2+c^2+a^2(1-v)^2+\right.$$
$$\left.+2ab\cos 2\pi u\right)\mathrm{d}w^2 + 8\pi^2 a(1-v)\left(a(1-v)\right.$$
$$\left.+b\cos 2\pi u\right)\mathrm{d}u\mathrm{d}w - 4\pi ab\sin 2\pi u\mathrm{d}v\mathrm{d}w.$$

Volume of the solid can be calculated from the formulae

$$\sqrt{D(u,v,w)} = 4\pi^2a^2c(1-v)$$

and

$$V = \int\int_0^1\int\sqrt{D(u,v,w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w = 2\pi^2a^2c.$$

Boundary of the created helical solid (*Fig. 7*) is a well-known circular helical surface 'winded column'. Applying a different helical movement to the same disc in the above example, a different helical solid can be created.

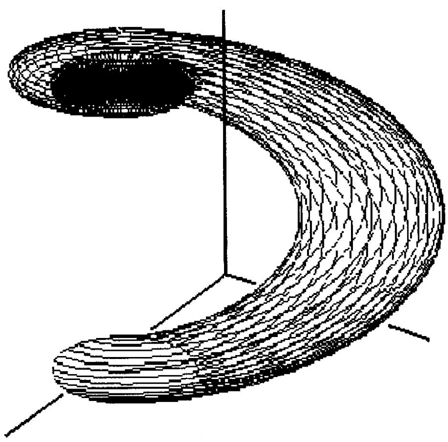

*Fig. 7.*

Let us choose the helical movement ${}^3\mathbf{T}_y(w)$ with the axis in the coordinate axis $y$ up to the angle $2\pi$ and the pitch $c \neq 0$ (*Fig. 9*). The helical solid illustrated in *Fig. 8* will be created with the boundary in the circular vaulted helicoid.

$${}^3\mathbf{T}_y(w) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\pi w & 0 & -\sin 2\pi w \\ 2\pi cw & 0 & 1 & 0 \\ 0 & \sin 2\pi w & 0 & \cos 2\pi w \end{pmatrix},$$

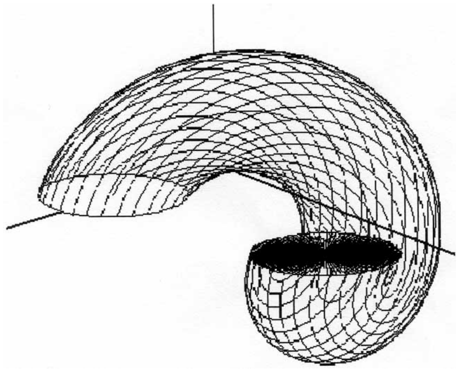

*Fig. 8.*

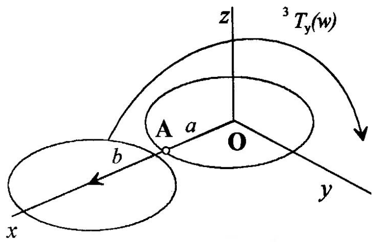


*Fig. 9.*

$$
{}^3\mathbf{T}'_y(w) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\pi\sin 2\pi w & 0 & -2\pi\cos 2\pi w \\ 2\pi c & 0 & 0 & 0 \\ 0 & 2\pi\cos 2\pi w & 0 & -2\pi\sin 2\pi w \end{pmatrix} .
$$

Solid point function for $(u, v, w) \in [0, 1]^3$ is in the form

$$
\mathbf{s}(u, v, w) =^3 \mathbf{T}_y(w)\mathbf{T}(b)^2\mathbf{T}(v)^1\mathbf{T}(u)\mathbf{A} =
$$

$$
= (1, (b + a(1 - v)\cos 2\pi u)\cos 2\pi w, 2\pi cw + a(1 - v)\sin 2\pi u,
$$
$$
(b + a(1 - v)\cos 2\pi u)\sin 2\pi w)^t .
$$

Tangent vectors to the solid isoparametric curve segments can be derived from functions

$$
\mathbf{s}_1(u, v, w) = 2\pi\ (0, -a(1 - v)\sin 2\pi u\cos 2\pi w, a(1 - v)\cos 2\pi u,
$$
$$
-a(1 - v)\sin 2\pi u\sin 2\pi w)^t ,
$$
$$
\mathbf{s}_2(u, v, w) = (0, -a\cos 2\pi u\cos 2\pi w, -a\sin 2\pi u, -a\cos 2\pi u\sin 2\pi w)^t ,
$$

$$\mathbf{s}_3(u, v, w) = 2\pi\ (0, (b + a(1 - v)\cos 2\pi u)\sin 2\pi w, c,$$
$$(b + a(1 - v)\cos 2\pi u)\cos 2\pi w)^t\ .$$

Solid first fundamental form

$$\Phi_1(u, v, w) = 4\pi^2 a^2 (1 - v)^2 \mathrm{d}u^2 + a^2 \mathrm{d}v^2 +$$
$$+4\pi^2 \left(c^2 + (b + a(1 - v)\cos 2\pi u)^2\right) \mathrm{d}w^2 +$$
$$+4\pi^2 a(1 - v)\ (-2\ (b + a(1 - v)\cos 2\pi u)\sin 2\pi u \sin 2\pi w \cos 2\pi w +$$
$$+c\cos 2\pi u)\ \mathrm{d}u\mathrm{d}w +$$
$$+4\pi a\ (-2\ (b + a(1 - v)\cos 2\pi u)\cos 2\pi u \sin 2\pi w \cos 2\pi w + c\sin 2\pi u)\ \mathrm{d}v\mathrm{d}w.$$

Solid volume can be calculated from the formulae

$$\sqrt{D(u, v, w)} = 4\pi^2 a^2 (1 - v)\ (b + a(1 - v)\cos 2\pi u)$$

$$V = \int\int_0^1\int \sqrt{D(u, v, w)}\mathrm{d}u\mathrm{d}v\mathrm{d}w = 2\pi^2 a^2 b.$$

Computer algebra softwares (e.g. Maple, MathCad) can help us in these computations.

## References

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[2] VELICHOVÁ, D. (1997): Modelling of Hyperpatches, *KoG – Scientific and Professional Information Journal,* No. 2, Zagreb pp. 9–13.