Periodica Polytechnica Mechanical Engineering, 66(3), pp. 237–243, 2022

Extending the Validity of Basic Equations for One-dimensional Flow in Tubes with Distributed Mass Sources and Varying Cross Sections

László Garbai[1*], Gábor Halász[2]

[1] Department of Building Services and Process Engineering, Faculty of Mechanical Engineering, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
[2] Department of Hydrodynamic Systems, Faculty of Mechanical Engineering, Budapest University of Technology and Economics, Műegyetem rkp. 3., H-1111 Budapest, Hungary
[*] Corresponding author, e-mail: garbai@epget.bme.hu

Received: 01 March 2022, Accepted: 09 June 2022, Published online: 29 June 2022

Abstract
Flow problems are solved using so-called fundamental equations and the corresponding initial and boundary conditions. The fundamental equations are the motion equation, the continuity equation, the energy conservation equation, and the state equations. In our paper, we extend the validity of the equation of motion used to describe one-dimensional, steady-state tubular flow to a case in which the mass flow of the medium changes along the tubular axis during the flow. Such flows occur in perforated and/or porous pipes and air ducts. The research in this direction was motivated by the fact that the extension and formulation of the equation of motion in this direction has not been carried out with completely general validity. In the equation of motion used to solve the problems, the isochoric and isotherm nature were assumed. In our paper, we present fundamental equations that formulate differential equations to describe polytrophic and expanding flows.

Keywords
fundamental flow equations, Navier-Stokes equation, energy equation, polytrophic state equation

1 Introduction

Space theory for one-dimensional flows within tubes has only been analyzed for the constant mass flow case. In this paper, we extend the basic equations to mass source spaces. This expansion is necessary to solve problems that occur during the design of ventilation and air duct systems, as well as to complete the theoretical study of such problems. The description should be applied for tubes with uniform porosity and perforation.

The basic equations consist of the equation of motion, the energy equation, the continuity equation and the state equation of the medium. We regard the flow in the tube as one-dimensional, and disregard the radial and tangential velocity components. We regard the flow to be uniform across the entire cross section of the tube. We assume that the flow is steady. We describe the motion equation as a balance of momentum flow rates. The reaction forces at the boundary surface are cancelled out by the compression forces. The enthalpy, translational kinetic energy and external heat convection appear in the energy equation. The flow is horizontal. The momentum loss or momentum gain is accounted for in the axial direction velocity component of the mass outflow. Mass outflow usually happens in the direction normal to the boundary surface. The translational kinetic energy is calculated from the absolute value of the mass outflow velocity. The geo-metric models appear in Fig. 1.

Related literature is available by Wang [1], Wang et al. [2], Garbai [3] and Czétány [4–7]. The application of the developed model can be utilized e.g. in the further development

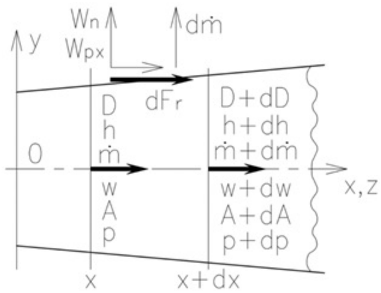

Fig. 1 Reaction force on the lateral surface

Cite this article as: Garbai, L., Halász, G. "Extending the Validity of Basic Equations for One-dimensional Flow in Tubes with Distributed Mass Sources and Varying Cross Sections", Periodica Polytechnica Mechanical Engineering, 66(3), pp. 237–243, 2022. https://doi.org/10.3311/PPme.20079

of the transpired solar collector's mathematical modeling, described by Fawaier et al. [8] Distributed mass flow source has also been described by Sánta [9] during the energy analysis of heat pump systems with different refrigerants.

2 Fundamental equations characterizing flow in tubes with mass and momentum sources

2.1 Motion equation for momentum flow rate and momentum source

The motion equation is a modified balance of momentums. The flow momentum balance is the sum of the work of normal forces, the change in momentum flow rate, the mass inflow or outflow momentum through the lateral surface of the tube, and the energy dissipation due to tube friction.

Therefore, the balance of momentum is as follows:

$$\frac{d}{dx}(pA) - p\frac{dA}{dx} + \frac{d}{dx}(\dot{m}w) = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D_e}\rho w^2, \tag{1}$$

further:

$$\begin{aligned} &A\frac{dp}{dx} + p\frac{dA}{dx} - p\frac{dA}{dx} \\ &+w\frac{d\dot{m}}{dx} + \dot{m}\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D_e}\rho w^2, \end{aligned} \tag{2}$$

$$A\frac{dp}{dx} + w\frac{d\dot{m}}{dx} + \dot{m}\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D_e}\rho w^2. \tag{3}$$

In a different form:

$$A\frac{dp}{dx} + w\frac{d\dot{m}}{dx} + A\rho w\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D}\rho w^2. \tag{4}$$

Dividing by $A\rho$:

$$\frac{1}{\rho}\frac{dp}{dx} + \frac{w}{A\rho}\frac{d\dot{m}}{dx} + w\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px}\frac{1}{A\rho} - \frac{\lambda}{2D}w^2. \tag{5}$$

Rearranging the equation:

$$w\frac{dw}{dx} = -\frac{1}{\rho}\frac{dp}{dx} - \frac{w}{A}v\frac{d\dot{m}}{dx} - \frac{1}{A\rho}\frac{d\dot{m}}{dx}w_{px} - \frac{\lambda}{2D}w^2. \tag{6}$$

Finally:

$$w\frac{dw}{dx} = -v\frac{dp}{dx} - \frac{v}{A}\frac{d\dot{m}}{dx}\left(w + w_{px}\right) - \frac{\lambda}{2D}w^2. \tag{7}$$

Special cases can be written as:

1. If
   $\dot{m}$ = constant;
   A = constant;
   $\rho \neq$ constant;
   $\dot{m} = A\rho w$.
   In this case the motion equation is Eq. (8):
   $$w\frac{dw}{dx} = -\frac{1}{\rho}\frac{dp}{dx} - \frac{\lambda}{2D}w^2, \tag{8}$$
   that's the typical form of the equation.
2. If
   $\dot{m}$ = constant;
   $A \neq$ constant;
   $\rho \neq$ constant.
   Then the motion equation is the same as in case of '1.' in the start of this listing:
   $$w\frac{dw}{dx} = -\frac{1}{\rho}\frac{dp}{dx} - \frac{\lambda}{2D}w^2, \tag{9}$$
   $$D = D_e = \frac{4 \cdot A}{P}, \tag{10}$$
   $$\frac{d\dot{m}}{dx} = 0, \tag{11}$$
   $$\dot{m} = A\rho w. \tag{12}$$
3. If
   $\dot{m}$ = constant;
   A = constant;
   ρ = constant.
   Then we again get case of '1.' in the start of this listing:
   $$w\frac{dw}{dx} = \frac{-1}{\rho}\frac{dp}{dx} - \frac{\lambda}{2D}w^2, \tag{13}$$
   $$\frac{d\dot{m}}{dx} = 0, \tag{14}$$
   $$\dot{m} = A\rho w. \tag{15}$$
4. If
   $\dot{m} \neq$ constant;
   $A \neq$ constant;
   ρ = constant.
   Then the motion equation is:
   $$A\frac{dp}{dx} + w\frac{d\dot{m}}{dx} + \dot{m}\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D}\rho w^2. \tag{16}$$

If ρ = constant:

$$A\frac{dp}{dx} + w\rho\frac{dAw}{dx} + A\rho w\frac{dw}{dx} = \frac{d\dot{m}}{dx}w_{px} - A\frac{\lambda}{2D}\rho w^2. \tag{17}$$

After the differentiating:

$$A\frac{dp}{dx}+w\rho w\frac{dA}{dx}+w\rho A\frac{dw}{dx}$$
$$+A\rho w\frac{dw}{dx}=\frac{d\dot{m}}{dx}w_{px}-A\frac{\lambda}{2D}\rho w^2. \quad (18)$$

After simplification:

$$A\frac{dp}{dx}+\rho w^2\frac{dA}{dx}+A\rho 2\cdot w\frac{dw}{dx}=\frac{d\dot{m}}{dx}w_{px}-A\frac{\lambda}{2D}\rho w^2. \quad (19)$$

Finally:

$$\frac{1}{\rho}\frac{dp}{dx}+\frac{w^2}{A}\frac{dA}{dx}+2w\frac{dw}{dx}=\frac{1}{A\rho}\frac{d\dot{m}}{dx}w_{px}-\frac{\lambda}{2D}w^2. \quad (20)$$

If $d\dot{m}/dx = Ap \times w_{py}/L$ then we obtain Wang's formula. However, we can define a function for $d\dot{m}/dx$, such as $d\dot{m}/dx = u$ = constant for uniform loss.

If:

$$\frac{d\dot{m}}{dx}=u, \quad (21)$$

then:

$$\dot{m}=\int_0^x u dx = ux+c, \quad (22)$$

$$\dot{m}=\dot{m}_0-\dot{q}x, \quad (23)$$

$$w(x)=\frac{\dot{m}_0-ux}{A(x)}. \quad (24)$$

If the momentum flow rate source = 0, because $w_{px} \approx 0$, and ρ = constant, then:

$$\frac{1}{\rho}\frac{dp}{dx}+\frac{w^2}{A}\frac{dA}{dx}+2w\frac{dw}{dx}=-\frac{\lambda}{2\cdot D}\cdot w^2. \quad (25)$$

In engineering practice, the above equation is the basic equation for sizing airducts for perforated tubes. If we want isobar flow then we can determine the change in the cross section along the tube by using this equation.

2.2 Deriving the energy equation

The energy equation for infinitesimal space in steady state is the balance of enthalpy and translational kinetic energy. The equation accounts for energy inflows and outflows at the front and rear of the space segment, as well as through the lateral surface. When the enthalpy of the inside flow and outflowing medium is the same h, the balance of the infinitesimal energy according to Fig. 1. is the following:

$$\dot{m}\left(h+\frac{w^2}{2}\right)-\left(\dot{m}+d\dot{m}\right)\left(\left(h+dh\right)+\frac{\left(w+dw\right)^2}{2}\right)$$
$$-d\dot{m}\left(h+\frac{w_n^{\ 2}}{2}\right)=0. \quad (26)$$

After multiplication:

$$\dot{m}\left(h+\frac{w^2}{2}\right)-\dot{m}\left(h+dh\right)-d\dot{m}h-\left(d\dot{m}dh\right)$$
$$-\left(\dot{m}+d\dot{m}\right)\left(\frac{w^2}{2}+\frac{2wdw}{2}+\frac{dw^2}{2}\right)-d\dot{m}h-d\dot{m}\frac{w_n^{\ 2}}{2}=0. \quad (27)$$

Further:

$$\dot{m}\left(h+\frac{w^2}{2}\right)-\dot{m}\left(h+dh\right)-d\dot{m}h-\dot{m}\frac{w^2}{2}$$
$$-\dot{m}wdw-d\dot{m}\frac{w^2}{2}-d\dot{m}wdw-d\dot{m}h-d\dot{m}\frac{w_n^{\ 2}}{2}=0. \quad (28)$$

Rearranging the equation:

$$\dot{m}h+\dot{m}\frac{w^2}{2}-\dot{m}h-\dot{m}dh-d\dot{m}h-\dot{m}\frac{w^2}{2}$$
$$-d\dot{m}\frac{w^2}{2}-\dot{m}wdw-d\dot{m}h-d\dot{m}\frac{w_n^{\ 2}}{2}=0. \quad (29)$$

After simplification:

$$\dot{m}\frac{w^2}{2}-\dot{m}dh-d\dot{m}h-\dot{m}\frac{w^2}{2}$$
$$-d\dot{m}\frac{w^2}{2}-\dot{m}wdw-d\dot{m}h-d\dot{m}\frac{w_n^{\ 2}}{2}=0. \quad (30)$$

After further simplification:

$$-\dot{m}\frac{dh}{dz}dz-\frac{d\dot{m}}{dz}dzh-\frac{d\dot{m}}{dz}dz\frac{w^2}{2}$$
$$-\dot{m}w\frac{dw}{dz}dz-\frac{d\dot{m}}{dz}dzh-\frac{d\dot{m}}{dz}dz\frac{w_n^{\ 2}}{2}=0. \quad (31)$$

Further:

$$-\dot{m}\frac{dh}{dz}-2\frac{d\dot{m}}{dz}h-\frac{d\dot{m}}{dz}\left(\frac{w^2}{2}+\frac{w_n^{\ 2}}{2}\right)-\dot{m}w\frac{dw}{dz}=0. \quad (32)$$

Furthermore:

$$\frac{dh}{dz}+w\frac{dw}{dz}=-\frac{2}{\dot{m}}\frac{d\dot{m}}{dz}h-\frac{d\dot{m}}{dz}\frac{1}{\dot{m}}\left(\frac{w^2}{2}+\frac{w_n^{\ 2}}{2}\right). \quad (33)$$

Finally:

$$\dot{m}\frac{d}{dz}\left(h+\frac{w^2}{2}\right)+\frac{d\dot{m}}{dz}\left(h+\frac{w^2}{2}\right)=-\frac{d\dot{m}}{dz}\left(\frac{w_n^{\ 2}}{2}\right)-\frac{d\dot{m}}{dz}h, \quad (34)$$

$$\frac{d}{dz}\left[\dot{m}\left(h+\frac{w^2}{2}\right)\right]=-\frac{d\dot{m}}{dz}\left(h+\frac{w_n^{\ 2}}{2}\right). \quad (35)$$

This is the typical energy equation, because usually flows exit rather than enter the air duct.

Developing the energy Eq. (35) further:

$$\frac{d}{dz}\left[\dot{m}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)\right]=-\frac{d\dot{m}}{dz}\left(h+\frac{w_n{}^2}{2}\right). \tag{36}$$

After differentiation:

$$\begin{aligned}&\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)+\dot{m}\left(\frac{\kappa}{\kappa-1}+\frac{\kappa}{\kappa-1}p\frac{dv}{dz}\right)\\&+\dot{m}\frac{dw}{dz}=-\frac{d\dot{m}}{dz}\left(h+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{37}$$

If there is heat intake or heat loss through the lateral surface, then the energy equation is Eq. (38):

$$\frac{d}{dz}\left[\dot{m}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)\right]=-\frac{d\dot{m}}{dz}\left(h+\frac{w_n{}^2}{2}\right)+\dot{q}. \tag{38}$$

3 Deriving the state equation for polytrophic flow

Our goal is to extend the validity of the pv^κ state equation from adiabatic and frictionless flow to polytrophic and frictional flow via the appropriate transformations.

Substituting motion Eq. (5) into motion Eq. (38):

$$w\frac{dw}{dz}=-v\frac{dp}{dz}-\frac{\lambda}{2D}w^2-\frac{v}{A}\frac{d\dot{m}}{dz}w-\frac{1}{A\rho}\frac{d\dot{m}}{dz}w_z. \tag{39}$$

After substituting this into the energy equation:

$$\begin{aligned}&\frac{1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)+\frac{\kappa}{\kappa-1}\left(\frac{dp}{dz}v+p\frac{dv}{dz}\right)\\&-v\frac{dp}{dz}-\frac{\lambda}{2D}w^2-\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\\&=\frac{\dot{q}}{\dot{m}}-\frac{1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{40}$$

Multiplying each element by (k-1):

$$\begin{aligned}&\frac{\kappa-1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)+\kappa\left(\frac{dp}{dz}v+p\frac{dv}{dz}\right)\\&-\left(\kappa-1\right)v\frac{dp}{dz}-\left(\kappa-1\right)\frac{\lambda}{2D}w^2\\&-\left(\kappa-1\right)\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\\&=\left(\kappa-1\right)\frac{\dot{q}}{\dot{m}}-\frac{\kappa-1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{41}$$

After expansion:

$$\begin{aligned}&\frac{\kappa-1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)+\kappa\frac{dp}{dz}v\\&+kp\frac{dv}{dz}-kv\frac{dp}{dz}+v\frac{dp}{dz}-\left(\kappa-1\right)\frac{\lambda}{2D}w^2\\&-\left(\kappa-1\right)\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\\&=\left(\kappa-1\right)\frac{\dot{q}}{\dot{m}}-\frac{\kappa-1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{42}$$

Multiplying both sides by v^{k-1}:

$$\begin{aligned}&\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right)+\kappa v^{\kappa-1}p\frac{dv}{dz}\\&+v^\kappa\frac{dp}{dz}-\left(\kappa-1\right)v^{\kappa-1}\frac{\lambda}{2D}w^2-\\&\left(\kappa-1\right)v^{\kappa-1}\frac{d\dot{m}}{dz}\frac{v}{A}\left(w+w_z\right)\\&=\left(\kappa-1\right)v^{\kappa-1}\frac{\dot{q}}{\dot{m}}-\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{43}$$

Rearranging the equation:

$$\begin{aligned}&\kappa v^{\kappa-1}p\frac{dv}{dz}+v^\kappa\frac{dp}{dz}\\&=\left(\kappa-1\right)v^{\kappa-1}\frac{\dot{q}}{\dot{m}}+\left(\kappa-1\right)v^{\kappa-1}\frac{\lambda}{2D}w^2\\&+\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right)\\&+\left(\kappa-1\right)\frac{v}{A}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(w+w_z\right)\\&+\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w^2}{2}\right).\end{aligned} \tag{44}$$

Finally:

$$\begin{aligned}&\frac{d}{dz}\left(pv^\kappa\right)=\left(\kappa-1\right)v^{\kappa-1}\left(\frac{\dot{q}}{\dot{m}}+\frac{\lambda}{2D}w^2\right)\\&+\frac{\kappa-1}{A}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(w+w_z\right)\\&+\frac{\kappa-1}{\dot{m}}\frac{v^{\kappa-1}}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right)\\&+\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}\left(\frac{\kappa}{\kappa-1}pv+\frac{w_n{}^2}{2}\right).\end{aligned} \tag{45}$$

Alternatively:

$$\begin{aligned}&\frac{d}{dz}\left(pv^\kappa\right)=\left(\kappa-1\right)v^{\kappa-1}\left(\frac{\dot{q}}{\dot{m}}+\frac{\lambda}{2D}w^2\right)\\&+2\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\frac{d\dot{m}}{dz}h+\frac{\kappa-1}{\dot{m}}v^{\kappa-1}\left(\frac{w^2}{2}+\frac{w_n{}^2}{2}\right)\\&+\frac{\kappa-1}{A}v^{\kappa-1}v\frac{d\dot{m}}{dz}\left(w+w_z\right).\end{aligned} \tag{46}$$

3.1 Polytrophic flow in porous tube with friction, other forms based on the previous equations

The following equations can be derived from Eq. (46) which are useful in engineering practice.

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\left[\begin{array}{l}\frac{\dot{q}}{\dot{m}}+\frac{\lambda}{2D}w^2+\frac{2}{\dot{m}}h\\+\frac{1}{\dot{m}}\frac{d\dot{m}}{dz}\left(\frac{w^2}{2}+\frac{{w_n}^2}{2}\right)+\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\end{array}\right], \tag{47}$$

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\frac{1}{\dot{m}}\left[\begin{array}{l}\dot{q}+\dot{m}\frac{\lambda}{2\cdot D}\cdot w^2\\+2\frac{d\dot{m}}{dz}h+\frac{d\dot{m}}{dz}\left(\frac{w^2}{2}+\frac{{w_n}^2}{2}\right)\\+\dot{m}\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\end{array}\right], \tag{48}$$

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\frac{1}{\dot{m}}\left[\begin{array}{l}\dot{q}+Aw\frac{\lambda}{2D}\rho w^2\\+2\frac{d\dot{m}}{dz}h+\frac{d\dot{m}}{dz}\left(\frac{w^2}{2}+\frac{{w_n}^2}{2}\right)\\+A\rho w\frac{v}{A}\frac{d\dot{m}}{dz}\left(w+w_z\right)\end{array}\right]. \tag{49}$$

For uniform loss:

$$w=\frac{\dot{m}_0-ux}{A\rho}. \tag{50}$$

3.2 Special cases

If the flow $\dot{m}$ = constant, and there is no source, then the state equation of the flow is Eq. (51):

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\frac{1}{\dot{m}}\left(\dot{q}+Aw\frac{\lambda}{2\cdot D}\rho w^2\right). \tag{51}$$

Alternatively:

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\left[\frac{\dot{q}}{\dot{m}}+\frac{1}{\dot{m}}\left(\frac{\lambda}{2\cdot D}\rho w^2\right)\dot{V}\right]. \tag{52}$$

If the flow is adiabatic, but frictional:

$$\frac{d}{dz}\left(pv^{\kappa}\right)=\left(\kappa-1\right)v^{\kappa-1}\frac{1}{\dot{m}}\left(\frac{\lambda}{2\cdot D}\rho w^2\right)\dot{V}. \tag{53}$$

If the flow is adiabatic, frictionless and isentropic:

$$\frac{d}{dz}\left(pv^{\kappa}\right)=0. \tag{54}$$

If the flow is politropic, frictional but isentropic:

$$\frac{d}{dz}\left(pv^{k}\right)=0,\ \text{and}\ \frac{\dot{q}}{\dot{m}}=\frac{1}{\dot{m}}\left(\frac{\lambda}{2D}\rho w^2\right)\dot{V}. \tag{55}$$

4 Isobar flow in airduct with uniform loss

Initial conditions are the following for isobar flow: $dp/dx = 0$, v = constant, $A \neq$ constant.

The velocity component of the outflow in the axial direction is zero, therefore:

$$w_{px}=0.$$

The motion equation with these parameters from Eq. (7):

$$w\frac{dw}{dx}+\frac{v}{A}\frac{d\dot{m}}{dx}w=-\lambda\frac{1}{2D}w^2. \tag{56}$$

After simplification:

$$\frac{dw}{dx}+\frac{v}{A}\frac{d\dot{m}}{dx}=-\lambda\frac{1}{2D}w. \tag{57}$$

Since $d\dot{m}/dx$ = constant = u, $\dot{m}=\dot{m}_0-ux$, and $w=(\dot{m}_0-ux)\times v/A$. Therefore:

$$\frac{dw}{dx}=\frac{d}{dx}\left(\dot{m}_0-ux\right)\frac{v}{A}=-u\frac{v}{A}-\left(\dot{m}_0-ux\right)v\frac{1}{A^2}\frac{dA}{dx}. \tag{58}$$

Substituting into Eq. (57):

$$\begin{aligned}&-u\frac{v}{A}-\left(\dot{m}_0-ux\right)v\frac{1}{A^2}\frac{dA}{dx}+u\frac{v}{A}\\&=-\lambda\frac{1}{2D}\left(\dot{m}_0-ux\right)\frac{v}{A}.\end{aligned} \tag{59}$$

After simplification:

$$\frac{-dA}{dx}=\frac{-\lambda}{2D}A. \tag{60}$$

For an air duct with a circular cross section:

$$\frac{dA}{dx}=\frac{-\lambda\sqrt{\pi}}{4}\sqrt{A}. \tag{61}$$

The differential equation can be solved by direct integration.

4.1 Proceeding from the energy equation

The state equation can be derived from the energy equation with the following assumptions:

$$\frac{dp}{dx}=0,\ \frac{dV}{dx}=0.$$

Therefore the energy equation from Eq. (35):

$$\dot{m}\frac{d}{dx}\frac{w^2}{2}=u\frac{w^2}{2}-uv\Delta p. \tag{62}$$

Since $\dot{m} = \dot{m}_0 - ux$, let $\dot{m}_0 - ux = x^*$, $dx = -dx^* 1/u$.
With these equations, Eq. (63) becomes:

$$-x^* u \frac{d}{dx^*}\frac{w^2}{2} - u\frac{w^2}{2} = -uv\Delta p. \tag{63}$$

After simplification:

$$\frac{d}{dx^*}\left(\frac{w^2}{2}\right) + \frac{1}{x^*}\left(\frac{w^2}{2}\right) = \left((uv)\Delta p\right)\frac{1}{x^*}. \tag{64}$$

The linear, inhomogeneous differential equation can be solved for ($w^2/2$).

For a known velocity w, the equation for the required cross section A is the following:

$$A(x) = \frac{x^* v}{w} = \frac{(\dot{m}_0 - ux)v}{w}. \tag{65}$$

Consider whether the results calculated from Eq. (61) and Eq. (65) are in agreement or differ.

4.2 Isobar air duct with non-constant specific volume, and non-constant *A*

Without deriving it, the Eq. (66) is the following:

$$\frac{dA}{dx}\cdot\frac{1}{A} - \frac{\lambda\cdot\sqrt{\pi}}{\sqrt{A}} = \frac{-1}{V}\cdot\frac{dV}{dx}. \tag{66}$$

The coupled energy equation:

$$\dot{m}\frac{\kappa}{\kappa-1} p \frac{dv}{dx} + \dot{m} w \frac{dw}{dx} = -u\left(\frac{w^2}{2} - \frac{w_n^2}{2}\right). \tag{67}$$

The velocity of the outflowing air:

$$w_n = \sqrt{\frac{2}{\rho}\cdot\Delta p},\ \frac{w_n^2}{2} = v\cdot\Delta p,\ \Delta p = \text{constant}.$$

Introducing a new variable, see Eq. (68):

$$\begin{aligned} &-\left(up\frac{\kappa}{\kappa-1}\right)\frac{dv}{dx^*} - u\left(\frac{x^* v}{A}\right)\frac{d}{dx^*}\left(\frac{x^* v}{A}\right) \\ &= u\frac{1}{x^*}\left(\frac{x^* v}{A}\right)^2 - \frac{v\Delta p}{x^*}, \end{aligned} \tag{68}$$

with:

$$\frac{x^* v}{A} = \dot{W},\ \dot{m} = \dot{m}_0 - ux = x^*.$$

The sizing of the airduct can be determined using Eqs. (66) and (68).

5 Summary

In this paper we extend the adiabatic state equation for politropic flow in tubes with varying cross sections and contiguous mass source by deriving the simultaneous differential equation system for the flow, the motion equation, the energy equation, the continuity equation, and the thermodynamic state equation. From these equations flow types without mass source can be derived. The politropic state equation together with the motion equation provide new computational possibilities for determining the state parameters of frictional, steady and politropic flows.

Nomenclature

A	cross section	[m^2]
p	pressure	[Pa]
x, z	space coordinates	
$\dot{m}$	mass flow	[kg/s]
h	enthalpy	[J/kg]
w, w_n	velocity, the absolute value of the outflow	[m/s]
w_{px}	the outflow's velocity component in axial direction through the lateral surface	[m/s]
D	tube diameter, equivalent tube diameter	[m]
$\dot{V}$	volume flow	[m^3/s]
ρ	density	[kg/m^3]
P	perimeter	[m]
κ	adiabatic exponent	[1]
v	kinematic viscosity	[m^2/s]

Acknowledgement

The research reported in this paper and carried out at BME has been supported by the NRDI Fund (TKP2020 IES, Grant No. BME-IE- MISC) based on the charter of bolster issued by the NRDI Office under the auspices of the Ministry for Innovation and Technology.

References

[1] Wang, J. "Theory of flow distribution in manifolds", Chemical Engineering Journal, 168(3), pp. 1331–1345, 2011. https://doi.org/10.1016/j.cej.2011.02.050

[2] Wang, J., Gao, Z., Gan, G., Wu, D. "Analytical solution of flow coefficients for a uniformly distributed porous channel", Chemical Engineering Journal, 84(1), pp. 1–6, 2001. https://doi.org/10.1016/S1385-8947(00)00263-1

[3] Garbai, L., Jasper, A., Bengery, E. "Flow in pipeline networks", Akadémiai Kiadó, 2017. ISBN: 9789630598910

[4] Czétány, L., Láng, P. "CFD Simulation of Three Different Geometrical Designs of a Conveyor Tempering Ventilation System", Energy Procedia, 78, pp. 2694–2699, 2015. https://doi.org/10.1016/j.egypro.2015.11.353

[5] Czetany, L., Szantho, Z., Lang, P. "Rectangular supply ducts with varying cross section providing uniform air distribution", Applied Thermal Engineering, 115, pp. 141–151, 2017. https://doi.org/10.1016/j.applthermaleng.2016.12.112

[6] Czetany, L., Lang, P. "Discharge Coefficients for Circular Side Outlets", Journal of Fluids Engineering, 140(7), 071205, 2018. https://doi.org/10.1115/1.4039117

[7] Czetany, L., Lang, P. "Impact of Inlet Boundary Conditions on the Fluid Distribution of Supply Duct", Applied Mechanics and Materials, 861, pp. 384–391, 2016. https://doi.org/10.4028/www.scientific.net/AMM.861.384

[8] Fawaier, M., Bokor, B., Horváth, M. "Wall heat loss recapture evaluation of transpired solar collectors for different climates: A European case study", Case Studies in Thermal Engineering, 24, 100836, 2021. https://doi.org/10.1016/j.csite.2020.100836

[9] Sánta, R. "Comparative Analysis of Heat Pump System with IHX Using R1234yf and R134a", Periodica Polytechnica Mechanical Engineering, 65(4), pp. 363–373, 2021. https://doi.org/10.3311/PPme.18390