

Numerical Analysis of the Collision of Bearing Balls with Different Integrators

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Abstract

This paper investigates the collision of different bearing balls with a massive steel cuboid in order to determine the restitution parameter experimentally. A test bench is constructed to measure the rebound distances of the different balls using slow-motion camera. The numerical analysis of the collision with energy dissipation assumes nonlinear viscoelastic property with Hertzian contact. The hysteresis damping factors are determined from the expression of the Flores model. First, second and fourth order numerical methods are compared to each other as regards the running times and accuracies for finding the most suitable technique.

Keywords

numerical methods, bearing balls, collision

1 Introduction

Numerous scientific fields use mathematical models to perform modeling and simulation analyses. However, in many cases the solution of the problems cannot be evaluated analytically. Thanks to the advances in computer technology and the available numerical methods, nowadays the numerical solution of the tasks can be achieved with adequate approximation.

Contact and impact of different objects are widely analyzed in a large number of papers [1–4]. Gorla et al. in [1] investigated numerically and experimentally a cantilever beam containing a particle damper. Pere et al. [2] mentioned the nonlinear nature of contact problems in the abstract of their paper and discussed the occurrence of spurious high-frequency oscillations in numerical solutions. Therefore, the selection of an appropriate numerical method is important in contact-impact simulations. De Albuquerque et al. [3] use impact damper as passive device to decrease unwanted vibrations in . They present a nonlinear modeling strategy and use the model of Flores et al. [4] to take into consideration the energy dissipation in the course of collision.

There are a great number of numerical methods [5], [6], which can be suitable to solve ordinary differential equations (ODEs) with no closed-form solution, or when the computation of an analytical solution is time consuming. The most well-known techniques in engineering applications include

second- or higher-order Runge–Kutta (RK) methods [7]. Explicit and implicit methods [8] can be distinguished, the latter generally requiring an iterative solution procedure, which may increase the computational cost of the simulation. The selection of an appropriate integration method is particularly important in engineering simulations involving contact and impact, where short-duration events, rapidly changing contact forces, and nonlinear responses may occur. Such requirements arise, e.g., in multibody dynamics, contact and impact analysis in finite element methods, and vibration and impact-damping applications. In these applications, the choice of the numerical integrator and time step can affect both the accuracy of the predicted response and the computational efficiency. Therefore, evaluating different integration methods for nonlinear contact problems is relevant from both numerical and practical engineering perspectives.

The main aim of this paper is to characterize the restitution behavior of steel bearing balls of different sizes and to quantitatively compare numerical integration methods for the corresponding nonlinear contact problem. While the applied Hertzian contact formulation and nonlinear viscoelastic damping model are established approaches, the present study focuses on their application to a well-defined steel ball – cuboid impact problem and on the systematic comparison of numerical integration methods. The methods

are evaluated not only in terms of numerical accuracy, but also with respect to rebound-time deviation and computational time. The objective is to identify the practical trade-off between accuracy and computational efficiency and to provide guidance for selecting an appropriate integration method for similar short-duration contact-impact simulations.

The article is organized as follows: Section 2 describes the mathematical model of the problem, which is divided to free falling and collision subsections. Furthermore, the section deals with numerical methods which are used to solve the task. Practical measurements are performed to determine the restitution of bearing balls with different sizes, which is detailed in Section 3. Section 4 contains the results of the simulations, which are related to the restitution and the collision parts. Different numerical methods for solving initial value problems are used to make comparison between the methods. The concluding remarks are placed in the last Section.

2 Definition of the mathematical model

In the following, the description of the mathematical model is detailed, which contains the free falling and contact parts. The system to be investigated consists of steel ball bearings with different diameters. The balls are released from a pre-defined height, after free falling the contact will be analyzed numerically. The results provided by different numerical methods will be compared in Section 4.

2.1 Free falling

The first stage of the process consists of free fall with air resistance taken into account. A vertical coordinate system is introduced, with the positive y-direction defined downward, i.e., in the direction of gravity. Consequently, the gravitational force is positive, while the air resistance force acts in the opposite direction to the motion. Newton's second law for the considered motion can therefore be written as:

$$m\ddot{y} = mg - F_r, \quad (1)$$

where m is the mass of the ball bearing and its acceleration is denoted with $\ddot{y}$, g is the gravitational acceleration, F_r is the force of air resistance:

$$F_r = \frac{1}{2} \rho \dot{y}^2 c_w A, \quad (2)$$

where ρ is the density of air, $c_w = 0.47$ is the drag coefficient for sphere and A is the cross-sectional area of the specimen. With the use of Eq. (1) the velocity of the balls at collision can be determined. In order to calculate the velocity, numerical methods for initial value problems (IVP) can be applied of which the second-order

Runge-Kutta and Newmark methods are detailed, see in subsection 2.3.

2.2 Investigation of the collision

Since contact takes place, the Hertz theory [9] is a good tool to determine the magnitude of penetration. With the use of [10] the penetration δ can be calculated as Eq. (3):

$$\delta = 1.2313 \sqrt[3]{\left(\frac{F^e(\delta)}{E} \right)^2 \frac{R_1 + R_2}{R_1 R_2}}. \quad (3)$$

In Eq. (3) the radius of each bearing ball is denoted by R_1 , the radius of the steel cuboid applied in the experiments is $R_2 = \infty$, therefore the δ is equal to

$$\delta = 1.2313 \sqrt[3]{\left(\frac{F^e(\delta)}{E} \right)^2 \frac{1}{R_1}}. \quad (4)$$

For better clarity, Fig. 1 illustrates a case before and after the collision, in accordance with the notations used in Eq. (3) and Eq. (4).

Expressing the force $F^e(\delta)$ for elastic collision from Eq. (4) can be written as:

$$F^e(\delta) = 0.7322E\sqrt{R_1}\delta^{3/2}. \quad (5)$$

According to [11] the analytical formulation can be given as:

$$\int_0^{\delta_{\max}} F^e(\delta) d\delta = \frac{1}{2} m v_c^2, \quad (6)$$

performing the integration $\delta_{\max}$ can be obtained:

$$\delta_{\max} = \sqrt[5/2]{\frac{1.25 m v_c^2}{0.7322 E \sqrt{R_1}}}. \quad (7)$$

Equation (7) can be used for checking the numerical solution presented in Section 4.

Voigt-Kelvin viscoelastic model [12] is widely used in order to treat energy loss in the course of contact, it consists

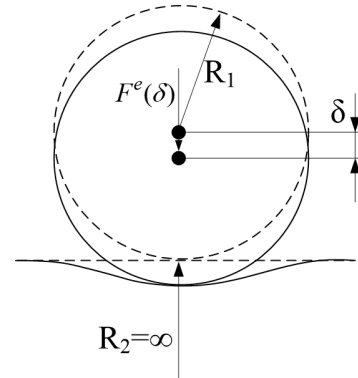


Fig. 1 A collision of a ball bearing

of linear spring and linear damper connected in parallel. In [11] the model was improved with nonlinear viscoelastic property, which utilizes the contact force calculated from Hertz theory containing elastic and dissipative parts:

$$F(\delta) = K\delta^{\frac{3}{2}} + D\delta^{\frac{3}{2}}\dot{\delta}, \quad (8)$$

where K is the generalized stiffness parameter and D is the hysteresis damping factor, which ensures the energy dissipation during contact. There are several models for dissipation in energy during collision, e.g., in [3]. The force can be written utilizing the hysteresis damping factor by Flores et al. [4] in the following:

$$F(\delta) = F^e(\delta) \left(1 + \frac{8(1-e)}{5ev_c} \dot{\delta} \right), \quad (9)$$

where e is the coefficient of restitution, and v_c is the impact velocity. The following ODE can be written for the collision with the use of the hysteresis damping factor based on [3]:

$$m\ddot{y} = -mg + F(\delta). \quad (10)$$

In order to handle the problem, the introduction of new variables is essential: $x_1 = y$, $x_2 = \dot{y}$, where y and $\dot{y}$ are the displacement and the velocity of different bearing balls, respectively. The two first order ODEs can be written as:

$$\dot{x}_1 = x_2, \quad (11)$$

$$\dot{x}_2 = -g + \frac{F(\delta)}{m}. \quad (12)$$

The initial values of the displacement and velocity depend on which ball is selected. The numerical solution of the penetration and the occurring rebound velocities can be handled by numerical methods for initial value problems in Section 4.

2.3 The applied numerical methods

In order to treat the free falling with air resistance and the collision, numerical methods will be applied. The classical fourth order Runge-Kutta method (ODE45), and second-order Runge-Kutta methods, e.g., Heun's, Midpoint, and Ralston's methods are the most frequently used techniques [13], [14]. In the following subsections the second-order Runge-Kutta and the Newmark methods are introduced.

2.3.1 Second-order Runge-Kutta method

The explicit RK methods are called two stages techniques because the first step is a predictor and the second one is a corrector step. The increments calculated from the tangent

lines are the following at the example of free falling, air resistance considered:

$$K_1 = \Delta t \left(g - \frac{1}{2m} \rho v_i^2 c_w A \right), \quad (13)$$

$$K_2 = \Delta t \left(g - \frac{1}{2m} \rho (v_i + \beta K_1)^2 c_w A \right), \quad (14)$$

where Δt is the time increment, v_i is the velocity determined for the i^{th} step, β is a coefficient, which depends on the method to be applied. The velocity for the next time step can be calculated as Eq. (15):

$$v_{i+1} = v_i + \alpha_1 K_1 + \alpha_2 K_2, \quad (15)$$

where α_1 and α_2 are coefficients (see Table 1).

The acceleration can be determined using the value of v_{i+1} :

$$a_{i+1} = g - \frac{1}{2m} \rho v_{i+1}^2 c_w A. \quad (16)$$

The actual position of the bearing ball is written as:

$$s_{i+1} = s_i - \left(v_{i+1} \Delta t + 0.5 a_{i+1} \Delta t^2 \right), \quad (17)$$

where s_i is the position calculated at i^{th} step.

2.3.2 Newmark method

This article utilizes the Newmark method [15] with a predictor and a corrector technique, i.e., iteration is not needed. The first step is to form the linear stiffness of the task, which is demonstrated for the collision part of the problem:

$$k_l = k + \frac{m}{\alpha \Delta t^2}, \quad (18)$$

where k is the first derivative of the contact force Eq. (9) with respect to the penetration, m is the mass of the bearing ball, α is a coefficient of the Newmark approach, which value is 0.25. The time increment is denoted with Δt . Thereafter in the predictor step, the effective load can be expressed as:

$$\hat{R} = F_{ext} + m \left(\frac{v_i}{\alpha \Delta t} + a_i \left(\frac{1}{2\alpha} - 1 \right) \right) + F(\delta), \quad (19)$$

where $F_{ext} = -mg$ represents the external force. The predicted displacement increment can be written as:

Table 1 The coefficients of RK2 methods

Coefficients	Heun	Ralston	Midpoint
α_1	0.5	1/4	0
α_2	0.5	3/4	1
β	1	2/3	0.5

$$\Delta \hat{s} = \frac{\hat{R}}{k_l}. \quad (20)$$

The approximation of acceleration, velocity, and displacement values:

$$a_{appr} = \frac{\Delta \hat{s}}{\alpha \Delta t^2} - \frac{v_i}{\alpha \Delta t} - a_i \left(\frac{1}{2\alpha} - 1 \right), \quad (21)$$

$$v_{appr} = \frac{\beta}{\alpha \Delta t} \Delta \hat{s} - v_i \left(\frac{\beta}{\alpha} - 1 \right) - \frac{a_i \Delta t \left(\frac{\beta}{\alpha} - 2 \right)}{2}, \quad (22)$$

$$s_{appr} = s_i + \Delta \hat{s}, \quad (23)$$

where $\beta = 0.5$ is a Newmark parameter. In the corrector step the approximated values are used to recalculate the contact force if penetration takes place. Thereafter the effective out-of-balance load and the displacement increment are the following:

$$R = F_{ext} - m a_{appr} + F^r(\delta), \quad (24)$$

$$\Delta s = \Delta \hat{s} + \frac{R}{k_l}, \quad (25)$$

where $F^r(\delta)$ is the calculated contact force with parameters s_{appr} and v_{appr} . The new acceleration, velocity, and displacement in time step $i+1$ after corrector step can be written as:

$$a_{i+1} = \frac{\Delta s}{\alpha \Delta t^2} - \frac{v_i}{\alpha \Delta t} - a_i \left(\frac{1}{2\alpha} - 1 \right), \quad (26)$$

$$v_{i+1} = v_i + a_i \Delta t (1 - \beta) + a_{i+1} \Delta t \beta, \quad (27)$$

$$s_{i+1} = s_i + \Delta s. \quad (28)$$

To assess the algorithmic dissipation characteristics of the employed time-integration schemes, an undamped single-degree-of-freedom oscillator was considered as a reference problem. The system was defined by a mass of 1 kg and a stiffness of 100 N/m, with an initial displacement of 1 m and an initial velocity of 0 m/s, yielding the analytical solution $x(t) = \cos(10t)$ and a constant total mechanical energy. The numerical response was evaluated over 50 s using time steps of $\Delta t = 0.005, 0.01$, and 0.02 s in Scilab [16]. For $\Delta t = 0.005$ s, the maximum relative deviation of the total mechanical energy from its initial value was 1.575×10^{-2} for the Heun scheme, 2.2×10^{-6} for RK4, 1.266×10^{-14} for the Newmark scheme, and 6.979×10^{10} for the Explicit Euler method. Thus, the Explicit Euler and Heun schemes exhibit artificial energy growth in the undamped oscillator, whereas the energy deviation of RK4 is negligible and the Newmark scheme conserves the mechanical energy to within numerical round-off. These results confirm that, for the time steps

investigated in this study, numerical dissipation does not constitute a significant source of error for the RK4 scheme.

3 Set-up a test bench

In order to determine the coefficient of restitution of different bearing balls, a test bench was created. The test bench consists of a steel cuboid, which serves as the impact surface for the bearing balls. The system is shown in Fig. 2. A screw is placed to hold and release the bearing balls in the course of drop tests.

A ruler was placed in the background of the setup as a length reference, allowing the rebound distances to be determined from the recorded images. Based on the image resolution and the visual identification of the ball position, the uncertainty associated with the height reading was estimated to be approximately ± 0.5 mm.

A camera recording slow-motion videos at 240 fps was used to determine the rebound distance of each ball. The 240 fps recording rate corresponds to a frame interval of approximately 4.17 ms. The finite frame rate introduces an uncertainty because the exact instant of the maximum rebound height may occur between two consecutive frames. However, close to the maximum rebound height, the vertical velocity of the ball approaches zero. Therefore, this contribution is considered negligible compared with the uncertainty associated with the visual determination of the rebound height. The camera was positioned approximately at the expected rebound height and perpendicular to the measurement plane in order to minimize the effect of parallax. The remaining effects associated with image-based position



Fig. 2 Photo of the test system

determination were therefore considered to be included in the ± 0.5 mm height-reading uncertainty.

The coefficient of restitution is defined as:

$$e = \frac{|v_{cl}|}{|v_c|}. \quad (29)$$

Neglecting air resistance, the coefficient of restitution can be expressed in terms of the drop and rebound heights:

$$e = \sqrt{\frac{h}{H}}, \quad (30)$$

where h is the rebound distance (see Fig. 3), and v_c, v_{cl} are the impact and rebound velocities, respectively.

Measurements were performed for four different bearing ball diameters. To assess the repeatability of the experiment, three independent measurements were performed for each ball diameter. The measured rebound heights are summarized in Table 2. Selected frames from the recorded measurements are shown in Figs. 4 and 5.

The average rebound height was calculated from the three measurements for each ball diameter. The repeatability was characterized by the sample standard deviation s_h . In addition to the statistical variation between repeated measurements, the estimated ± 0.5 mm uncertainty associated with the visual determination of the rebound height was taken into account. A conservative estimate was used by combining the sample standard deviation of the individual measurements with the image-reading uncertainty:

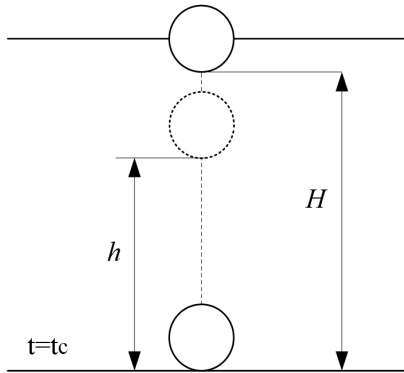


Fig. 3 Determining the restitution from distances H and h

Table 2 Measured rebound heights of the bearing balls

Ball d [mm]	Measurement 1 [mm]	Measurement 2 [mm]	Measurement 3 [mm]
14	148	148	148.5
7	151	149	149.5
6	149	149	149.5
5.5	150.5	149.5	150

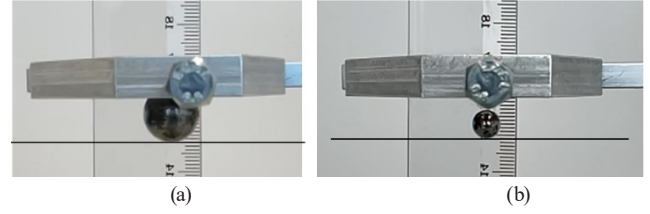


Fig. 4 Pictures of the rebound heights (a) $d=14$ mm ball; (b) $d=7$ mm ball

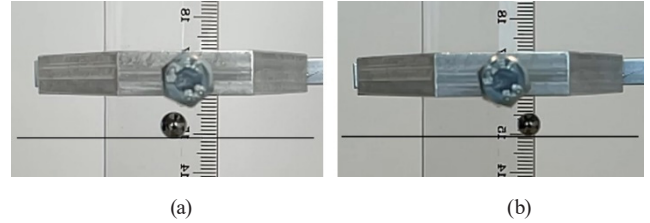


Fig. 5 Pictures of the rebound heights (a) $d=6$ mm ball; (b) $d=5.5$ mm ball

$$u_h = \sqrt{s_h^2 + u_{read}^2}, \quad (31)$$

where $u_{read} = 0.5$ mm represents the estimated uncertainty of the image-based height reading. Since the initial height H was considered known and its uncertainty was neglected, first-order uncertainty propagation gives:

$$u_e = \left| \frac{\partial e}{\partial h} \right| u_h = \frac{e}{2h_{avg}} u_h, \quad (32)$$

where h_{avg} is the average rebound height. The calculated average rebound heights, sample standard deviations, combined height uncertainties, coefficients of restitution, and corresponding uncertainties are summarized in Table 3.

The resulting coefficients of restitution remain within a relatively narrow range of 0.9623–0.9683. The experimentally determined coefficients of restitution are therefore very similar for all investigated ball diameters.

4 Performing simulations with different balls

The input data of the simulation contains the geometry of the bearing ball, and the elastic material properties of the contact bodies: the Poisson ratio is $\mu = 0.3$, the Young's modulus is $E = 200$ GPa. Eq. (3) can be used if Young's modulus E of the contacting bodies is equal and the Poisson ratio is $\mu = 0.3$ [10].

Table 3 Average rebound heights, repeatability, and coefficients of restitution

d [mm]	H [mm]	h_{avg} [mm]	s_h [mm]	u_h [mm]	e	u_e
14	160	148.167	0.289	0.577	0.96231	± 0.0019
7	160	149.833	1.041	1.155	0.96771	± 0.0037
6	160	149.167	0.289	0.577	0.96555	± 0.0019
5.5	160	150	0.5	0.707	0.96825	± 0.0023

A special-purpose program was developed in Scilab [16] to determine the impact and rebound velocities of the steel bearing balls when air resistance is considered, and it is neglected. First, the experimentally measured drop height presented in Section 3 was prescribed as the initial height, and the impact velocity was calculated. Subsequently, the experimentally average rebound height was prescribed as the initial height, and the corresponding velocity was calculated, representing the rebound velocity. The impact and rebound velocities obtained in this way were then used to determine the coefficient of restitution. Thus, the experimentally measured and calculated average heights from Section 3 provide the input data for the velocity calculations in Section 4 and establish the basis for determining the restitution parameter used in the contact simulations. The penetration of each ball is simulated with different numerical methods using the Flores model. The methods will be compared to each other.

4.1 Determining the restitution parameter

The calculated velocity, and position for free falling with air resistance are shown in Fig. 6 and Fig. 7 in the case of a $d = 14$ mm ball diameter with initial height $H = 0.16$ m, initial velocity $v_0 = 0$ m/s, $\rho = 1.2$ kg/m³.

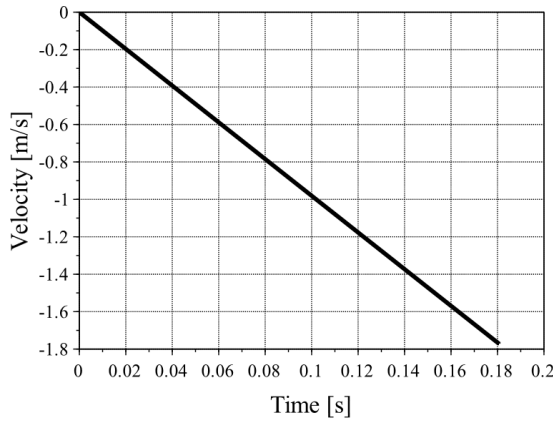


Fig. 6 Velocity determined by Heun's method

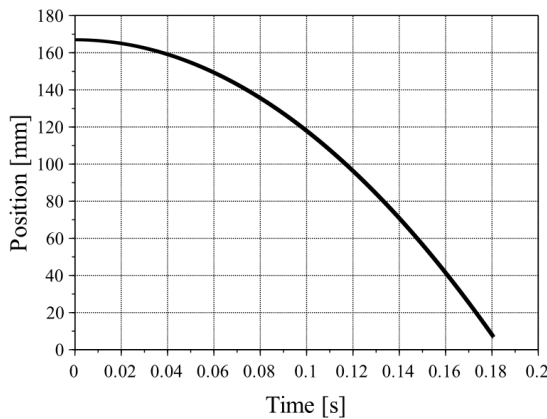


Fig. 7 The calculated position in the function of time

To verify the suitability of the selected time step $\Delta t = 10^{-6}$ s, a time-step sensitivity analysis was performed for a drop height of 160 mm. The calculated impact velocity was evaluated using time steps of 10^{-5} s, 10^{-6} s, and 10^{-7} s. The corresponding impact velocities were 1.771155 m/s, 1.7712236 m/s, and 1.7712324 m/s, respectively. The difference between the results obtained with 10^{-6} s and 10^{-7} s was 8.82×10^{-6} m/s, corresponding to approximately 0.0005% of the calculated velocity. Therefore, $\Delta t = 10^{-6}$ s was considered sufficiently small for the numerical calculation of the impact and rebound velocities.

The determined ball velocities and restitutions are listed in Table 4, which considers air resistance. Heun's method is used with equidistant $\Delta t = 10^{-6}$ s time increments. Table 5 consists of the data, when air resistance is neglected.

It is worth to mention that free falling of bearing balls from low heights with the consideration of air resistance has not significant deviance from the case of neglectation of air resistance, since the discrepancies occur only at the fourth or fifth decimal place in the restitution parameters. Based on the comparison of the experimentally determined and calculated restitution values, the restitution parameter for the subsequent contact simulations was set to $e = 0.97$.

4.2 Simulation of the collisions

Five methods will be compared, which are the explicit Euler, Heun's, fourth order Runge-Kutta (RK4) with fixed time step, Newmark methods, and the adaptive Runge-Kutta of order 4, which is a built-in solver in Scilab [16].

Determining the maximum penetration (see Table 6) for the balls, with the use of Eq. (7) and v_c values from Table 4 in the case of elastic collision.

Table 4 Velocities determined to different bearing ball diameters, air resistance is considered

Ball diameter	Velocity before collision	Velocity after collision	Restitut.
$d = 14$ mm	$v_c = -1.771224$ m/s	$v_{cl} = 1.704516$ m/s	$e \approx 0.9623$
$d = 7$ mm	$v_c = -1.77068$ m/s	$v_{cl} = 1.713568$ m/s	$e \approx 0.9677$
$d = 6$ mm	$v_c = -1.7705$ m/s	$v_{cl} = 1.709602$ m/s	$e \approx 0.9656$
$d = 5.5$ mm	$v_c = -1.770385$ m/s	$v_{cl} = 1.714256$ m/s	$e \approx 0.9683$

Table 5 Velocities if air resistance is neglected

Ball diameter	Velocity before collision	Velocity after collision	Restitut.
$d = 14$ mm	$v_c = -1.77177$ m/s	$v_{cl} = 1.704998$ m/s	$e \approx 0.9623$
$d = 7$ mm	$v_c = -1.77177$ m/s	$v_{cl} = 1.714553$ m/s	$e \approx 0.9677$
$d = 6$ mm	$v_c = -1.77177$ m/s	$v_{cl} = 1.710746$ m/s	$e \approx 0.9656$
$d = 5.5$ mm	$v_c = -1.77177$ m/s	$v_{cl} = 1.715514$ m/s	$e \approx 0.9682$

Table 6 The calculated maximum penetrations

Ball diameter	$\delta_{\max}$
$d = 14$ mm	0.0265 mm
$d = 7$ mm	0.0132 mm
$d = 6$ mm	0.0113 mm
$d = 5.5$ mm	0.0104 mm

According to [17] the duration of the contact can be calculated if the radius of the steel cuboid is $R_2 = \infty$ and the material of the two contact bodies are same:

$$\tau_c = 2.94 \left(\frac{m_{red}^2}{64E^2} \frac{R_1 v_c}{225(1-\mu^2)^2} \right)^{1/5}, \quad (33)$$

where E is the Young's modulus, μ is the Poisson ratio, v_c is the velocity before collision, m_{red} is the reduced mass:

$$m_{red} = \frac{m_1 m_2}{m_1 + m_2} \cong m_1, \quad (34)$$

since the mass of the steel cuboid is several orders of magnitude larger than each bearing ball. With the use of (33) the τ_c duration of the contact for the largest bearing ball is: 4.1888×10^{-5} s if the bearing ball mass $m_1 = 10.5$ g. Aware of the results, the time step should be chosen smaller. Eq. (9) is used to compute the penetrations and velocities when dissipation takes place, i.e., the collision is not an elastic one. The simulation program uses the velocities before collision from Table 4 to compute the contact force. The results for $d = 14$ mm bearing ball are shown in Figs. 8–11, equidistant time increments. The final integration time is $t_f = 4.42 \times 10^{-5}$ s. The RMS error of the other methods at the discrete time steps is determined compared to the adaptive RK method:

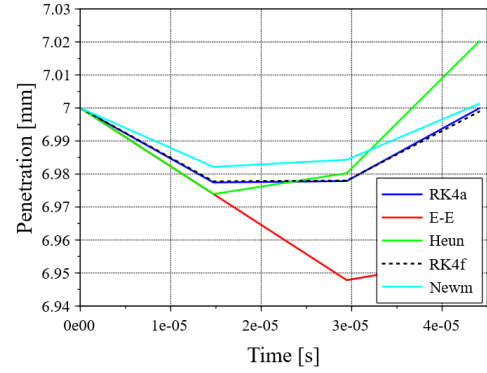
$$RMS_e = \sqrt{\frac{\sum_{i=1}^n (RK_i^a - other_i)^2}{n}}, \quad (35)$$

where n is the number of the time steps, $other_i$ is the value of the other methods at the i^{th} time step.

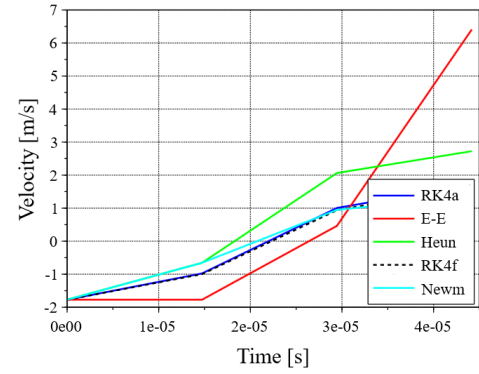
The RMS error is determined for time steps $\Delta t = t_f/3$ s, $\Delta t = t_f/6$ s, $\Delta t = t_f/12$ s, $\Delta t = t_f/24$ s. The RMS values of the positions are placed in Table 7. The RMS error of velocities can be found in Table 8.

Differences in positions and velocities at the final time step are determined compared to the adaptive RK method, which is considered as reference (see Fig. 12, 13).

The comparison of the running times is detailed in Table 9. The computer configuration is the following:



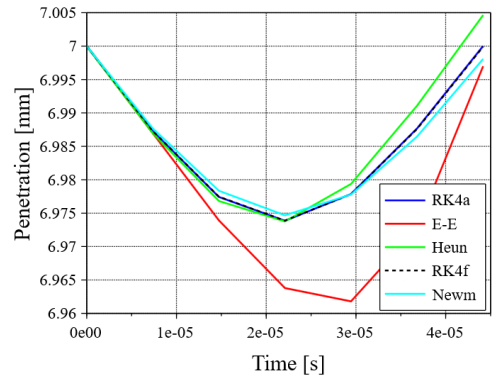
(a)



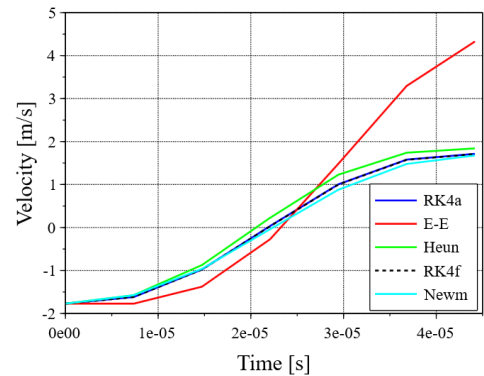
(b)

Fig. 8 Collision of $d = 14$ mm ball with equidistant time step $\Delta t = t_f/3$ s

(a) Penetration; (b) Velocity



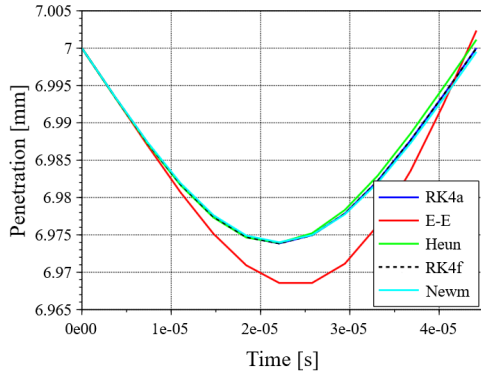
(a)



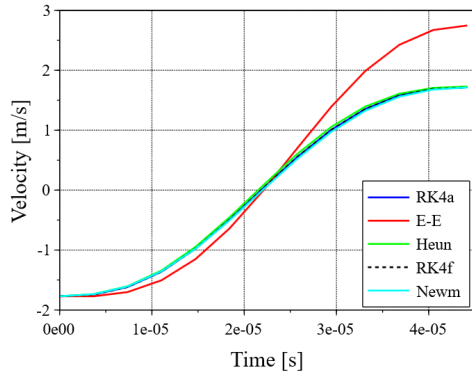
(b)

Fig. 9 Collision of $d=14$ mm ball with equidistant time step $\Delta t = t_f/6$ s

(a) Penetration; (b) Velocity

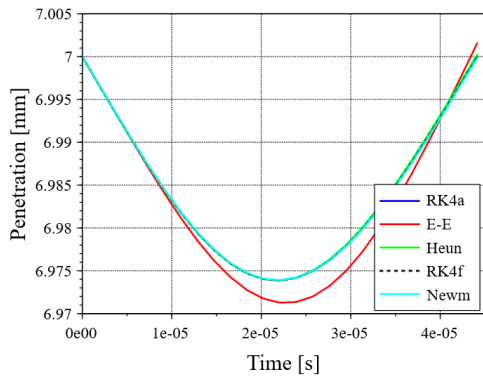


(a)

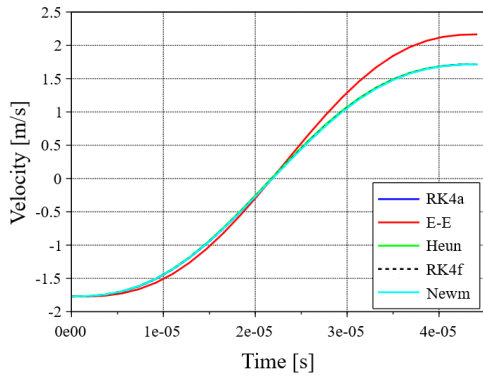


(b)

Fig. 10 Collision of $d=14$ mm ball with equidistant time step $\Delta t = t_f/12$ s
(a) Penetration; (b) Velocity



(a)



(b)

Fig. 11 Collision of $d=14$ mm ball with equidistant time step $\Delta t = t_f/24$ s
(a) Penetration; (b) Velocity

Table 7 RMS errors of the position

Time incr.	Expl. Euler	Heun	RK4	Newmark
$t_f/3$ s	0.0000272	0.0000104	0.0000005	0.0000041
$t_f/6$ s	0.0000093	0.0000023	$4.619 \cdot 10^{-8}$	0.000001
$t_f/12$ s	0.0000039	0.0000006	$3.492 \cdot 10^{-9}$	0.0000003
$t_f/24$ s	0.0000018	0.0000001	$3.337 \cdot 10^{-10}$	$6.437 \cdot 10^{-8}$

Table 8 RMS errors of the velocity

Time incr.	Expl. Euler	Heun	RK4	Newmark
$t_f/3$ s	2.396848	0.7490337	0.0475579	0.2378713
$t_f/6$ s	1.2117876	0.1421383	0.0034313	0.070631
$t_f/12$ s	0.5117138	0.032052	0.0002644	0.0178621
$t_f/24$ s	0.2305452	0.0076472	0.0000298	0.0045899

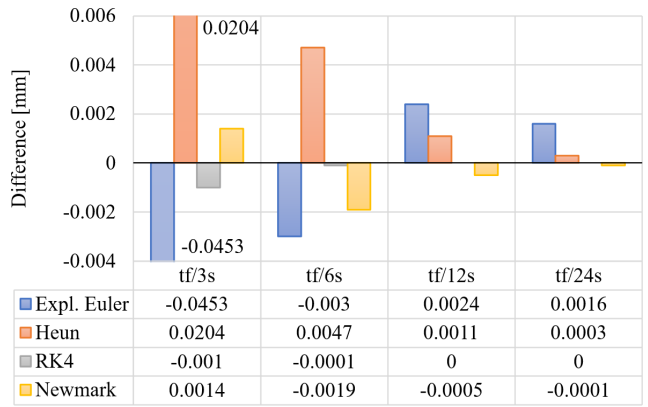


Fig. 12 Deviances of the end-point positions compared to adaptive RK

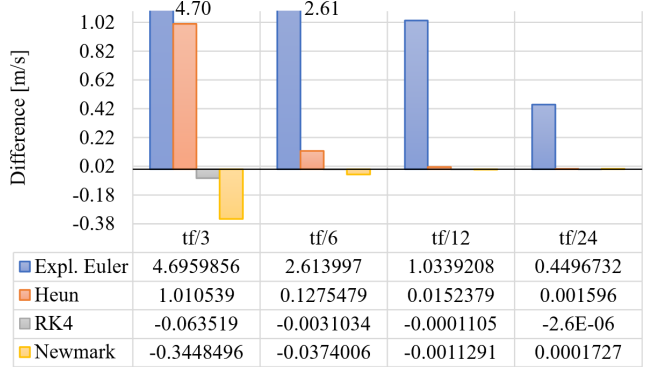


Fig. 13 Deviances of the velocities compared to adaptive RK results

Table 9 Running time percentages compared to the adaptive RK method

Time incr. [s]	Adapt. RK [ms]	Expl. Euler	Heun	RK4	Newmark
$t_f/3$	8.3736	1.99%	1.89%	4.53%	3.61%
$t_f/6$	8.8026	2.14%	2.81%	7%	5.4%
$t_f/12$	9.0873	2.79%	4.77%	9.3%	9.76%
$t_f/24$	11.0803	4.08%	7.36%	13.64%	20.6%

Intel(R) Core™ i3-8300 CPU, RAM: 8.00 GB, SSD,
operating system: Windows 10 Home, ver. 21H2.

The adaptive RK method has the highest running time. Since the Newmark method and the RK4 contain more time-consuming mathematical operations than the Heun's and Explicit Euler methods, therefore they have higher running time. The rebound times of the $d = 14$ mm bearing balls with different time steps are determined by the bisection method. The rebound time deviances compared to the adaptive RK method are shown in Table 10.

Summarizing the results, the Explicit Euler method has the shortest running time with the worst accuracy since it is a first order method. The Newmark method has better accuracy compared to the other second-order ones, but its running time is higher than its double and quite the same as RK4. Based on the results of the simulations, the classical RK4 method has a high accuracy and comparable to the adaptive RK method.

5 Conclusions

The paper dealt with the analysis of collisions utilizing different size bearing balls. The nonlinear mathematical model of the problem was described in the first part of the article. To treat the collision of the steel balls between the steel cuboid mathematically, the determination of the restitution parameter of the steel bearing balls was essential. Therefore, a test bench was built utilizing a slow-motion camera option. The contact force computation was performed with Hertz

theory extending with a nonlinear viscoelastic property to consider the dissipation energy.

In engineering practice, modeling and simulation are required in many cases. Therefore, it is advisable to be aware of which numerical method can be used efficiently. The main essence of the paper was to solve the initial value problem of contact with five numerical methods to compare the techniques based on the accuracy and running time of each one.

The results showed that, for a time increment of $t_f/3s$, the RMS error of the RK4 method for the position compared with the adaptive RK method was 5×10^{-7} . A comparable error magnitude was achieved by the second-order methods using a smaller time increment of $t_f/12s$, while the velocity RMS errors showed a similar tendency. The running time of RK4 with a time increment of $t_f/3s$ was 0.379 ms, whereas achieving comparable accuracy with the second-order methods increased the running time by 14.2% for Heun's method and 134% for the Newmark method. Although Newmark's method was slightly more accurate than Heun's method in terms of RMS errors and rebound-time deviations for comparable error magnitudes, its running time was approximately twice as long. From an engineering perspective, the classical RK4 method therefore provides the most favorable balance between accuracy and computational efficiency for the investigated contact problem and range of time increments, while Heun's method can be considered a computationally efficient second-order alternative. The additional assessment of algorithmic dissipation using an undamped single-degree-of-freedom reference system showed that the numerical energy deviation of RK4 remained negligible over the investigated time-step range. Therefore, the above conclusions regarding the favorable accuracy-efficiency balance of RK4 remain valid when its algorithmic dissipation characteristics are taken into account.

Table 10 Rebound time deviations in percentage

Time incr. [s]	Adapt. RK	Expl. Euler	Heun	RK4	Newmark
$t_f/3$		41.4%	-13.95%	1.45%	11.19%
$t_f/6$	Rebound time:44.2 μs	1.95%	-5.16%	0.135%	2.697%
$t_f/12$		-1.96%	-1.44%	0.011%	0.68%
$t_f/24$		-1.68%	-0.36%	0%	0.17%
$t_f/48$		-1.02%	-0.09%	0%	0.045%

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