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THE NEWEST RESULTS OF HEAT CONDUCTION THEORY

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Abstract

This paper gives a detailed system theoretical treatment of the heat flux theory in the linear heat conduction based on the Laplace transformation method. By restricting the investigations to the simplest geometrical structures occurring in the practice, the authors prove the criteria guaranteeing the existence of the convolutional representations of the heat flux depending on the known temperature.

Keywords: heat conduction, heat flux, Laplace transform.

I. The System Theory of Heat Flux

I.1. Introduction

Let us consider the linear heat equation in one space variable x

$$\Delta\vartheta(x,t) = \frac{1}{\kappa}\frac{\partial\vartheta(x,t)}{\partial t}, \qquad t > 0, \quad x \in I, \tag{1}$$

where I denotes a finite, or a semi-infinite interval, Δ, κ denote the Laplace operator, and the thermal diffusivity, respectively. We shall assume in the sequel that the initial condition equals zero

$$\vartheta(x,0) = 0, \tag{2}$$

for every inner point of the interval I. The unicity of the solution of (1) is guaranteed by the initial condition (2) and the boundary conditions. However, from the viewpoint of the theory and applications of the heat flux, the knowledge of the boundary conditions is generally superfluous and uninteresting.

The main problems of the theory of the heat flux can be formulated as follows. Let an arbitrary linear heat conduction process be given satisfying (1), (2), moreover let $x, x_0, x_1, x_2, (x_1 \neq x_2)$ be arbitrary points of I.

Problem I. What is the connection between the heat flux at the point x, and the temperature at the point x_0 on the time interval $0 < t < \infty$, provided that the temperature determines the heat flux uniquely.

Problem II. What is the connection between the heat flux at the point x, and the temperatures at the points x_1, x_2, on the time interval $0 < t < \infty$, provided that the temperatures determine the heat flux uniquely.

Problem III. What is the connection between the heat flux at the point x, and the temperature and heat flux at the points x_1, x_2, respectively, on the time interval $0 < t < \infty$, provided that the latter determine the previous quantity uniquely.

We shall call Problem II the pure problem and Problem III the mixed problem of the theory of heat flux, respectively.

The heat flux is by definition:

$$j(x,t) = -K \frac{\partial \vartheta(x,t)}{\partial x}, \tag{3}$$

where K denotes the thermal conductivity. In the sequel we assume that the quantities κ, K are constants not depending on position, time and temperature.

By restricting ourselves to the simplest geometrical structures, we shall solve the above problems by the application of the Laplace transformation method using a system theoretical treatment. We assume that the functions under consideration are Laplace transformable and that the time functions, which are obtained by the inverse Laplace transformation, describe the concrete heat flux problem.

I.2. The Solution of Problem I

By transforming (1), and taking into account (2), we obtain

$$\Delta\Theta(x,s) - \frac{s}{\kappa}\Theta(x,s) = 0, \tag{4}$$

where

$$\Theta(x,s) = \int_0^\infty \vartheta(x,t)e^{-st}\,\mathrm{d}t. \tag{5}$$

Let $\Theta_1(x,s)$, $\Theta_2(x,s)$ be two linearly independent solutions of (5). The general solution is of the for

$$\Theta(x,s) = \alpha(s)\Theta_1(x,s) + \beta(s)\Theta_2(x,s), \tag{6}$$

where $\alpha(s)$, $\beta(s)$ are arbitrary functions of the complex variable s.

We have by (6)

$$\Theta(x_0,s) = \alpha(s)\Theta_1(x_0,s) + \beta(s)\Theta_2(x_0,s), \tag{7}$$
$$\Theta'(x,s) = \alpha(s)\Theta_1'(x,s) + \beta(s)\Theta_2'(x_0,s). \tag{8}$$

($'$ denotes the derivative $\frac{\mathrm{d}}{\mathrm{d}x}$). It is easily seen that the quantity $\Theta(x_0,s)$ does not determine uniquely the value of $\Theta'(x,s)$ in general. In this paragraph we shall restrict ourselves to such structures, where only one of the linearly independent solutions of (5) should be considered. Let us denote this solution by $f(x, s)$. So we have

$$\begin{aligned}\Theta(x, s) &= \alpha(s) f(x, s),\\ \Theta(x_0, s) &= \alpha(s) f(x_0, s),\\ \Theta'(x, s) &= \alpha(s) f'(x, s),\end{aligned} \tag{9}$$

and

$$\Theta'(x, s) = \Theta(x_0, s)\frac{f'(x, s)}{f(x_0, s)}, \tag{10}$$

$$-K\Theta'(x, s) = -K\Theta(x_0, s)\frac{f'(x, s)}{f(x_0, s)}. \tag{11}$$

By introducing the notations

$$H(x, x_0, s) = -K\frac{f'(x, s)}{f(x_0, s)} \tag{12}$$

(11) can be written as

$$J(x, s) = \Theta(x_0, s)H(x, x_0, s). \tag{13}$$

The equation (13) describes a transmission system, the scheme of which is illustrated in *Fig. 1*.

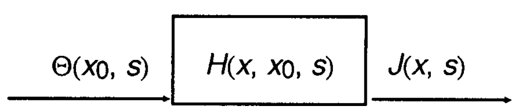

Fig. 1. Transmission system model of the heat flux

This scheme symbolises the connection between the input (temperature) and the output (heat flux). The function $H(x, x_0, s)$ being the quotient of the Laplace transforms of the output and input, is called the transfer function of the system. (see FODOR [2], KAPLAN [3]).

It follows from the convolution theorem of the Laplace transformation that if there exists the time function $h(x, x_0, t)$ having the Laplace transform $H(x, x_0, s)$

then by inverting (1), (6), the heat flux can be written in the form of the convolution integral

$$j(x,t) = \int_0^t \vartheta(x_0, t-\tau) h(x, x_0, \tau)\, d\tau, \tag{14}$$

having a great practical importance.

If the transfer function has no inverse in the time domain, then, as we shall see in special cases, the function

$$\frac{1}{H(x, x_0, s)}$$

will be invertable. Denoting its inverse by $h^*(x, x_0, t)$, (13) is equivalent to the following convolution type integral equation of the first kind

$$\int_0^t j(x,\tau) h^* x, x_0, t-\tau)\, d\tau = \vartheta(x_0, t). \tag{15}$$

We cannot give the explicit form of the solution of (15) in general, since (15) cannot be reduced to an integral equation of the second kind, the solution of which is represented by Neumann series. However, in special cases we give the explicit solution of (15), but not in the form of a convolution type integral.

So the knowledge of the criteria deciding about the two cases above is very important in the practice. We shall prove these simple criteria for the following geometrical structures

- The semi-infinite rod (or wall) $I = (0, \infty)$.
- The region bounded internally by a sphere $I = [a, \infty)$, $a > 0$.
- The sphere $I = (0, a)$, $a > 0$.
- The region bounded internally by an infinite circular cylinder $I = [a, \infty)$, $a > 0$.
- The infinite circular cylinder $I = [0, a)$, $a > 0$.

I.2.1. The Semi-infinite Rod (or Wall)

$$f(x,s) = e^{-\sqrt{\frac{s}{k}}x}, \tag{16}$$

(see FODOR [2], DOETSCH [7]). We have by (12) that

$$H(x, x_0, s) = K\sqrt{\frac{s}{k}} e^{-\sqrt{\frac{s}{k}}(x-x_0)} \tag{17}$$

holds. Let $x > x_0$. Then

$$h(x, x_0, t) = \frac{K}{2t\sqrt{\pi k t}} \left[\frac{(x-x_0)^2}{2kt} - 1\right] e^{-\frac{(x-x_0)^2}{4kt}}, \tag{18}$$

see for example (DITKIN – PRUDNIKOV [14]). It follows from (14) that

$$j(x,t) = \frac{K}{2}\int_0^t \vartheta(x_0, t-\tau)\frac{1}{\tau\sqrt{\pi k\tau}}\left[\frac{(x-x_0)^2}{2k\tau} - 1\right] e^{-\frac{(x-x_0)^2}{4kt}}\,\mathrm{d}\tau. \tag{19}$$

Let $x \le x_0$. Then the inverse Laplace transform of (17) does not exist, since

$$\lim_{s\to\infty}\sqrt{\frac{s}{k}}e^{-\sqrt{\frac{s}{k}}(x-x_0)} \neq 0 \qquad \text{(see [3])}. \tag{20}$$

The inverse of the function $\dfrac{1}{H(x,x_0,s)}$ exists. We have by [4]

$$h^*(x,x_0,t) = \frac{1}{K}\sqrt{\frac{k}{\pi t}}\exp\left[-\frac{(x-x_0)^2}{4kt}\right], \tag{21}$$

and taking into account (15) the following integral equation will be obtained

$$\int_0^t j(x,\tau)\frac{\exp\left[-\dfrac{(x-x_0)^2}{4k(t-\tau)}\right]}{\sqrt{t-\tau}}\,\mathrm{d}\tau = K\sqrt{\frac{\pi}{k}}\vartheta(x_0,t). \tag{22}$$

The kernel of (8) and its derivatives of arbitrary high order vanish for $t = 0$, if $x < x_0$. So (22) cannot be reduced to an integral equation of the second kind and the explicit solution of (22) cannot be given. (see FENYŐ–STOLLE [5]) For $x = x_0$ we obtain from (13), (15):

$$J(x_0,s) = K\sqrt{\frac{s}{k}}\Theta(x_0,s) = K\frac{1}{\sqrt{ks}}s\Theta(x_0,s). \tag{23}$$

Let x_0 be an arbitrary inner point of the domain I. Since $\vartheta(x_0,t)$ is absolutely continuous and $\vartheta(x_0,0) = 0$, by inverting (23) we obtain

$$j(x_0,t) = \frac{K}{\sqrt{\pi k}}\int_0^t \frac{\partial\vartheta(x_0,\tau)}{\partial\tau}\cdot\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau. \tag{24}$$

The convolution occurring on the right-hand side of (24) contains the derivative of the temperature (not the temperature itself). So we rewrite this formula as follows. Let $0 < \varepsilon < t$.

An integration by parts gives,

$$\begin{aligned}
\int_0^t \frac{\frac{\partial \vartheta(x_0,\tau)}{\partial \tau}}{\sqrt{t-\tau}}\,\mathrm{d}\tau &= \lim_{\varepsilon\to 0}\int_0^{t-\varepsilon} \frac{\frac{\partial \vartheta(x_0,\alpha\tau)}{\partial \tau}}{\sqrt{t-\tau}}\,\mathrm{d}\tau \\
&= \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)}{\sqrt{\varepsilon}} - \frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] \\
&= \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)}{\sqrt{3}} - \frac{1}{2}\vartheta(x_0,t)\int_0^{t-\varepsilon}\frac{\mathrm{d}\tau}{(t-\tau)^{\frac{3}{2}}}\right. \\
&\quad \left. +\frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] \\
&= \lim_{\varepsilon\to 0}\left[\frac{\vartheta(x_0,t-\varepsilon)-\vartheta(x_0,t)}{\sqrt{3}} + \frac{\vartheta(x_0,t)}{\sqrt{t}}\right. \\
&\quad \left. +\frac{1}{2}\int_0^{t-\varepsilon}\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] \\
&= \frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^{t}\frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau. \qquad (25)
\end{aligned}$$

Finally we have

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right]. \qquad (26)$$

In other form

$$j(x,t) = \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x,t)}{\partial t^{\frac{1}{2}}}, \qquad t > 0. \qquad (27)$$

Let now $x_0 = 0$ and let $\vartheta(0,t)$ be absolutely continuous. Then by inverting the formula (23)

$$j(0,t) = \frac{K}{\sqrt{\pi\kappa}}\int_0^t \frac{\partial\vartheta(0,\tau)}{\partial\tau}\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau + \frac{K\vartheta(0,0)}{\sqrt{\pi\kappa t}} \qquad (28)$$

is obtained. Analogously to the previous case a simple calculation shows that $\frac{K\vartheta(0,0)}{\sqrt{\pi\kappa t}}$ falls out and

$$j(0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(0,t)-\vartheta(0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right]. \qquad (29)$$

In other form

$$j(x,t) = \frac{K}{\sqrt{\kappa}} \frac{\partial^{\frac{1}{2}} \vartheta(x,t)}{\partial t^{\frac{1}{2}}}, \qquad t > 0. \tag{30}$$

This formula can be found in OLDHAM – SPANIER [6], the conditions of the validity of the above formula, however, are not given in [6].

I.2.2. The Region Bounded Internally by a Sphere

$$f(x,s) = \frac{e^{-\sqrt{\frac{s}{\kappa}}x}}{x} \tag{31}$$

(see [1]) and we obtain

$$H(x,x_0,s) = \frac{Kx_0\sqrt{\frac{s}{\kappa}}}{x} e^{-\sqrt{\frac{s}{\kappa}}(x-x_0)} + \frac{Kx_0}{x^2} e^{-\sqrt{\frac{s}{\kappa}}(x-x_0)}. \tag{32}$$

We get from [4] that

$$\begin{aligned} h(x,x_0,t) &= \frac{Kx_0}{2t\sqrt{\pi\kappa t}x}\left[\frac{(x-x_0)^2}{2\kappa t} - 1\right] e^{-\frac{(x-x_0)^2}{4\kappa t}} \\ &+ \frac{Kx_0(x-x_0)}{2x^2 t\sqrt{\kappa\pi t}} \exp\left[-\frac{(x-x_0)^2}{4\kappa t}\right], \qquad x > x_0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} h^*(x,x_0,t) &= \frac{\sqrt{\kappa}x}{Kx_0\sqrt{\pi t}} \exp\left[-\frac{(x-x_0)^2}{4\kappa t}\right] \\ &- \frac{2\kappa}{K\sqrt{\pi}x_0} \exp\left(\frac{x_0}{x} - 1 + \frac{\kappa t}{x^2}\right) \int_{\frac{x_0-x}{2\sqrt{\kappa t}}+\frac{\sqrt{\kappa t}}{x}}^{\infty} e^{-u^2}\,\mathrm{d}u, \qquad x_0 \geq x \end{aligned} \tag{34}$$

hold. By the aid of (34) we obtain the corresponding integral equation related to the heat flux.

We have by (32) that

$$H(x_0,x_0,s) = K\sqrt{\frac{s}{\kappa}} + \frac{k}{x_0} \tag{35}$$

holds. By taking into account (13)

$$J(x_0,s) = K\sqrt{\frac{s}{\kappa}}\Theta(x_0,s) + \frac{K}{x_0}\Theta(x_0,s) = K\frac{s}{\sqrt{\kappa s}}\Theta(x_0,s) + \frac{K}{x_0}\Theta(x_0,s) \tag{36}$$

will be obtained. Finally, by an inverse Laplace transformation we get the formula

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}} \int_0^t \frac{\frac{\partial\vartheta(x_0,\tau)}{\partial\tau}}{\sqrt{t-\tau}}\,\mathrm{d}\tau + \frac{K}{x_0}\vartheta(x_0,t) \tag{37}$$

for every inner point x_0 of the domain.

Moreover, by (24), (26)

$$j(x_0,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0,t)-\vartheta(x_0,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] + \frac{K}{x_0}\vartheta(x_0,t) \tag{38}$$

holds. Similarly we obtain that, if (a,t) is absolutely continuous, then the validity of (38) holds true also for the limit point $x_0 = a$.

I.2.3. The Sphere

$$f(x,s) = \frac{\operatorname{sh}\sqrt{\frac{s}{\kappa}}x}{x}, \tag{39}$$

and

$$H(x,x_0,s) = \frac{Kx_0\left(\operatorname{sh}\sqrt{\frac{s}{\kappa}}x - \sqrt{\frac{s}{\kappa}}x\,\operatorname{ch}\sqrt{\frac{s}{\kappa}}x\right)}{x^2\operatorname{sh}\sqrt{\frac{s}{\kappa}}x_0}. \tag{40}$$

The case $x = 0$ can be excluded from the discussion, since the heat flux equals zero for $x = 0$. (40) has no inverse for $x \geq x_0$ since $\lim_{s\to\infty} H \neq 0$. The inverse of (40) exists for $x < x_0$. Applying

$$\left(1 - e^{-2\sqrt{\frac{s}{\kappa}}x_0}\right)^{-1} = \sum_{\nu=0}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}, \tag{41}$$

we have

$$\begin{aligned} H(x,x_0,s) = K\frac{x_0}{x^2}\Bigg[&\left(1-\sqrt{\frac{s}{\kappa}}x\right)\sum_{\nu=0}^{\infty}\exp\left(-\sqrt{\frac{s}{\kappa}}[(1+2\nu)x_0 - x]\right) \\ &- \left(1+\sqrt{\frac{s}{\kappa}}x\right)\sum_{\nu=0}^{\infty}\exp\left(-\sqrt{\frac{s}{\kappa}}[(1+2\nu)x_0 + x]\right)\Bigg]. \end{aligned} \tag{42}$$

By the application of a theorem of DOETSCH [7] (page 206) it is easily seen that the term by term inversion (42) is admissible.

So applying [4] we get

$$\begin{aligned}
h(x, x_0, t) &= \frac{-Kx_0}{2xt\sqrt{\pi\kappa t}}\left[\sum_{\nu=0}^{\infty}\left(\frac{[(1+2\nu)x_0 - x]^2}{2\kappa t} - 1\right)e^{-\frac{[(1+2\nu)x_0-x]^2}{4\kappa t}}\right.\\
&\quad\left.+\sum_{\nu=0}^{\infty}\left(\frac{[(1+2\nu)x_0 + x]}{2\kappa t} - 1\right)e^{-\frac{[(1+2\nu)x_0+x]^2}{4\kappa t}}\right]\\
&\quad-\frac{Kx_0}{2x^2t\sqrt{\pi\kappa t}}\left[\sum_{\nu=0}^{\infty}((1+2\nu)x_0 + x)e^{-\frac{[(1+2\nu)x_0+x]^2}{4\kappa t}}\right.\\
&\quad\left.-\sum_{\nu=0}^{\infty}((1+2\nu)x_0 - x)e^{-\frac{[(1+2\nu)x_0-x]^2}{4\kappa t}}\right]. \qquad (43)
\end{aligned}$$

For $x \geq x_0$ we apply Heaviside's Expansion Theorem and obtain

$$h^*(x, x_0, t) = \frac{-2\kappa}{Kx_0}\sum_{n=0}^{\infty}\frac{\sin\alpha_n\frac{x_0}{x}}{\sin\alpha_n}e^{-\frac{\alpha_n^2}{x^2}\kappa t}, \qquad t \geq 0, \quad x_0 \neq 0, \qquad (44)$$

(see CARSLAW-JAEGER [1]). I here α_n denotes the n-th positive root of the equation

$$\alpha = \operatorname{tg}\alpha. \qquad (45)$$

Important special cases:

$$x = x_0 \neq 0, \qquad h^*(x_0, x_0, t) = \frac{-2\kappa}{Kx_0}\sum_{n=1}^{\infty}e^{-\frac{\alpha_n^2}{x^2}\kappa t}, \qquad (46)$$

$$t > 0$$

$$x_0 = 0, \qquad h^*(x, 0, t) = -\frac{2\kappa}{Kx}\sum_{n=1}^{\infty}\frac{\alpha_n e^{-\frac{\alpha_n^2}{x^2}\kappa t}}{\sin\alpha_n}. \qquad (47)$$

Let $x = x_0 \neq 0$. The explicit form of the heat flux can be obtained in the following way. By (40) we have

$$H(x_0, x_0, s) = \frac{K}{x_0} - K\sqrt{\frac{s}{k}}\frac{\operatorname{ch}\sqrt{\frac{s}{\kappa}}x_0}{\operatorname{sh}\sqrt{\frac{s}{\kappa}}x_0} \qquad (48)$$

and

$$
\begin{aligned}
H(x_0, x_0, s) &= \frac{K}{x_0} - K\sqrt{\frac{s}{k}}\frac{1+e^{-2\sqrt{\frac{s}{\kappa}}x_0}}{1-e^{-2\sqrt{\frac{s}{\kappa}}x_0}} \\
&= \frac{K}{x_0} - K\sqrt{\frac{s}{\kappa}}\left(1+e^{-2\sqrt{\frac{s}{\kappa}}x_0}\right)\sum_{\nu=0}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0} \\
&= \frac{K}{x_0} - K\sqrt{\frac{s}{\kappa}} - K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0} \\
&\quad -K\sqrt{\frac{s}{\kappa}}\sum_{\nu=0}^{\infty} e^{-2(\nu+1)\sqrt{\frac{s}{\kappa}}x_0} \\
&= \frac{K}{x_0} - K\sqrt{\frac{s}{\kappa}} - 2K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0}. \qquad (49)
\end{aligned}
$$

(13) gives

$$
J(x_0, s) = \frac{K}{x_0}\Theta(x_0, s) - K\sqrt{\frac{s}{\kappa}}\Theta(x_0, s) - 2K\sqrt{\frac{s}{\kappa}}\sum_{\nu=1}^{\infty} e^{-2\nu\sqrt{\frac{s}{\kappa}}x_0\Theta(x_0,s)}. \qquad (50)
$$

Taking into account (36), (38) and applying [4], we obtain by the application of a Laplace invertation the formula

$$
\begin{aligned}
j(x_0, t) &= \frac{K\vartheta(x_0, t)}{x_0} - \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_0, t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_0, t) - \vartheta(x_0, \tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] \\
&\quad -\vartheta(x_0, t)^{*}\frac{K}{t}\sum_{\nu=0}^{\infty}\frac{1}{\sqrt{\pi\kappa t}}e^{-\frac{\nu^2 x_0^2}{\kappa t}}\left(\frac{2\nu^2 x_0^2}{\kappa t} - 1\right) \qquad (51)
\end{aligned}
$$

provided that either x_0 is an inner point of I or $x_0 = a$ and $\vartheta(a, t)$ is absolutely continuous. (We denoted here the convolution by*.)

In other form

$$
j(x_0, t) = \frac{K\vartheta(x_0, t)}{x_0} - \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x, t)}{\partial t^{\frac{1}{2}}} - \vartheta(x_0, t)^{*}\frac{K}{t}\sum_{\nu=1}^{\infty}\frac{1}{\sqrt{\pi\kappa t}}e^{\frac{\nu^2 x_0^2}{\kappa t}\left(\frac{2\nu^2 x_0^2}{\kappa t}-1\right)}. \qquad (52)
$$

I.2.4. *The Region Bounded Internally by an Infinite Circular Cylinder*

We have

$$
f(x, s) = K_0\left(\sqrt{\frac{s}{\kappa}}x\right), \qquad (53)
$$

where K_0 denotes the modified Bessel function of the second kind of order zero. So it is

$$H(x, x_0, s) = K\sqrt{\frac{s}{\kappa}}\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)}, \tag{54}$$

where K_1 denotes the modified first order Bessel function of the second kind. From the asymptotic expansion of the Bessel functions it follows that

$$\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)} \sim \sqrt{\frac{x_0}{x}}e^{\sqrt{\frac{s}{\kappa}}(x_0-x)} \tag{55}$$

holds for $s \to \infty$.

If $x \leq x_0$ then $\lim_{s\to\infty} = \infty$ and (54) has no inverse Laplace transform. We show that (54) has the inverse for $x > x_0$ and we determine this.

Eq. (54) has the following properties for $x > x_0$. Let $\gamma > 0$ be arbitrary. Then

1. $H(x, x_0, s)$ is analytic in the half plane Re $s \geq \gamma$.
2. $$\int_{\gamma-i\infty}^{\gamma+i\infty} |H(x, x_0, s)| \, \mathrm{d}s < \infty. \tag{56}$$
3. In the half plane Re $s \geq \gamma$ $H(x, x_0, s)$ tends uniformly to zero with respect to arg s if $|s| \to \infty$. Then an easy application of a theorem in DOETSCH [3] (p. 236) or BERG [8] (p. 27) shows that $H(x, x_0, s)$ has its inverse in the above half plane and

$$h(x, x_0, t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} H(x, x_0, s)e^{st} \, \mathrm{d}s, \tag{57}$$

moreover, $h(x, x_0, t)$ is a continuous function of t and $h(x, x_0, 0)$. Applying the Fourier–Mellin inversion integral

$$h(x, x_0, t) = \frac{K}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\sqrt{\frac{\lambda}{\kappa}}\frac{K_1\left(\sqrt{\frac{\lambda}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{\lambda}{\kappa}}x_0\right)} \mathrm{d}\lambda. \tag{58}$$

The integrand has a branch point in $\lambda = 0$, so we choose the following contour on the complex plane (see *Fig. 2*).

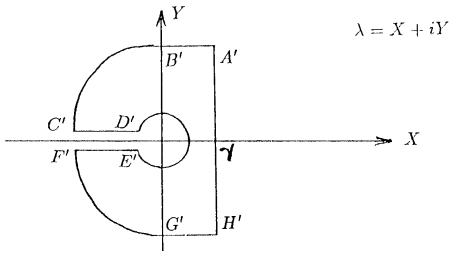

Fig. 2. Applying of Fourier-Mellin inversion integral on the complex plane

By omitting the details, the evaluation of the inversion integral above gives the following results.

$$
\begin{aligned}
h(x, x_0, t) &= \frac{2\kappa K}{\pi} \int_0^\infty e^{-\kappa u^2 t} u^2 \\
&\quad \times \frac{J_1(xu)Y_0(x_0 u) - Y_1(xu)J_0(x_0 u)}{J_0^2(x_0 u) + Y_0^2(x_0 u)} \,\mathrm{d}u, \qquad t > 0, \\
h(x, x_0, t) &= 0.
\end{aligned} \tag{59}
$$

Here J_0, Y_0 denote the nullth order Bessel functions of the first and second kind, J_1, Y_1 denote the first order Bessel functions of the first and second kind, respectively.

Finally let $x \leq x_0$. If $x < x_0$, then the above properties 1,2,3 are satisfied for the function $\dfrac{1}{H}$. However, the inverse of $\dfrac{1}{H}$ also exists for $x = x_0$ (see the procedure in CARSLAW–JAEGER [1] p. 388).

The following results are obtained:

$$
\begin{aligned}
h^{\otimes}(x, x_0, t) &= \frac{2\kappa}{K\pi} \int_0^\infty e^{-\kappa u^2 t} \\
&\quad \times \frac{J_1(xu)Y_0(x_0 u) - Y_1(xu)J_0(x_0 u)}{J_1^2(xu) + Y_1^2(xu)} \,\mathrm{d}u, \; t > 0,
\end{aligned} \tag{60}
$$

$$
h^{\otimes}(x, x_0, 0) = 0. \tag{61}
$$

For $x = x_0$

$$
h^{\otimes}(x_0, x_0, 0) = \infty \tag{62}
$$

$$
h^{\otimes}(x_0, x_0, t) = \frac{4\kappa}{K\pi^2 x_0} \int_0^\infty \frac{e^{-\kappa u^2 t} \,\mathrm{d}u}{u[J_1^2(x_0 u) - Y_1^2(x_0 u)]}, \qquad t > 0, \tag{63}
$$

which follows from (60) by the application of the relation

$$
J_1(z)Y_0(z) - J_0(z)Y_1(z) = \frac{2}{\pi z}. \tag{64}
$$

GOLDSTEIN [10] proves that the inverse of

$$s^{\alpha} K_{\mu}(\sqrt{s}\lambda) \tag{65}$$

exists and can be represented by the aid of Whittaker functions. So, for $\alpha = \frac{1}{2}$, $\mu = 1$, our result can be considered as a generalisation of [10].

I.2.5. The Infinite Circular Cylinder

We have

$$H(x, x_0, s) = -K\sqrt{\frac{s}{\kappa}} \frac{I_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{I_0\left(\sqrt{\frac{s}{\kappa}}x\right)}, \tag{66}$$

where I_0, I_1 denote the modified nullth, and first order Bessel functions of the first kind, respectively. By the application of the inversion formula we obtain the following:

Let $x < x_0$, then

$$h(x, x_0, t) = \frac{2\kappa K}{x_0} \sum_{n=1}^{\infty} \alpha_n^2 e^{-\kappa\alpha_n^2 t} \frac{J_1(\alpha_n x)}{J_1(\alpha_n x_0)}, \qquad t > 0, \tag{67}$$

$$h(x, x_0, 0) = 0, \tag{68}$$

where α_n denotes the positive roots of the equation

$$J_0(\alpha x_0) = 0. \tag{69}$$

Let $x > x_0$, then

$$h^*(x, x_0 t) = -\frac{2\kappa}{Kx}\left(1 + \sum_{n=1}^{\infty} \frac{J_0(\beta_n x_0)}{J_0(\beta_n x)} e^{-\kappa\beta_n^2 t}\right), \quad t > 0, \tag{70}$$

$$h^{\otimes}(x, x_0, 0) = 0. \tag{71}$$

Let $x = x_0$, then

$$h^{\otimes}(x, x_0, 0) = -\infty, \tag{72}$$

$$h^*(x_0, x_0 t) = -\frac{2\kappa}{Kx_0}\left(1 + \sum_{n=1}^{\infty} e^{-\kappa\beta_n^2 t}\right), \quad t > 0, \tag{73}$$

where β_n denotes the positive roots of the equation

$$J_1(\beta x) = 0. \tag{74}$$

The following statement holds:

Statement. Let us consider the cases A, B, D. The heat flux can be represented as a convolution integral if and only if $x > x_0$. For $x \leq x_0$ the heat flux satisfied a convolution type integral equation of the first kind. Let us consider the cases C, E. The heat flux can *be represented as a convolution integral if and only if* $x < x_0$. *For* $x \geq x_0$*, the heat flux satisfies a convolution type integral equation of the first kind.*

Moreover, if $x = x_0$, then the solutions of the corresponding integral equations can be given in explicit forms in the cases A, B, C provided that the point x_0 is either an inner point of the domain I, or is the limit point of I, where the temperature is absolutely continuous.

Remarks. 1.) In the discussion of the case of a region bounded internally by an infinite circular cylinder, we obtained

$$J(x,s) = K\sqrt{\frac{s}{\kappa}}\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right)}\Theta(x_0 s). \tag{75}$$

GARBAI [11] gets an integral equation for the heat flux as follows. Since

$$K_0\left(\sqrt{\frac{s}{\kappa}}x_0\right) J(x,s) = K\sqrt{\frac{s}{\kappa}}K_1\left(\sqrt{\frac{s}{\kappa}}x\right)\Theta(x_0,s). \tag{76}$$

By inverting both sides of this equation and applying the convolution theorem of the Laplace transformation, the integral equation.

$$\int_0^t j(x,\tau)\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{t-\tau}\,\mathrm{d}\tau = \frac{Kx}{2\kappa}\int_0^t \vartheta(x_0,\tau)\frac{e^{-\frac{x_0^2}{4\kappa(t-\tau)}}}{(t-\tau)^2}\,\mathrm{d}\tau \tag{77}$$

is obtained. (77) holds for every pair (x, x_0) and its kernel function is simpler than the corresponding ones given by (60), (63). The disadvantage of (77) lies in the fact that there occurs a convolution on the right-hand side of it.

It is surprising that (77) has no analogue in the case of the infinite circular cylinder.

2.) Our results can be well applied in the practice, if the heat flux has a convolutional representation. Then by measuring the temperature in discrete time intervals, the convolution can be evaluated by known numerical methods. On the other hand, there are numerical methods also for solving convolutional integral equations. We shall deal with these methods in a following paper.

3.) The condition of the absolute continuity of the temperature in the limit points is a sufficient condition, which holds in the practice. It is, however, not necessary.

I.3. Harmonic Processes

It follows from the theory of the linear systems that the results related to the harmonic processes are simple consequences of our results discussed above (see [2], [3]). If we substitute $s = i\omega$ in (13) in Part I, where ω is the angular frequency of the harmonic oscillation, and replace the Laplace transforms by the notations $\overline{\Theta}(x, i\omega)$, $\overline{J}(x, i\omega)$ then the equation

$$\overline{J}(x, i\omega) = H(x, x_0, i\omega)\overline{\Theta}(x_0, i\omega) \tag{78}$$

will be obtained. $\overline{\Theta}(x, i\omega)$, $\overline{J}(x, i\omega)$ are the complex amplitudes of the harmonic input (temperature), and harmonic output (heat flux), respectively. $H(x, x_0, i\omega)$ is the complex transfer characteristics of the system. *Eq.* (78) describes this transmission system, the scheme of which is illustrated in *Fig. 3*.

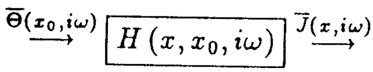

Fig. 3. Transmission system model of heat flux for harmonic processes

Practically, the most important quantity is the amplitude characteristics. $A(x, x_0, \omega)$ is the absolute value of the transfer characteristics $H(x, x_0, i\omega)$.

The amplitude characteristics describe the frequence dependency of the quotient of the amplitudes of the output and input (reasonance curve). Let us determine these in the structures discussed above.

I.3.1. The Case of the Half Space

By (17) in we have

$$H(x, x_0, i\omega) \quad = \quad K\sqrt{\frac{i\omega}{\kappa}}\exp\left[-\sqrt{\frac{i\omega}{\kappa}}(x - x_0)\right] \tag{79}$$

and

$$A(x, x_0, \omega) \quad = \quad K\sqrt{\frac{\omega}{\kappa}}\exp\left[-\sqrt{\frac{\omega}{2\kappa}}(x - x_0)\right]. \tag{80}$$

For $x > x_0$ there exists one resonance frequency

$$\omega_\tau = \frac{2\kappa}{(x - x_0)^2} \tag{81}$$

and

$$A\max = \frac{K\sqrt{2}}{e(x-x_0)}. \tag{82}$$

I.3.2. The Region Bounded Internally by the Sphere

By (32)

$$H(x, x_0, i\omega) = K\frac{x_0}{x}\left(\sqrt{\frac{i\omega}{\kappa}} + \frac{1}{x}\right)e^{-\sqrt{\frac{i\omega}{\kappa}}(x-x_0)}, \tag{83}$$

and by calculating its absolute value we get

$$A(x, x_0, \omega) = \frac{Kx_0}{x}\sqrt{\frac{\omega}{\kappa} + \frac{1}{x}\sqrt{\frac{2\omega}{\kappa}} + \frac{1}{x^2}}\exp\left[-\sqrt{\frac{\omega}{2\kappa}}(x-x_0)\right] \tag{84}$$

having one resonance frequency for $x > x_0$:

$$\omega_\tau = \frac{\kappa x_0}{x(x-x_0)^2}\left[1 + \frac{x_0}{x}\sqrt{\frac{2x}{x_0} - 1}\right]. \tag{85}$$

I.3.3. The Sphere

By (40) we have

$$H(x, x_0, i\omega) = K\frac{x_0\left(\sqrt{\frac{i\omega}{\kappa}}x \text{ ch}\sqrt{\frac{i\omega}{\kappa}}x - \text{sh}\sqrt{\frac{i\omega}{\kappa}}x\right)}{x^2 \text{ sh}\sqrt{\frac{i\omega}{\kappa}}x_0}, \qquad x_0 \neq 0. \tag{86}$$

Let us introduce the notations

$$\alpha = x\sqrt{\frac{\omega}{2\kappa}}, \qquad \alpha_0 = x_0\sqrt{\frac{\omega}{2\kappa}}, \tag{87}$$

so after some calculations we obtain the formula

$$\begin{aligned} A(x, x_0, \omega) &= \frac{Kx_0}{x^2} \\ &\times\sqrt{\frac{2\alpha^2(\text{ch}^2\alpha - \sin^2\alpha) - \alpha\text{ sh}2\alpha - \alpha\sin 2\alpha\text{ ch }2\alpha + \text{sh}^2\alpha + \sin^2\alpha}{\text{sh}^2\alpha_0 + \sin^2\alpha_0}}, \end{aligned} \tag{88}$$

and for $x \to 0$ we have

$$A(x,0,\omega) = \frac{K}{x^2}\sqrt{\frac{\kappa}{\omega}} \times\sqrt{2\alpha^2(\mathrm{ch}^2\alpha - \sin^2\alpha) - \alpha\,\mathrm{sh}2\alpha - \alpha\sin 2\alpha\,\mathrm{ch}\,2\alpha + \mathrm{sh}^2\alpha + \sin^2\alpha}. \tag{89}$$

I.3.4. The Region Bounded Internally by the Infinite Circular Cylinder

We have by (54)

$$H(x,x_0,i\omega) = K\sqrt{\frac{i\omega}{\kappa}}\frac{K_1\left(e^{i\frac{\pi}{4}}\sqrt{\frac{\omega}{\kappa}}x\right)}{K_0\left(e^{i\frac{\pi}{4}}\sqrt{\frac{\omega}{\kappa}}x_0\right)}, \tag{90}$$

the transfer characteristics can be expressed by Kelvin functions. Since

$$\mathrm{ker}_\nu z + i\,\mathrm{kei}_\nu z = e^{-\frac{1}{2}\nu\pi i}K\nu\left(ze^{i\frac{\pi}{4}}\right) \qquad z \geq 0, \qquad \nu \geq 0, \tag{91}$$

(see ABRAMOVITZ–STEGUN [12]), we get

$$A(x,x_0,\omega) = K\sqrt{\frac{\omega}{\kappa}}\sqrt{\frac{\mathrm{ker}_1^2\sqrt{\frac{\omega}{\kappa}}x + \mathrm{kei}_1^2\sqrt{\frac{\omega}{\kappa}}x}{\mathrm{ker}^2\sqrt{\frac{\omega}{\kappa}}x_0 + \mathrm{kei}^2\sqrt{\frac{\omega}{\kappa}}x_0}}. \tag{92}$$

(We omit the lower index notation for $v = 0$).

I.3.5. The Infinite Circular Cylinder

By (66) we have

$$H(x,x_0,i\omega) = K\sqrt{\frac{i\omega}{\kappa}}\frac{I_1\left(e^{i\frac{\pi}{4}}\sqrt{\frac{\omega}{\kappa}}x\right)}{I_0\left(e^{i\frac{\pi}{4}}\sqrt{\frac{\omega}{\kappa}}x_0\right)}, \tag{93}$$

since

$$\mathrm{ber}_v z + i\;\mathrm{bei}_v z = e^{-\frac{1}{2}v\pi i}I_v\left(ze^{i\frac{\pi}{4}}\right), \tag{94}$$

(see [9]), consequently

$$A(x, x_0, \omega) = K\sqrt{\frac{\omega}{\kappa}}\sqrt{\frac{\operatorname{ber}_1^2\sqrt{\frac{\omega}{\kappa}}x + \operatorname{bei}_1^2\sqrt{\frac{\omega}{\kappa}}x}{\operatorname{ber}^2\sqrt{\frac{\omega}{\kappa}}x_0 + \operatorname{bei}^2\sqrt{\frac{\omega}{\kappa}}x_0}}, \tag{95}$$

(where the notation is omitted for $v = 0$).

The formulas (92), (95) are useful for numerical calculations, since the values of the square sums can be found in the table KIRK-YOUNG [13].

I.4. The Solution of Problem II

Let

$$\Theta(x, s) = \alpha(s)\Theta_1(x, s) + \beta(s)\Theta_2(x, s). \tag{96}$$

If the temperature is known in the points x_1, x_2, then

$$\Theta(x_1, s) = \alpha(s)\Theta_1(x_1, s) + \beta(s)\Theta_2(x_1, s), \tag{97}$$

$$\Theta(x_2, s) = \alpha(s)\Theta_1(x_2, s) + \beta(s)\Theta_2(x_2, s) \tag{98}$$

forms an equation system for the unknowns $\alpha(s)$, $\beta(s)$. On the other hand, we get from (96)

$$J(x, s) = -K\alpha(s)\Theta_1'(x, s) - K\beta(s)\Theta_2'(x, s). \tag{99}$$

Determining the operators $\alpha(s)$, $\beta(s)$ from (3.4) and substituting their values to (99) we get:

$$J(x, s) = H_1(x, x_1, x_2, s)\Theta(x_1, s) + H_2(x, x_1, x_2, s)\Theta(x_2, s), \tag{100}$$

$$H_1(x, x_1, x_2, s) = -K\frac{\Theta_2(x_2, s)\Theta_2'(x, s) - \Theta_1(x_2, s)\Theta_2'(x, s)}{\Theta_1(x_1, s)\Theta_2(x_2, s) - \Theta_1(x_2, s)\Theta_2(x_1, s)}, \tag{101}$$

$$H_2(x, x_1, x_2, s) = -K\frac{\Theta_1(x_1, s)\Theta_2'(x, s) - \Theta_2(x_1, s)\Theta_1'(x, s)}{\Theta_1(x_1, s)\Theta_2(x_2, s) - \Theta_1(x_2, s)\Theta_2(x_1, s)}. \tag{102}$$

(100) describes a transmission system represented in the scheme *Fig.1*.

If $H_1(x, x_1, x_2, s)$, $H_2(x, x_1, x_2, s)$ have Laplace inverses, then by inverting both sides of (102), we obtain that the heat flux can be represented as the sum of two convolution integrals.

The operators H_1, H_2 are called the pure transfer functions of the system. In the following we show the application of the theory to the case of a finite rod (or an infinite wall of finite thickness). We shall see that in special limit cases, the flux can be expressed explicitly by the temperatures, but not by the sum of two convolutions.

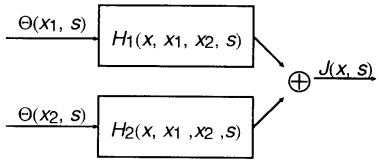

Fig. 4. Transmission system model of the heat flux. The pure problem.

I.4.1. The Case of the Finite Rod

Assume $x_2 > x_1$. Then it holds the following.

Statement 1: The heat flux can be written as the sum of two convolutions if and only if $x_1 < x < x_2$.

If $x = x_1, x = x_2$, then the heat flux can be expressed explicitly by the temperatures, provided that the points x_1, x_2 are inner points of the domain.

In our case

$$\Theta(x_1, s) = e^{\sqrt{\frac{s}{\kappa}}x}, \qquad \Theta(x_2, s) = e^{-\sqrt{\frac{s}{\kappa}}x}. \tag{103}$$

By determining the expressions (101), (102) we have

$$H_1(x, x_1, x_2, s) = K\sqrt{\frac{s}{\kappa}}\frac{e^{-\sqrt{\frac{s}{\kappa}}(2x_2-x_1-x)} + e^{-\sqrt{\frac{s}{\kappa}}(x-x_1)}}{1 - e^{-2\sqrt{\frac{s}{\kappa}}(x-x_1)}}, \tag{104}$$

$$H_2(x, x_1, x_2, s) = -K\sqrt{\frac{s}{\kappa}}\frac{e^{-\sqrt{\frac{s}{\kappa}}(x+x_2-2x_1)} + e^{-\sqrt{\frac{s}{\kappa}}(x_2-x_1)}}{1 - e^{-2\sqrt{\frac{s}{\kappa}}(x_2-x_1)}}. \tag{105}$$

If $x \leq x_1$, or $x \geq x_2$, then (104), (105) cannot be inverted simultaneously. Obviously let $x \leq x_1$, then (104) does not tend to zero for $s \to \infty$. Let $x \geq x_2$ then (105) does not tend to zero for $s \to \infty$.

By expanding the expression

$$\frac{1}{1 - e^{-2\sqrt{\frac{s}{\kappa}}(x_2-x_1)}}$$

in a geometric series we have

$$H_1(x, x_1, x_2, s) = K\sqrt{\frac{s}{\kappa}}\left[\sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[2(v+1)x_2-(2v+1)x_1-x]} + \sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[2vx_2-(2v+1)x_1-x]}\right], \tag{106}$$

$$H_2(x, x_1, x_2, s) = -K\sqrt{\frac{s}{\kappa}}\left[\sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[2(v+1)x_2-(2v+1)x_1-x]} + \sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[(2v+1)x_2-2vx_1-x]}\right]. \tag{107}$$

If $x_1 < x < x_2$, then the arguments of the exponential functions are negative and the infinite series can be inverted term by term. (The validity of this procedure can be easily seen by a theorem of MIKUSINSKI's operational calculus [14].)

Denoting the inverse of (106) by $h_1(x, x_1, x_2, t)$, and the inverse of (107) by $h_2(x, x_1, x_2, t)$, we get

$$\begin{aligned} h_1(x, x_1, x_2, t) = \frac{K}{2t\sqrt{\pi\kappa t}}\Bigg\{&\sum_{v=0}^{\infty}\left(\frac{[2(v+1)x_2-(2v+1)x_1-x]^2}{2\kappa t}-1\right) \\ &\times\exp\left[-\frac{[2(v+1)x_2-(2v+1)x_1-x]^2}{4\kappa t}\right] \\ &+\sum_{v=0}^{\infty}\left(\frac{[2vx_2-(2v+1)x_1-x]^2}{2\kappa t}-1\right) \\ &\times\exp\left[-\frac{[2vx_2-(2v+1)x_1-x]^2}{4\kappa t}\right]\Bigg\}, \end{aligned} \tag{108}$$

$$h_1(x, x_1, x_2, t) = -h_2(x, x_1, x_2, t). \tag{109}$$

We have for heat flux

$$j(x, t) = \vartheta(x_1, t)^{*} h_1(x, x_1, x_2, t) + \vartheta(x_2, t)^{*} h_2(x, x_1, x_2, t), \tag{110}$$

where the convolution is denoted by *.

Let $x = x_1$, and x_1 be an inner point of I. Then we can write:

$$H_1(x, x_1, x_2, s) = K\sqrt{\frac{s}{\kappa}} + 2K\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[2v(x_2-x_1)]}, \tag{111}$$

$$H_2(x, x_1, x_2, s) = -2K\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty} e^{-\sqrt{\frac{s}{\kappa}}[(2v+1)(x_2-x_1)]}. \tag{112}$$

Let us denote the inverse of $H_1 - K\sqrt{\frac{s}{\kappa}}$ by $\rho_1(t)$, the inverse of H_2 by $\rho_2(t)$. By taking the inversion procedure we get

$$\begin{aligned}
\rho_1(t) &= \frac{K}{t\sqrt{\pi\kappa t}}\sum_{v=1}^{\infty}\left(\frac{2v^2(x_2-x_1)^2}{\kappa t}-1\right)\exp\left[-\frac{v^2(x_2-x_1)^2}{\kappa t}\right],\\
\rho_2(t) &= -\frac{K}{t\sqrt{\pi\kappa t}}\sum_{v=1}^{\infty}\left(\frac{(2v+1)^2(x_2-x_1)^2}{2\kappa t}-1\right)\\
&\quad\times\exp\left[-\frac{(2v+1)^2(x_2-x_1)^2}{4\kappa t}\right].
\end{aligned}\tag{113}$$

Taking into account (110), it can be written that

$$\begin{aligned}
J(x_1,s) &= K\sqrt{\frac{s}{\kappa}}\Theta(x_1,s)+2K\Theta(x_1,s)\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty}e^{-\sqrt{\frac{s}{\kappa}}[2v(x_2-x_1)]}\\
&\quad-2K\Theta(x_2,s)\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty}e^{-\sqrt{\frac{s}{\kappa}}[(2v+1)(x_2-x1)]}.
\end{aligned}\tag{114}$$

By inverting both sides of (114) and taking into account (23), (26) we get the heat flux as

$$\begin{aligned}
j(x_1,t) &= P\frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_1,t)}{\sqrt{t}}+\frac{1}{2}\int_0^t\frac{\vartheta(x_1,t)-\vartheta(x_1,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right]\\
&\quad+\int_0^t\rho_1(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau\\
&\quad+\int_0^t\rho_2(\tau)\vartheta(x_2,t-\tau)\,\mathrm{d}\tau.
\end{aligned}\tag{115}$$

In the other form

$$\begin{aligned}
j(x_1,t) &= \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x_1,t)}{\partial t^{\frac{1}{2}}}+\int_0^t\rho_1(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau\\
&\quad+\int_0^t\rho_2(\tau)\vartheta(x_2,t-\tau)\,\mathrm{d}\tau.
\end{aligned}\tag{116}$$

Let $x = x_2$, then by omitting the details we have

$$\begin{aligned} j(x_2,t) &= -\frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_2,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_2,t)-\vartheta(x_2,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right] \\ &\quad - \int_0^t \rho_1(\tau)\vartheta(x_2,t-\tau)\,\mathrm{d}\tau \\ &\quad - \int_0^t \rho_2(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{117}$$

In other form

$$\begin{aligned} j(x_2,t) &= -\frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x_2,t)}{\partial t^{\frac{1}{2}}} - \int_0^t \rho_1(\tau)\vartheta(x_2,t-\tau)\,\mathrm{d}\tau \\ &\quad - \int_0^t \rho_2(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau. \end{aligned} \tag{118}$$

So the statement has been proved.

I.4.2. The Case of a Region Bounded by Two Concentric Infinite Circular Cylinders

Then

$$\Theta_1(x,s) = I_0\left(\sqrt{\frac{s}{\kappa}}x\right), \qquad \Theta_2(x,s) = K_0\left(\sqrt{\frac{s}{\kappa}}x\right). \tag{119}$$

From the theory it follows that

$$\begin{aligned} H_1(x,x_1,x_2,s) &= K\sqrt{\frac{s}{\kappa}} \\ &\quad\times\frac{K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)I_1\left(\sqrt{\frac{s}{\kappa}}x\right) + I_0\left(\sqrt{\frac{s}{\kappa}}x_2\right)K_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_0\left(\sqrt{\frac{s}{\kappa}}x_2\right) - I_0\left(\sqrt{\frac{s}{\kappa}}x_2\right)K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)}, \end{aligned} \tag{120}$$

$$\begin{aligned} H_2(x,x_1,x_2,s) &= K\sqrt{\frac{s}{\kappa}} \\ &\quad\times\frac{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_1\left(\sqrt{\frac{s}{\kappa}}x\right) + K_0\left(\sqrt{\frac{s}{\kappa}}x_2\right)I_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_0\left(\sqrt{\frac{s}{\kappa}}x_2\right) - I_0\left(\sqrt{\frac{s}{\kappa}}x_2\right)K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)}. \end{aligned} \tag{121}$$

Therefore the heat flux can be represented by the sum of two convolutions if and only if $x_1 < x < x_2$.

Obviously, if the condition is not satisfied, then from the asymptotic expansions of the Bessel functions

$$I_v(z) \sim \frac{e^z}{\sqrt{2\pi z}}, \qquad K_v(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \qquad z \to \infty,$$

it follows that, apart of constant factors, the asymptotic representations of H_1, H_2 are equal to (104) and (105), respectively.

If the condition holds, then H_1, H_2 have Laplace inverses, since it can be easily shown that the properties 1.2.3 are valid for H_1, H_2. Moreover H_1, H_2 are even and single-valued function of $\sqrt{\frac{s}{\kappa}}$, so they can be inverted by Heaviside's Expansion Theorem.

The following results are obtained for $t > 0$:

$$h_1 = 2K\kappa \sum_{n=1}^{\infty} \eta_n^2 \tag{122}$$

$$\times \frac{[J_0(\eta_n x_2)Y_1(\eta_n x) - J_1(\eta_n x)Y_0(\eta_n x_2)]e^{-\kappa\eta_n^2 t}}{x_1[J_0(\eta_n x_2)Y_1(\eta_n x_1) - J_1(\eta_n x_1)Y_0(\eta_n x_2)] + x_2[J_1(\eta_n x_2)Y_0(\eta_n x_1) - J_0(\eta_n x_1)Y_1(\eta_n x_2)]}$$

and

$$h_2 = -2K\kappa \sum_{n=1}^{\infty} \eta_n^2 \tag{123}$$

$$\times \frac{[J_0(\eta_n x_1)Y_1(\eta_n x) - J_1(\eta_n x)Y_0(\eta_n x_1)]e^{-\kappa\eta_n^2 t}}{x_1[J_0(\eta_n x_2)Y_1(\eta_n x_1) - J_1(\eta_n x_1)Y_0(\eta_n x_2)] + x_2[J_1(\eta_n x_2)Y_0(\eta_n x_1) - J_0(\eta_n x_1)Y_1(\eta_n x_2)]}$$

and

$$\lim_{t\to 0} h_1 = \lim_{t\to 0} h_2 = 0. \tag{124}$$

The number η_n are the roots of the equation

$$J_0(\eta x_1)Y_0(\eta x_1) - J_0(\eta x_2)Y_0(\eta x_1) = 0 \qquad (n = 1, 2, \ldots). \tag{125}$$

I.5. The Solution of Problem III

The temperature and the heat flux are known in the points x_1 and x_2, respectively.

We have the equation system

$$\Theta(x_1, s) = \alpha(s)\Theta_1(x_1, s) + \beta(s)\Theta_2(x_1, s), \tag{126}$$

$$J(x_2, s) = -K\alpha(s)\Theta_1'(x_2, s) - K\beta(s)\Theta_2'(x_2, s). \tag{127}$$

By substituting the solution of (126), (127) to (101) we get

$$J(x,s) = V_1(x,x_1,x_2,s)\Theta(x_1,s) + V_2(x,x_1,x_2,s)J(x_2,s). \tag{128}$$

$$V_1 = \frac{K\Theta_1'(x_2)\Theta_2'(x) - K\Theta_2'(x_2)\Theta_1'(x)}{\Theta_1(x_1)\Theta_2'(x_2) - \Theta_2(x_1)\Theta_1'(x_2)}, \tag{129}$$

$$V_2 = \frac{\Theta_1(x_1)\Theta_2'(x) - \Theta_2(x_1)\Theta_1'(x)}{\Theta_1(x_1)\Theta_2'(x_2) - \Theta_2(x_1)\Theta_1'(x_2)}. \tag{130}$$

These equations describe a transmission system (see *Fig. 2*).

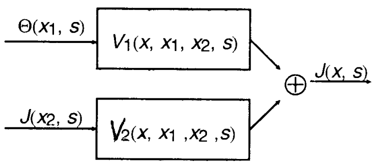

Fig. 5. Transmission system model of the heat flux. The mixed problem

If the operators V_1, V_2 have Laplace inverses, then by inverting (34), we obtain the desired representation of the heat flux. The operators V_1, V_2 are called the mixed transfer functions.

I.5.1. The Case of Finite Rod

The following statements hold.

Statement 2: The heat flux can be represented as the sum of two convolution integrals if and only if $x_1 < x < x_2$. If $x = x_1$, and x_1 is an inner point of domain, then the heat flux has an explicit representation.

Indeed, we have

$$V_1(x,x_1,x_2,s) = K\sqrt{\frac{s}{\kappa}}\,\frac{e^{-\sqrt{\frac{s}{\kappa}}(2x_2-x_1-x)} + e^{-\sqrt{\frac{s}{\kappa}}(x-x_1)}}{1+e^{-2\sqrt{\frac{s}{\kappa}}(x_2-x_1)}}, \tag{131}$$

$$V_2(x,x_1,x_2,s) = \frac{e^{-\sqrt{\frac{s}{\kappa}}(x+x_2-2x_1)} + e^{-\sqrt{\frac{s}{\kappa}}(x_2-x)}}{1+e^{-2\sqrt{\frac{s}{\kappa}}(x_2-x_1)}}. \tag{132}$$

Repeating briefly the idea of solution of the Problem II we get that (131), (132) are simultaneously invertible only in the case of $x_1 < x < x_2$. Denoting the inverses by v_1, v_2, the following expression will be obtained:

$$\begin{aligned} v_1(x, x_1, x_2, t) &= \frac{K}{2t\sqrt{\pi\kappa t}}\Bigg\{-\sum_{v=0}^{\infty}(-1)^v\left(\frac{[2(v+1)x_2-(2v+1)x_1-x]^2}{2\kappa t}\right) \\ &\quad\times \exp\left[-\frac{[2(v+1)x_2-(2v+1)x_1-x]^2}{4\kappa t}\right] \\ &\quad+\sum_{v=0}^{\infty}(-1)^v\left(\frac{[2vx_2-(2v+1)x_1-x]^2}{2\kappa t}-1\right) \\ &\quad\times \exp\left[-\frac{[2vx_2-(2v+1)x_1-x]^2}{4\kappa t}\right]\Bigg\}, \qquad (133) \\ v_2(x, x_1, x_2, t) &= \frac{K}{2t\sqrt{\pi\kappa t}}\Bigg\{\sum_{v=0}^{\infty}(-1)^v[2(v+1)x_2-(2v+1)x_1-x] \\ &\quad\times \exp\left[-\frac{[2(v+1)x_2-(2v+1)x_1-x]^2}{4\kappa t}\right] \\ &\quad+\sum_{v=0}^{\infty}(-1)^v[(2v+1)x_2-2vx_1-x] \\ &\quad\times \exp\left[-\frac{[(2v+1)x_2-2vx_1-x]^2}{4\kappa t}\right]\Bigg\}, \qquad (134) \end{aligned}$$

For $x = x_1$ we have by (131), (132)

$$\begin{aligned} V_1 &= K\sqrt{\frac{s}{\kappa}} + 2K\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty}(-1)^v e^{-2\sqrt{\frac{s}{\kappa}}(x_2-x_1)}, \\ V_2 &= 2\sum_{v=0}^{\infty}(-1)^v e^{-\sqrt{\frac{s}{\kappa}}[(2v+1)(x_2-x_1)]}, \qquad x = x_1. \qquad (135) \end{aligned}$$

Denoting the inverse of $V_1 - K\sqrt{\frac{s}{\kappa}}$ by $\xi_1(t)$, the inverse of V_2 by $\xi_2(t)$ and taking their inversions

$$\begin{aligned} \xi_1(t) &= \frac{K}{t\sqrt{\pi\kappa t}}\sum_{v=1}^{\infty}(-1)^v\left(\frac{2v^2(x_2-x_1)^2}{\kappa t}-1\right)\exp\left[-\frac{v^2(x_2-x_1)^2}{\kappa t}\right], \\ \xi_2(t) &= -\frac{x_2-x_1}{t\sqrt{\pi\kappa t}}\sum_{v=1}^{\infty}(-1)^v(2v+1)\exp\left[-\frac{(2v+1)^2(x_2-x_1)^2}{4\kappa t}\right] \qquad (136) \end{aligned}$$

are obtained. Taking into account (135) we have

$$J(x_1,s) = K\sqrt{\frac{s}{\kappa}}\Theta(x_1,s) + 2K\Theta(x_1,s)\sqrt{\frac{s}{\kappa}}\sum_{v=0}^{\infty}(-1)^v e^{-\sqrt{\frac{s}{\kappa}}[2v(x_2-x_1)]}$$
$$+2J(x_2,s)\sum_{v=0}^{\infty}(-1)^v e^{-\sqrt{\frac{s}{\kappa}}(2v+1)(x_2-x_1)}. \qquad (137)$$

Applying again a Laplace inversion, by (23), (26) (in Part I) the heat flux can be represented as

$$j(x_1,t) = \frac{K}{\sqrt{\pi\kappa}}\left[\frac{\vartheta(x_1,t)}{\sqrt{t}} + \frac{1}{2}\int_0^t \frac{\vartheta(x_1,t)-\vartheta(x_1,\tau)}{(t-\tau)^{\frac{3}{2}}}\,\mathrm{d}\tau\right]$$
$$+\int_0^t \xi_1(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau + \int_0^t \xi_2(\tau)j(x_2,t-\tau)\,\mathrm{d}\tau. \qquad (138)$$

In other form

$$j(x_1,t) = \frac{K}{\sqrt{\kappa}}\frac{\partial^{\frac{1}{2}}\vartheta(x_1,t)}{\partial t^{\frac{1}{2}}} + \int_0^t \xi_1(\tau)\vartheta(x_1,t-\tau)\,\mathrm{d}\tau + \int_0^t \xi_2(\tau)\vartheta(x_2,t-\tau)\,\mathrm{d}\tau. \qquad (139)$$

I.5.2. The Case of a Region Bounded by Two Concentric Infinite, Circular Cylinders

The mixed transfer functions can be calculated from (129), (130) and are of the form:

$$V_1 = K\sqrt{\frac{s}{\kappa}}\frac{K_1\left(\sqrt{\frac{s}{\kappa}}x\right)I_1\left(\sqrt{\frac{s}{\kappa}}x_2\right) - I_1\left(\sqrt{\frac{s}{\kappa}}x\right)K_1\left(\sqrt{\frac{s}{\kappa}}x_2\right)}{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_1\left(\sqrt{\frac{s}{\kappa}}x_2\right) + K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)I_1\left(\sqrt{\frac{s}{\kappa}}x_2\right)}, \qquad (140)$$

$$V_2 = \frac{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_1\left(\sqrt{\frac{s}{\kappa}}x\right) + K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)I_1\left(\sqrt{\frac{s}{\kappa}}x\right)}{I_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)K_1\left(\sqrt{\frac{s}{\kappa}}x_2\right) + K_0\left(\sqrt{\frac{s}{\kappa}}x_1\right)I_1\left(\sqrt{\frac{s}{\kappa}}x_2\right)}. \qquad (141)$$

The heat flux can be written as the sum of two convolutions if and only if $x_1 < x < x_2$.

This can be shown quite similarly to the corresponding case of the Problem II. The following results are obtained:

$$v_1 = 2K\kappa \sum_{n=1}^{\infty} \eta_n^2 \tag{142}$$

$$\times \frac{[Y_1(\lambda_n x_2) J_1(\lambda_n x) - Y_1(\lambda_n x) J_0(\lambda_n x_2)] e^{-\kappa \eta_n^2 t}}{x_1[Y_1(\lambda_n x_2) J_1(\lambda_n x_1) - Y_1(\lambda_n x_1) J_1(\lambda_n x_2)] + x_2[J_0(\lambda_n x_2) Y_0(\lambda_n x_1) - J_0(\lambda_n x_1) Y_0(\lambda_n x_2)]},$$

$$v_2 = -2\kappa \sum_{n=1}^{\infty} \lambda_n \times \tag{143}$$

$$\times \frac{[J_0(\lambda_n x_1) Y_1(\lambda_n x) - J_1(\lambda_n x) Y_0(\lambda_n x_1)] e^{-\kappa \eta_n^2 t}}{x_1[J_0(\lambda_n x_1) Y_1(\lambda_n x_2) - J_1(\lambda_n x_2) Y_0(\lambda_n x_1)] + x_2[J_0(\lambda_n x_2) Y_0(\lambda_n x_1) - J_0(\lambda_n x_1) Y_1(\lambda_n x_2)]}$$

and

$$\lim_{t \to 0} v_1 = \lim_{t \to 0} v_2 = 0. \tag{144}$$

$\lambda_n > 0$ are the roots of the equation

$$J_0(\lambda x_1) Y_1(\lambda x_2) - J_1(\lambda x_2) Y_0(\lambda x_1) = 0. \tag{145}$$

I.6. Alternative Solution Methods

The theory of the fractional calculus has been applied first by GARBAI [11] in the investigation of the heat flux. By using time Laplace transformation it has been shown in [11] that the connection between the temperature and heat flux can be put down by the application of a semiderivative and its inverse. In the case of a semi-infinite rod, this connection can be represented by the semidifferential conducting law

$$j(x,t) = \frac{K}{\sqrt{\kappa}} \frac{\partial^{\frac{1}{2}} \vartheta(x,t)}{\partial t^{\frac{1}{2}}}, \qquad t > 0. \tag{146}$$

The Laplace transform of the general solution of a heat conduction problem for zero initial value in an infinite wall of finite thickness (or in a finite rod) is of the form

$$\Theta(x,s) = \alpha(s) e^{-\sqrt{\frac{s}{\kappa}} x} + \beta(s) e^{\sqrt{\frac{s}{\kappa}} x}, \tag{147}$$

so

$$\Theta'(x,s) = -\sqrt{\frac{s}{\kappa}} \alpha(s) e^{-\sqrt{\frac{s}{\kappa}} x} + \sqrt{\frac{s}{\kappa}} \beta(s) e^{\sqrt{\frac{s}{\kappa}} x}, \tag{148}$$

from which the formula

$$\Theta'(x,s) - \sqrt{\frac{s}{\kappa}} \Theta(x,s) = \left[\Theta'(o,s) - \sqrt{\frac{s}{\kappa}} \Theta(o,s) \right] e^{-\sqrt{\frac{s}{\kappa}} x} \tag{149}$$

can be deduced, representing the connection between the Laplace transforms of the temperatures and heat flux, at the points $(0 - x)$, respectively.

By the definition of the heat flux, taking the inversion of (55) we have

$$j(x,t) = -\sqrt{K\rho c}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\vartheta(x,t) + \frac{1}{2}\left[j(o,t) + \sqrt{K\rho c}\frac{\partial^{\frac{1}{2}}}{\partial t^{\frac{1}{2}}}\vartheta(o,t)\right] * \frac{x}{\sqrt{\kappa\pi t^3}}e^{-\frac{x^2}{4\kappa t}} \tag{150}$$

(see [11]).

II. Transient Heat Conduction in Composite Systems

II.1. Introduction

In technical practice one often encounters transient heat conduction problems in composite systems consisting of solid layers, e.g., walls of buildings, walls of furnaces, heat insulation of pipelines, etc.

Investigating these involves solving the simultaneous system of differential equations

$$\Delta\vartheta_i = \frac{1}{\kappa_i}\frac{\partial\vartheta_i}{\partial t}, \qquad i = 1, 2, \ldots, N \tag{151}$$

under prescribed initial and boundary conditions, where:

If we assume that the temperature depends only on one space coordinate, x, in addition to time, and that the temperature of the system at the time $t = 0$ was zero, then problems of the type indicated above can be redefined mathematically in the following way:

From among the solutions of the system of heat equations

$$\Delta\vartheta_i(x,t) = \frac{l}{\kappa_i}\frac{\partial\vartheta_i}{\partial t}, \qquad l_{i-1} < x < l_i < \infty, \qquad i = 1, 2, \ldots, N, \tag{152}$$

we are to determine the one that satisfies the zero initial condition at the time $t = 0$ and the continuity conditions at the separating surfaces (or points) with co-ordinates l_i

$$\vartheta_i(l_i,t) = \vartheta_{i+1}(l_i,t),$$

$$-K_i\frac{\partial\vartheta_i(x,t)}{\partial x}\bigg|_{x=l_i} = -K_{i+1}\frac{\partial\vartheta_{i+1}(x,t)}{\partial x}\bigg|_{x=l_i}, \qquad i = 1, 2, \ldots, N-1.$$

The simplest composite systems in practice are the following:

I. Composite plane walls of N layers.
II. Composite hollow spheres of N layers.
III. Composite hollow circular cylinders of N layers.

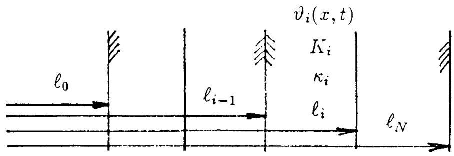

Fig. 6. The scheme of transient heat conduction in composite heat systems

The following figure indicates the domain x for each of the three cases.

We offer solutions for these three basic problems in our paper. In the first structure we choose $l_0 = 0$, but in structures II and III $l_0 = a > 0$, as we do not consider the process in the interior of the system in the domain $(0 \leq x \leq l_0)$. Temperatures will only be considered in the points l_i dividing the individual conductors, at the beginning of the system (l_0) and at its end (l_N), which are important special cases in technical practice, and whose investigation, as we are going to see, makes a clear system-theoretic approach possible.

Problems of the types shown above are discussed in the literature using the method of Laplace transformation (see [1], LIKOV [16]). However, even in the case $N = 2$, the Laplace transforms of the temperatures become complex expressions whose inversion – with the exception of some special cases – poses insurmountable difficulties. This fact was also stated by JAEGER in [3].

In this paper we are going to present the PAPOULIS-BERG inversion method [8], [17] in the system - theoretic investigation of heat conduction problems of the type shown above. The greatest advantage of this method is that it is fairly easy to apply for arbitrarily large values of N, so the number of heat conducting layers with different physical properties is not limited.

II.2. Determining the Laplace Transform of Temperatures

After applying the Laplace transformation to *Eq.* (151), considering the zero initial condition, we obtain the following transformed expression

$$\Delta\Theta_i(x,s) = q_i^2\Theta_i(x,s), \qquad q_i\sqrt{\frac{s}{\kappa_i}}, \qquad i = 1, 2, \ldots, N. \tag{153}$$

In the special structures investigated by us:

Plane walls

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2} = q_i^2\Theta_i(x,s). \tag{154}$$

Sphere

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2} + \frac{2}{x}\frac{\partial\Theta_i(x,s)}{\partial x} = q_i^2\Theta_i(x,s). \tag{155}$$

Cylinders

$$\frac{\partial^2\Theta_i(x,s)}{\partial x^2}+\frac{1}{x}\frac{\partial\Theta_i(x,s)}{\partial x}=q_i^2\Theta_i(x,s). \tag{156}$$

(See, e.g., [1].)

Let us introduce the Laplace transforms of the heat fluxes $j_i(x,t) = -K_i\frac{\partial\Theta_i(x,t)}{\partial t}$

$$J_i(x,s)=\int_0^\infty j_i(x,t)e^{-st}\,\mathrm{d}t, \qquad i=1,2,\ldots,N. \tag{157}$$

For the sake of simplicity, let us use the notations $\vartheta(l_i,t)$, $j(l_i,t)$, $\Theta(l_i)$, $J(l_i)$ for the temperatures, heat fluxes and their Laplace transforms in the points l_i. It can be demonstrated that the relationship between the transforms of the temperatures and heat fluxes occurring at the input with coordinate l_i and exit with coordinate l_{i-1} of the i-th heat conductor can be established using the so-called transfer matrix

$$\mathbf{A_i}(\mathbf{s})=\begin{pmatrix} A_i(s) & B_i(s)\\ C_i(s) & D_i(s)\end{pmatrix}$$

of the i-th layer in the following form:

$$\begin{pmatrix}\Theta(l_i)\\ J(l_i)\end{pmatrix}=\begin{pmatrix} A_i(s) & B_i(s)\\ C_i(s) & D_i(s)\end{pmatrix}\begin{pmatrix}\Theta(l_{i-1})\\ J(l_{i-1})\end{pmatrix} \qquad i=1,2,\ldots,N. \tag{158}$$

For structure I the entries of the transfer matrix can be found in [1]. For structures II and III we have computed the values, and we will get back to them later.

Let us now consider the system consisting of N heat conducting layers. Then the following matrix relationship prevails between the temperatures and heat fluxes at the input of the system and at the exit of the i-th heat conductor

$$\begin{pmatrix}\Theta(l_i)\\ J(l_i)\end{pmatrix}=\begin{pmatrix} \overline{A}_i(s) & \overline{B}_i(s)\\ \overline{C}_i(s) & \overline{D}_i(s)\end{pmatrix}\begin{pmatrix}\Theta(l_0)\\ J(l_0)\end{pmatrix}, \tag{159}$$

where

$$\mathbf{H_i}(\mathbf{s})=\begin{pmatrix} \overline{A}_i(s) & \overline{B}_i(s)\\ \overline{C}_i(s) & \overline{D}_i(s)\end{pmatrix}=\prod_{j=0}^{i-1}A_{i-j}(s) \qquad i=1,2,\ldots,N, \tag{160}$$

and $\mathbf{H_l}(\mathbf{s})=A_l(s)$, and in particular, for $i=N$

$$\begin{pmatrix}\Theta(l_N)\\ J(l_N)\end{pmatrix}=\begin{pmatrix} \overline{A}_i(s) & \overline{B}_i(s)\\ \overline{C}_i(s) & \overline{D}_i(s)\end{pmatrix}\begin{pmatrix}\Theta(l_0)\\ J(l_0)\end{pmatrix}, \tag{161}$$

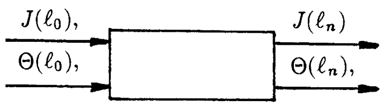

Fig. 7. Linear transmission system of transient heat conduction in composite systems

which describes the operator relationship between the input and exit of the system of heat conductors, and which can be schematically represented as a linear transmission system as follows:

It is apparent that the combined transfer matrix of the two systems is equal to the product of the two transfer matrices. Two boundary conditions must be given in order to solve the problem. One of the two pertains to the beginning (input) of the system of heat conductors, the other to the end (output). If two of the operators $\Theta(l_0)$, $\Theta(l_N)$, $J(l_0)$, $J(l_N)$, are known, then the other two can be determined from (160).

Now we are going to write the Laplace transforms of the temperature we are most interested in far the most important boundary condition occurring in technical practice.

1. Temperature is given at both ends of the system. Then

$$\Theta(l_i) = \Theta(l_0)\frac{\overline{A}_i\overline{B}_N - \overline{B}_i\overline{A}_N}{\overline{B}_N} + \Theta(l_N)\frac{\overline{B}_i}{\overline{B}_N}, \qquad i = 1, 2, \ldots, N. \quad (162)$$

2. Heat flux is given at both ends of the system. Then from (160)

$$\Theta(l_0) = -J(l_0)\frac{\overline{D}_N}{\overline{C}_N} + \frac{J(l_N)}{\overline{C}_N},$$

$$\Theta(l_i) = J(l_0)\frac{\overline{B}_i\overline{C}_N - \overline{A}_i\overline{D}_N}{\overline{C}_N} + J(l_N)\frac{\overline{A}_i}{\overline{C}_N}, \qquad i = 1, 2, \ldots, N. \quad (163)$$

3. Temperature is given at l_0, the input of the system, and the heat flux is given at l_N, the output.
   Similarly to the preceding cases,

$$\Theta(l_i) = \Theta(l_0)\frac{\overline{A}_i\overline{D}_N - \overline{B}_i\overline{C}_N}{\overline{D}_N} + J(l_N)\frac{\overline{B}_i}{\overline{D}_N}, \qquad i = 1, 2, \ldots, N. \quad (164)$$

   The case involving the roles of l_0 and l_N reversed can be written analogously.
4. Temperature is given at l_0, the input of the system, and the heat flux is proportional to the temperature at l_N, the output.

The latter boundary condition in the form of a Laplace transform is (I):

$$J(l_N) = \gamma \Theta(l_N). \tag{165}$$

Then

$$\Theta(l_i) = \Theta(l_0) \frac{\gamma(\overline{B}_i \overline{A}_N - \overline{A}_i \overline{B}_N) + \overline{A}_i \overline{D}_N - \overline{B}_i \overline{C}_N}{\overline{D}_N - \gamma \overline{B}_N}, \qquad i = 1, 2, \ldots N. \tag{166}$$

5. The heat flux is given at l_0, the input of the system, and the heat flux is proportional to the temperature at (l_N), the output.
   In this case

$$\Theta(l_0) = J(l_0) \frac{\overline{D}_N - \gamma \overline{B}_N}{\gamma \overline{A}_N - \overline{C}_N}, \tag{167}$$

$$\Theta(l_i) = J(l_0) \frac{\gamma(\overline{A}_i \overline{B}_N - \overline{B}_i \overline{A}_N) - \overline{A}_i \overline{D}_N + \overline{B}_i \overline{C}_N}{\overline{C}_N - \gamma \overline{A}_N}. \tag{168}$$

Let us now write the entries of the transition matrix $\mathbf{A_i}(\mathbf{s})$ for structures I, II and III:

I. Then, according to CARSLAW-JAEGER [1]

$$\varphi_i^{(1)} = \text{ch}\, q_i l_i, \qquad \varphi_i^{(2)} = \text{sh}\, q_i l_i,$$
$$A_i(s) = D_i(s) = \text{ch}\,(l_i - l_{i-1}) q_i,$$

$$\begin{aligned} B_i(s) &= -\frac{1}{K_i q_i}\, \text{sh}\,(l_i - l_{i-1}) q_i, \\ C_i(s) &= -K_i q_i\, \text{sh}\,(l_i - l_{i-1}) q_i, \qquad i = 1, 2, \ldots, N \end{aligned} \tag{169}$$

and

$$\det \mathbf{A_i}(\mathbf{s}) = 1$$

which implies

$$\det \mathbf{H_i}(\mathbf{s}) = 1.$$

II.

$$\varphi_i^{(1)} = \frac{\text{ch}\, q_i l_i}{l_i}, \qquad \varphi_i^{(2)} = \frac{\text{sh}\, q_i l_i}{l_i}.$$

Computing the entries of the transfer matrix, we obtain

$$A_i(s) = \frac{l_{i-1}}{l_i}\, \text{ch}\, q_i (l_i - l_{i-1}) + \frac{l}{q_i l_i}\, \text{ch}\, q_i (l_i - l_{i-1}).$$

We assume that the surrounding temperature is zero $\vartheta(x,t) = 0$, if $x < l_0$, or $x > l_N$.

$$B_i(s) = -\frac{l_{i-1}}{K_i q_i l_i}\,\mathrm{sh}\, q_i(l_i - l_{i-1}),$$

$$C_i(s) = -K_i \frac{l_i - l_{i-1}}{l_i^2}\,\mathrm{ch}\, q_i(l_i - l_{i-1}) + K_i\left(\frac{l}{q_i l_i^2} - \frac{l_{i-1}}{l_i} q_i\right)\mathrm{sh}\, q_i(l_i - l_{i-1}),$$

$$D_i(s) = \frac{l_{i-1}}{l_i}\,\mathrm{ch}\, q_i(l_i - l_{i-1}) - \frac{l_{i-1}}{q_i l_i^2}\,\mathrm{sh}\, q_i(l_i - l_{i-1}), \qquad \det \mathbf{A_i(s)} = \frac{l_{i-1}^2}{l_i^2},$$

thus

$$\det \mathbf{H_i}(\mathbf{s}) = \frac{l_0^2}{l_i^2}, \qquad i = 1, 2, \ldots, N. \tag{170}$$

III. In this case

$$\varphi_i^{(1)} = I_0(q_i l_i), \qquad \varphi_i^{(2)} = K_0(q_i l_i),$$

where I_0 and K_0 are the Bessel functions of the first and second kind, of order 0. The entries of the transfer matrix are found to be

$$\begin{aligned}
A_i(s) &= l_{i-1} q_i [I_0(q_i l_i) K_1(q_i l_{i-1}) + K_0(q_i l_i) I_1(q_i l_{i-1})],\\
B_i(s) &= \frac{l_{i-1}}{K_i}[I_0(q_i l_{i-1}) K_0(q_i l_i) - K_0(q_i l_{i-1}) I_0(q_i l_i)],\\
C_i(s) &= K_i l_{i-1} q_i^2 [K_1(q_i l_i) I_1(q_i l_{i-1}) - I_1(q_i l_i) K_1(q_i l_{i-1})],\\
D_i(s) &= l_{i-1} q_i [I_0(q_i l_{i-1}) K_1(q_i l_i) + K_0(q_i l_{i-1}) I_1(q_i l_i)],
\end{aligned} \tag{171}$$

where I_1 and K_1 are the modified Bessel functions of the first and second kind, of order one. Furthermore,

$$\det \mathbf{A_i} = \frac{l_{i-1}}{l_i}, \qquad i = 1, 2, \ldots, N,$$

consequently

$$\det \mathbf{H_i}(\mathbf{s}) = \frac{l_0}{l_i}.$$

Now the Laplace transforms of the temperatures at the points l_i can be written explicitly with the help of the formulae obtained thus far. Unfortunately, owing to the products of the function matrices in (170), the entries of the matrix $\mathbf{H_i}(\mathbf{s})$ and, consequently, the formulae generating transforms of the temperatures are so complex even for $N > 2$ that inverting them with classical methods is practically impossible.

The following PAPOULIS–BERG inversion method fixes this problem. Next, we are going to explain the method briefly, then we are going to invert the Laplace transforms of the temperatures using the method.

II.3. Applying the Papoulis–Berg Inversion Method to Solving the Heat Conduction Problem

Let $f(t)$ be a continuous function of bounded variation defined for $t \geq 0$ and Laplace transformable, $f(0) = 0$ and let

$$F(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t, \qquad s > s_0. \tag{172}$$

PAPOULIS [17] obtains the inverse of $F(s)$ without applying the Fourier–Mellin inversion integral in the following way.

Let $\sigma > s_0$ be an arbitrary positive number and let us substitute

$$e^{-\sigma t}\cos(x) \tag{173}$$

into (172). Then by denoting $f(t) = g(x)$, we obtain

$$F(s) = \frac{1}{\sigma}\int_0^{\pi/2} g(x)\cos^{\frac{s}{\sigma}-1} x \cdot \sin x\,\mathrm{d}x. \tag{174}$$

Let

$$s = (2v+1)\sigma, \qquad v = 1, 2, 3, \ldots,$$

then

$$F[(2v+1)\sigma] = \frac{1}{\sigma}\int_0^{\pi/2} g(x)\cos^{2v} x \cdot \sin x\,\mathrm{d}x. \tag{175}$$

Let us now define the function $g(x)$ to the domain $\frac{\pi}{2} < x \leq \pi$ by way of the formula $g(x) = g(\pi - x)$. From the theory of Fourier series it is known that the function $f(t)$ can be expanded into a Fourier series of the form

$$f(t) = \sum_{n=0}^\infty c_n \sin(2n+1)x = \sum_{n=0}^\infty c_n \sin[(2n+1)\arccos(e^{-\sigma t})], \tag{176}$$

$$c_n = \frac{4}{\pi}\int_0^{\pi/2} g(x)\sin(2n+1)x\,\mathrm{d}x. \tag{177}$$

The following formula is easily verified by mathematical induction:

$$\sin(2n+1)x = \sin x \sum_{v=0}^\infty (-1)^{n-v} 4^v \binom{n+v}{n-v}\cos^{2v} x. \tag{178}$$

Substituting this into (178), taking (176) into account, we obtain

$$c_n = \frac{\sigma}{\pi}\sum_{v=0}^n (-1)^{n-v} 4^{v+1}\binom{n+v}{n-v} F[\sigma(2v+1)]. \tag{179}$$

Thereby the inverse Laplace transform of $F(s)$ is obtained. Indeed, the coefficients c_n are easy to compute from (179) once $F(s)$ is known, so $f(t)$ can be determined from (176) for an arbitrary t. For fixed values of t, the speed of convergence of (176) depends on the choice of σ, of course.

Considering that BERG [8] has suggested that in order to accelerate the convergence of (176), one should not use a constant value of σ, but rather – given the asymptotic relationship between $f(t)$ and $F(s)$ – the product σt should be chosen to be constant (i.e., for small values of t, σ should be large and vice versa). From (173) it is apparent that the value of x is constant this way. Choosing this constant to be the midpoint of the basic interval, $\left(0, \frac{\pi}{2}\right)$, $x = \frac{\pi}{4}$ the following formulae are obtained:

$$\sigma = \frac{\log 2}{2t}, \qquad \sin(2n+1)\frac{\pi}{4} = \frac{(-1)^{\left[\frac{n}{2}\right]}}{\sqrt{2}}$$

and

$$f(t) = \frac{\sqrt{2}\log 2}{\pi t}\sum_{n=0}^{\infty}(-1)^{n+\left[\frac{n}{2}\right]} \times \sum_{v=0}^{n}(-1)^v 4^v \binom{n+v}{n-v} F\left[\frac{\log 2}{2t}(2v+1)\right]. \tag{180}$$

Here $\left[\frac{n}{2}\right]$ is the least integer of n and $t > 0$. (See BERG [8].)

It is apparent that the time t takes the role of the complex variable s in the Laplace transform $F(s)$ in the form (180) of the inverse transform $f(t)$, meaning that it is to be evaluated numerically for arbitrary fixed t.

Let us now apply formula (180) for inverting the Laplace transforms of the temperatures determined in the preceding section. For the sake of better understanding, let us introduce the following notations:

$$\begin{gathered}
\Theta(l_N, s) = \Theta(l_N), \Theta(l_0, s) = \Theta(l_0), J(l_N, s) = J(l_N), J(l_0, s) = J(l_0), \\
A_i^{(v)} = A_i^{(v)}(t) = \overline{A}_i\left[\frac{\log 2}{2t}(1+2v)\right], \quad B_i^{(v)} = B_i^{(v)}(t) = \overline{B}_i\left[\frac{\log 2}{2t}(1+2v)\right], \\
C_i^{(v)} = C_i^{(v)}(t) = \overline{C}_i\left[\frac{\log 2}{2t}(1+2v)\right], \quad D_i^{(v)} = D_i^{(v)}(t) = \overline{D}_i\left[\frac{\log 2}{2t}(1+2v)\right].
\end{gathered} \tag{181}$$

Then, on the basis of (180), we obtain the following formulae f or the individual inverses.

The inverse of (162):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v} \times\left\{\Theta\left[l_0,\frac{\log 2}{2t}(2v+1)\right]\cdot\frac{A_i^{(v)}B_N^{(v)}-B_i^{(v)}A_N^{(v)}}{B_N^{(v)}} +\Theta\left[l_N,\frac{\log 2}{2t}(2v+1)\right]\frac{B_i^{(v)}}{B_N^{(v)}}\right\}. \quad (182)$$

The inverse of (163):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v} \times\left\{J\left[l_0,\frac{\log 2}{2t}(2v+1)\right]\cdot\frac{B_i^{(v)}C_N^{(v)}-A_i^{(v)}D_N^{(v)}}{C_N^{(v)}} +J\left[l_N,\frac{\log 2}{2t}(2v+1)\right]\frac{A_i^{(v)}}{C_N^{(v)}}\right\}. \quad (183)$$

The inverse of (164):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v} \times\left\{\Theta\left[l_0,\frac{\log 2}{2t}(2v+1)\right]\cdot\frac{A_i^{(v)}D_N^{(v)}-B_i^{(v)}C_N^{(v)}}{D_N^{(v)}} +J\left[l_N,\frac{\log 2}{2t}(2v+1)\right]\frac{B_i^{(v)}}{D_N^{(v)}}\right\}. \quad (184)$$

The inverse of (170):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v}\Theta\left[l_0,\frac{\log 2}{2t}(2v+1)\right] \times\frac{\gamma\left(B_i^{(v)}A_n^{(v)}-A_i^{(v)}B_N^{(v)}\right)+A_i^{(v)}D_N^{(v)}-B_i^{(v)}C_N^{(v)}}{D_N^{(v)}-\gamma B_N^{(v)}}. \quad (185)$$

The inverse of (168):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v} \times J\left[l_0,\frac{\log 2}{2t}(2v+1)\right]\times\frac{\gamma B_N^{(v)}-D_N^{(v)}}{C_N^{(v)}-\gamma A_N^{(v)}}. \qquad (186)$$

The inverse of (168):

$$\vartheta(l_i,t) = \frac{\sqrt{2}\log 2}{t}\sum_{n=0}^{\infty}(-1)^{n+[\frac{n}{2}]}\sum_{v=0}^{n}(-1)^v 4^v\binom{n+v}{n-v} J\left[l_0,\frac{\log 2}{2t}(2v+1)\right] \times\frac{\gamma\left(A_i^{(v)}B_n^{(v)}-B_i^{(v)}A_N^{(v)}\right)-A_i^{(v)}D_N^{(v)}-B_i^{(v)}C_N^{(v)}}{C_N^{(v)}-\gamma A_N^{(v)}}. \qquad (187)$$

From formulae (158), (160) and (181) it is apparent that

$$\begin{pmatrix} A_i^{(v)}(t) & B_i^{(v)}(t) \\ C_i^{(v)}(t) & D_i^{(v)}(t)\end{pmatrix} = \prod_{j=0}^{i-1}\begin{pmatrix} A_{i-j}\left[\frac{\log 2}{2t}(1+2v)\right] & B_{i-j}\left[\frac{\log 2}{2t}(1+2v)\right] \\ C_{i-j}\left[\frac{\log 2}{2t}(1+2v)\right] & D_{i-j}\left[\frac{\log 2}{2t}(1+2v)\right]\end{pmatrix}, \qquad (188)$$

where the quantities A_{i-j}, B_{i-j}, C_{i-j}, D_{i-j} in the matrix product on the right-hand side of the equation can be obtained from formulae (170), (171) or (172), depending on the geometric structure being investigated, putting the expression $\frac{\log 2}{2t}(1+2v)$ in the place of s.

Thus, if we wish to compute the temperatures (182) . . . (187) for a fixed time, $t > 0$, then we must substitute the numerical values of the time t into (188), whereby the multiplication of function matrices is reduced to the multiplication of numeric matrices.

This is the main advantage of the PAPOULIS–BERG inversion, which is the consequence of the fact that the time t replaces the complex variable s in the Laplace transform of formula (180).

Conclusions

The system theoretical treatment given in this paper presents a new approach of the heat flux problem and the results can be well applied in the engineering practice.

In those simpler cases when the heat flux can be given explicitly lay convolutional or other type of integrals, these integrals may be computed by the application of well-known numerical techniques. On the other hand, in the cases when the determination of the heat flux is reduced to the solution of convolutional integral equations, simple approximate methods are available in the mathematical literature.

Symbols

j – heat flux
J – Laplace transformed form of the heat flux (j)
K – thermal conductivity
s – complex variable
t – time
x – space variable
Δ – Laplace operator
ϑ – temperature
κ – thermal diffusivity
τ – time

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