PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 43, NO. 1, PP. 55–62 (1999)

DYNAMICAL SYSTEMS IN MATERIAL INSTABILITY OF CRACKED MEDIUM[1]

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Received: Feb. 1, 1999

Abstract

The paper deals with rate-dependent and rate-independent smeared crack models by considering them as a dynamical system. In such approach one of the most important mode of material instability, the strain localization can be studied as a static bifurcation of a steady-state solution. Then the uniqueness of the solution is lost at the loss of (Liapunov) stability. In terms of the theory of dynamical system for a rate-independent crack model this phenomenon happens in an ungeneric structurally unstable way. In the paper we show that by adding a rate-dependent term the system of the basic equations regains structural stability.

Keywords: smeared cracks, dynamical systems.

1. Introduction

The numerical studies on the strain localization of smeared crack models show that in rate-independent cases the results are essentially dependent on the finite element discretization [12], [13]. This deficiency can be corrected by using several methods. One of them is to add rate-dependent terms [8]. By considering a continuous material as a dynamical system [14] material instability is closely related to the Liapunov stability [11] of some state of the material. There are two basic possibilities for a dynamical system of the loss of stability, the dynamic and static bifurcations. The first one means the onset of a self-sustained oscillation, the second means a change in the number of solutions. Rate-independence for continua causes nongeneric dynamic behavior [2], [3]. This means that the loss of stability of a rate-independent continuum cannot be classified into the generic types.

The paper aims to treat smeared crack models as dynamical systems and study what happens in the case when the numerical investigation finds mesh dependence. In the first part the basic equations of the cracked medium are derived by using both rate-independent and rate-dependent smeared crack models. The next part shows how the dynamical system concept can be introduced for cracked media. Then the nature of the nongeneric behavior is studied in the rate-independent case at strain localization and the effect of rate-dependence is presented. At last a uniaxial

[1] This work was supported by the Hungarian Development and Research Fund (OTKA F0175331).

problem is studied to demonstrate how the method of part three works. We obtain a mathematical interpretation of an internal material length, too.

2. The Basic Equations of a Cracked Medium

At first the basic equations of the cracked medium should be obtained. In the case of small strain the kinematic equation is

$$\epsilon = \frac{1}{2}\left(\mathbf{u} \circ \nabla + \nabla \circ \mathbf{u}\right), \tag{1}$$

where ϵ is the strain tensor, $\mathbf{u}$ is the displacement vector and $\circ$ denotes diadic product. The equation of motion without body forces is

$$\rho \ddot{\mathbf{u}} = \mathbf{T}\nabla, \tag{2}$$

where ρ is the density and $\mathbf{T}$ denotes the symmetric Cauchy stress tensor. In the smeared crack models the strain rate $\dot{\epsilon}$ can be divided into an elastic and crack strain rate part

$$\dot{\epsilon} = \dot{\epsilon}_e + \dot{\epsilon}_{cr}. \tag{3}$$

By using matrix $\mathbf{D}_e$ of the elastic moduli and (3) the stress-strain relation is

$$\dot{\sigma} = \mathbf{D}_e\left(\dot{\epsilon} - \dot{\epsilon}_{cr}\right). \tag{4}$$

Let the crack model be a rate-independent one

$$\dot{\sigma} = \mathbf{D}_{cr}\dot{\epsilon}_{cr}. \tag{5}$$

Now like in [13] the constitutive equation can easily be obtained. From (4) and (5)

$$\dot{\epsilon}_{cr} = \left(\mathbf{D}_{cr} + \mathbf{D}_e\right)^{-1}\mathbf{D}_e\dot{\epsilon},$$

then by substituting into (5) the constitutive equation is

$$\dot{\sigma} = \mathbf{D}_{cr}\left(\mathbf{D}_{cr} + \mathbf{D}_e\right)^{-1}\mathbf{D}_e\dot{\epsilon},$$

or

$$\dot{\sigma} = \mathbf{D}\dot{\epsilon}. \tag{6}$$

The basic equations of the cracked medium are (1), (2) and (6). Let a Cartesian coordinate system with basis vectors $\mathbf{g}_j$, $(j = 1, 2, 3)$ be introduced,

$$\mathbf{u} = u_j\mathbf{g}_j.$$

By using the rate form of (1)

$$2\dot{\epsilon}_{ij} = v_{i,j} + v_{j,i}, \tag{7}$$

where v_i is the velocity, and notation

$$v_{i,j} = \frac{\partial v_i}{\partial x_j}$$

is used for the velocity gradient.

While for small strain σ can be used instead of $\mathbf{T}$, the equation of motion in rate form is

$$\rho \ddot{v}_i = \dot{\sigma}_{ij,j}. \tag{8}$$

To get a convenient form for the following part, the basic equations should be given on the velocity field. Such form can be obtained from (8) with (6) and (7)

$$\rho \dddot{v}_i = D_{ijkl} v_{k,lj}, \tag{9}$$

because $D_{ijkl} = D_{ijlk}$.

The rate-dependent smeared crack model of [12] and [13] can be obtained from (5) by adding a rate-dependent term in form

$$\dot{\sigma} = \mathbf{D}_{cr} \dot{\epsilon}_{cr} + \mathbf{M} \ddot{\epsilon}_{cr}, \tag{10}$$

where $\mathbf{M}$ is the matrix of the rate-sensitivity parameters. From (4)

$$\dot{\epsilon}_{cr} = \left(\dot{\epsilon} - \mathbf{D}_e^{-1} \dot{\sigma}\right).$$

Then by substituting into (10) the constitutive equation in form

$$\dot{\sigma} = \mathbf{D}_{cr} \left(\dot{\epsilon} - \mathbf{D}_e^{-1} \dot{\sigma}\right) + \mathbf{M} \left(\ddot{\epsilon} - \mathbf{D}_e^{-1} \ddot{\sigma}\right) \tag{11}$$

is obtained. Now the basic equations of the cracked medium are (1), (2) and (11). The system of the fundamental equations should be written on the velocity field. Such form can be obtained from equations (1), (2) and (11)

$$\mathbf{M}\mathbf{D}_e^{-1} \dddot{\mathbf{v}} + \left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right) \ddot{\mathbf{v}} - \frac{1}{\rho} \mathbf{M} \left(\dot{\mathbf{v}} \circ \nabla + \nabla \circ \dot{\mathbf{v}}\right) \nabla -$$

$$\frac{1}{\rho} \mathbf{D}_{cr} \left(\mathbf{v} \circ \nabla + \nabla \circ \mathbf{v}\right) \nabla = 0. \tag{12}$$

3. Dynamical Systems

In an abstract form *Eq.* (12) reads

$$\dddot{v} = F^1 v + F^2 \dot{v} + F^3 \ddot{v}. \tag{13}$$

Here $v = (v_1, v_2, v_3)$ is a vector of the coordinates of the velocity field satisfying the boundary conditions and F^1, F^2 and F^3 are linear differential operators defined by the right-hand side of (12). *Eq.* (13) can be regarded as an infinite dimensional dynamical system.

The stability of a state of the continuum (S^0 for example) is defined by the Liapunov stability of a solution $v^0(t)$ of (13). That is, a state represented by $v^0(t)$ is stable when the perturbed velocity field $v^0(t)+\bar{v}(t)$ remains sufficiently close to the unperturbed one. Such definitions are also used in solid mechanics [5], [6], [9]. The stability investigation of the solution $v^0(t)$ starts with a transformation into a local form at that solution by substituting

$$v(t) = v^0(t) + \bar{v}(t)$$

into (13),

$$\dddot{v}^0 + \dddot{\bar{v}} = F^1\left(v^0 + \bar{v}\right) + F^2\left(\dot{v}^0 + \dot{\bar{v}}\right) + F^3\left(\ddot{v}^0 + \ddot{\bar{v}}\right). \tag{14}$$

While v^0 is a solution of (13) and F^1, F^2 are linear operators, the first terms of each part in (14) are equal, thus the equation of motion (14) of the perturbation $\bar{v}(t)$ has the same form as (13). Then (14) should be transformed into a system of first order equations by introducing new variables

$$y_1 = \bar{v}_1, \quad \dots, \quad y_3 = \bar{v}_3, \quad y_4 = \dot{\bar{v}}_1, \quad \dots, \quad y_6 = \dot{\bar{v}}_3,$$

and vectors

$$y_\alpha, \quad (\alpha = 1, \dots, 3), \quad y_\beta, \quad (\beta = \alpha + 3), \quad y_\psi, \quad (\psi = \alpha + 6).$$

The transformed equations are

$$\dot{y}_\alpha = y_\beta, \tag{15}$$
$$\dot{y}_\beta = y_\psi, \tag{16}$$
$$\dot{y}_\psi = F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi. \tag{17}$$

Now the stability properties are determined by the eigenvalues of the linear operator $\hat{F}$ defined by the right-hand sides of (15), (16) and (17),

$$\hat{F}(y_\alpha, y_\beta, y_\psi) = (y_\beta, y_\psi, F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi).$$

By using Liapunov's indirect method [4], the solution v^0 is asymptotically stable, when the real parts of all eigenvalues of $\hat{F}$ are negative. In the case of zero real parts, the system is on the stability boundary. The characteristic equation of$\hat{F}$ reads

$$\begin{aligned} \lambda y_\alpha &= y_\beta, \\ \lambda y_\beta &= y_\psi, \\ \lambda y_\psi &= F^1 y_\alpha + F^2 y_\beta + F^3 y_\psi. \end{aligned} \tag{18}$$

By substituting the first two groups of (18) into the third equation

$$\lambda^3 y_\alpha - \lambda^2 F^3 y_\alpha - \lambda F^2 y_\alpha - F^1 y_\alpha = 0 \tag{19}$$

is obtained. The condition of stability is Re $\lambda_i \leq 0, \ \ i = 1\ldots$ for all λ_i satisfying (19).

The typical way of the loss of stability is the case when (a) a real λ_c or (b) the real part of a pair of complex conjugate λ_{c1} and $\lambda_{c2}(=\bar{\lambda}_{c1})$ changes sign, while all the others satisfy Re $\lambda_i < 0, \quad i \neq c$ or $i \neq c1, c2$, respectively. Thus the loss of stability can either be a generic static (a) or dynamic (b) bifurcation [3]. In the case (a) (19) has a (real) eigenvalue $\lambda_c = 0$. Then the condition of the static bifurcation is

$$F^1 y_\alpha = 0. \tag{20}$$

Note that this phenomenon is called in mechanics the divergence instability or the onset of strain localization [10]. In this case also the uniqueness of the solution ϑ is lost and other, nontrivial solutions can appear.

In abstract form (9) reads

$$\frac{d^2 v}{dt^2} = F v. \tag{21}$$

Then the characteristic equation of $\hat{F}$ contains only the squares of λ,

$$\lambda^2 y_\varphi = F y_\varphi.$$

Thus equations like (9) cannot give strict results for stability because the set of eigenvalues consists of pairs $\pm\sqrt{\beta}$. If $\beta > 0$, there is a positive real part, consequently the state is unstable. If $\beta < 0$, the real part of the eigenvalues is zero. Such kind of behavior is ungeneric for dynamical systems because typically it should have eigenvalues with nonzero real parts. In this sense (9) is called structurally unstable [1].

For a rate-dependent crack model by introducing new variables

$$\mathbf{y_1} = \mathbf{v}, \mathbf{y_2} = \dot{\mathbf{v}}, \mathbf{y_3} = \ddot{\mathbf{v}}$$

(12) can be transformed into a system of differential equations

$$\begin{aligned} &\dot{\mathbf{y}}_\mathbf{1} = \mathbf{y_2}, \\ &\dot{\mathbf{y}}_\mathbf{2} = \mathbf{y_3}, \\ &\dot{\mathbf{y}}_\mathbf{3} = -\mathbf{D}_e\mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right)\mathbf{y_3} + \tfrac{1}{\rho}\mathbf{M}\left(\mathbf{y_2}\circ\nabla + \nabla\circ\mathbf{y_2}\right)\nabla \\ &\qquad + \tfrac{1}{\rho}\mathbf{D}_{cr}\left(\mathbf{y_1}\circ\nabla + \nabla\circ\mathbf{y_1}\right)\nabla. \end{aligned} \tag{22}$$

After proper rearrangements the eigenvalue equation is

$$\left(\lambda^3 + \mathbf{D}_e\mathbf{M}^{-1}\left(\mathbf{I} + \mathbf{D}_{cr}\mathbf{D}_e^{-1}\right)\lambda^2\right)\mathbf{y_1} = \frac{1}{\rho}\mathbf{D}_e\left(\lambda\mathbf{I} + \right.$$

$$+\mathbf{M}^{-1}\mathbf{D}_{cr})\,(\mathbf{y_1}\circ\nabla+\nabla\circ\mathbf{y_1})\,\nabla. \tag{23}$$

Now a λ satisfying (23) can have nonzero real part, thus, the rate-dependent smeared crack model is structurally stable.

Now the condition for the (Liapunov) stability of a state of the rate-dependent cracked medium can be formulated. A state of this medium is asymptotically stable when all λ satisfying (23) have negative real parts. When there is a zero λ, the system is on the stability boundary. Then from (23) the condition of strain localization is

$$\frac{1}{\rho}\mathbf{D}_e\mathbf{M}^{-1}\mathbf{D}_{cr}\,(\mathbf{v}\circ\nabla+\nabla\circ\mathbf{v})\,\nabla=0$$

for all velocities $\mathbf{v}$ satisfying the boundary conditions.

4. One-Dimensional Case

In this part the stability of a rod of length L is studied. Then instead of the vector $\mathbf{y}_1$ a scalar v is used. Firstly, the case of the rate-independent crack model is considered. Then instead of the matrices in (4) and (5) scalar material parameters are used, the Young modulus E and $h = D_{cr\,11}$. Introducing notation $c_e^2 = \frac{E}{\rho}$ the one-dimensional form of the rate-independent constitutive equation is

$$\dot{\sigma} = c_e^2\frac{h}{E+h}\dot{\epsilon},$$

and of (8) is

$$\ddot{v} = c_e^2\frac{h}{E+h}\frac{\partial^2 v}{\partial x^2}.$$

The eigenvalue equation reads

$$\lambda^2 v = c_e^2\frac{h}{E+h}\frac{\partial^2 v}{\partial x^2}. \tag{24}$$

In the case of homogeneous boundary conditions

$$v = e^{i\alpha_k x},\ \text{where}\alpha_k = \frac{k\pi}{L}\ (k=1,\ldots),$$

should be substituted into (20) and for the eigenvalues

$$\lambda_k^2 = -\alpha_k^2 c_e^2\frac{h}{E+h} \tag{25}$$

is obtained.

In the case of stability all λ_k of (25) are imaginary numbers, thus $h > 0$. The loss of stability happens at $h = 0$. Then all the eigenvalues are zero, which means an additional degeneracy because in a typical loss of stability only one (real) eigenvalue or one pair of conjugate complex eigenvalues cross the imaginary axis.

For a rate-dependent crack model the results of the fourth part can be applied. Then *Eq.* (23) reads

$$\lambda^2 \left(\frac{E+h}{m} + \lambda \right) v = c_e^2 \left(\frac{h}{m} + \lambda \right) \frac{\partial^2 v}{\partial x^2}, \tag{26}$$

where $m = M_{11}$. In the case of homogeneous boundary conditions from (26),

$$\lambda^3 + \frac{E+h}{m}\lambda^2 + \alpha_k^2 c_e^2 \lambda + \frac{h}{m}\alpha_k^2 c_e^2 = 0. \tag{27}$$

From the Routh–Hurwitz criterion [7] concludes that when

$$\frac{E+h}{m} > 0, \quad \alpha_k^2 c_e^2 > 0, \quad \frac{h}{m} > 0$$

all the real parts of the solutions λ of (27) are negative.

The loss of stability happens at $h = 0$. Then there is a $\lambda = 0$ solution of (27), thus this is a static bifurcation or localization. The other eigenvalues are

$$\lambda_{2,3,k} = -\frac{E}{2m} \pm \sqrt{\frac{E^2}{4m^2} - \alpha_k^2 c_e^2} \qquad (k = 1, \ldots). \tag{28}$$

The real parts of all eigenvalues $\lambda_{2,3,k}$ are negative.

There is a change in the type of λ in (28) when the expression under the root changes sign. Introducing notation

$$\alpha_* = \frac{E}{2mc_e}$$

the types of the roots can be given. When $\alpha \leq \alpha_*$ the eigenvalues are real, when $\alpha > \alpha_*$ they are complex numbers. While the solution of (26) is a combination of functions

$$e^{\lambda t} e^{i\alpha x},$$

when α is large, there is an oscillatory behavior and when α is small, there is no oscillation. From α_* an internal length can be defined

$$l_* = \frac{2mc_e\pi}{E},$$

which depends only on the material parameters. The results show that there is no oscillation with length $l > l_*$.

5. Conclusion

When the cracked medium is described with a rate-independent model, the resulting dynamical system is unproper in dynamical sense. A kind of degeneracy can also be found at the mesh dependence of the numerical investigations described in literature [12], [13]. This behavior disappears by adding rate-dependence both in the numerical studies of the literature and in the present analysis based on dynamical systems theory.

In the uniaxial case also an additional degeneracy is found at the rate-independent crack model, because all eigenvalues coincide at the loss of stability. By using rate-dependent model the equations are nondegenerate. Moreover, as with [12], the dynamical systems theory also results in a kind of 'dynamic' internal length having the same value.

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