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ANALYSIS AND SYNTHESIS OF OPEN-ENDED TENDON STRUCTURES

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Abstract

The article deals with the analysis and synthesis of open-ended tendon transmissions, which are widely used in robotics, mainly at dextrous hands. A general theoretical approach is introduced, feasibility and isomorphism of arbitrary tendon structures are investigated with the help of defining normalized, schematic and canonical form of manipulators. Kinematic description is discussed based on the structure matrix, and solution methods of inverse tasks are presented. For the purpose of constructional synthesis, minimum and maximum number of tendons and pulleys are analysed. To enable practical use of the results, several algorithms are developed. As an application example for system optimization, transmission design of the TUB-PC multifingered robot hand is presented.

Keywords: kinematics, open-ended tendons, dextrous robot hands.

1. Introduction

Tendons are common elements widely used in the field of constructional synthesis of various machines. Most frequently tendons are *endless*, for example in common applications of various belts or chains. In other cases, tendons are *open-ended*. Mathematically, the speciality of open-ended tendon transmission lays in the *unidirectional* character of the system, i.e. a tendon can transmit energy only in the direction of pulling.

Multifingered robot hands are often operated with open-ended tendons, because effective pay-load increase and manipulator structure becomes lighter, compared to other transmissions (e.g. direct drive). The reason is that actuators can be installed on the ground and the transmission system has relatively low mass and inertia. On the other hand, tendon transmission has some practical drawbacks, the non-linear system behaviour, the limited tensile strength and fatigue of tendons must be compensated with proper control methods and careful selection of tendon material (see SALISBURY, 1982; JACOBSEN et al., 1984).

A manipulator with n degrees-of-freedom (DOF) and with m open-ended tendons can be classified as an $n \times m$ type of manipulator. In practice, usually two types of tendon routing are applied. One approach is the

$m \times 2n$ type of manipulator, for example at the UTAH/MIT robot hand (JACOBSEN et al., 1984). Although, transmission of such manipulators is redundant, the operation principle is simple and quite similar to the tendon arrangement of a natural finger. Another possibility is the $m \times (n+1)$ type of manipulator, which aims to obtain relatively low actuator mass due to the minimum number of motors. Quite many dextrous grippers have transmissions of the latter type, for example fingers of the Okada Hand (OKADA, 1979), the Stanford/JPL Hand (SALISBURY, 1982), or the TUB-PC Hand (LUDVIG, 1997). It must be noted, there are other types of universal grippers (e.g. Soft Gripper, see PHAM and HEGINBOTHAM, 1986), robots arms (e.g. Microbot, see HILL and CLEMENT, 1982), and other devices (e.g. endoscopes, see STURGES and LAOWATTANA, 1993) too, which have open-ended transmission.

In general, there are many other possibilities to construct open-ended tendon transmissions. MORECKI et al. (1980) presented a theoretic approach to the problem, and that was criticized and developed later by LEE and TSAI (1991). Although the latter outstanding work introduced several useful concepts, the analysis was focused only on the theory of pseudo-triangular structure matrices, thus many common tendon structures were not covered. This article generalizes results of LEE and TSAI, with the aim to provide a wider theoretical background for the analysis and design synthesis of open-ended tendon structures.

2. Operation Principle

For readers not familiar with the topic, the operation principle of open-ended transmissions will be introduced through some simple examples.

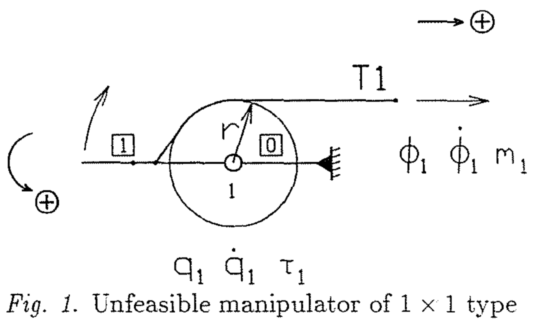

Fig. 1. Unfeasible manipulator of 1×1 type

EXAMPLE 1 Assume, given a simple 1 DOF mechanism with 1 tendon, as shown in *Fig. 1.* The pulley can rotate freely around the axis of the joint, its distal end is jointed to the distal segment, the tendon is conducted around the pulley, then its other end is fastened to a motor pulley (not presented).

If a positive force acts on the proximal end of the tendon, it generates a negative joint torque. On the other hand, the manipulator is not feasible, because the mechanism cannot be operated in the negative direction with the tendon.

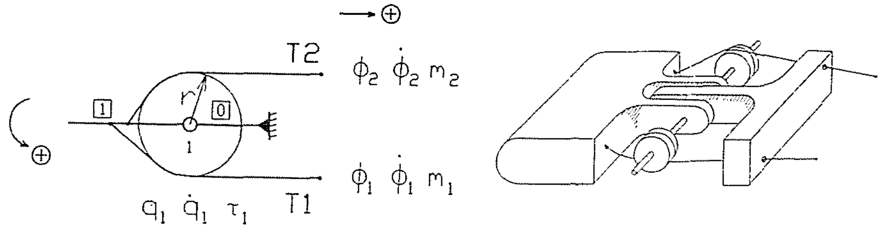

Fig. 2. Feasible manipulator of 1×2 type a) planar b) spatial

In order to enable the operation of the above mechanism downwards too, a further tendon and pulley (behind the first one) must be applied, which results an 1×2 type of manipulator, see *Fig. 2.a.* Let us define the following kinematic variables: ϕ_1, ϕ_2 motor angular displacements; m_1, m_2 motor torque, q_1 joint angular position; τ_1 joint torque. Assume that pulley radii are of unit length.

Motor displacements $(\Delta\phi_1, \Delta\phi_2)$ can be calculated from a given value of joint displacement (Δq_1) with the following equation:

$$\begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Delta q_1 . \tag{1}$$

Notice that transmission ratio depends only on pulley radii, but distances between joint axis and tendon fixing points have no effect. The above transformation matrix is called the structure matrix of the transmission system ($\mathbf{A}$). In the inverse position task, joint displacement must be calculated from motor displacements. A solution exists if the following condition is fulfilled:

$$\Delta\phi_1 = -\Delta\phi_2 , \tag{2}$$

which expresses that a tendon cannot loosen or be torn. A possible solution is:

$$\Delta q_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi_1 \\ \Delta\phi_2 \end{bmatrix} , \tag{3}$$

where the transformation matrix $(\mathbf{A}^{-1})$ is a pseudo inverse (generalized inverse). Similar equations apply for joint and motor torque. The direct torque equation is calculated with the transposing of the transformation matrix:

$$\boldsymbol{\tau} = \mathbf{A}^T \mathbf{m} . \tag{4}$$

In the inverse task, motor torque must be calculated from joint torque:

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tau_1 + \kappa_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \tag{5}$$

where the transformation matrix is $\mathbf{A}^{-T}$. There are infinite solutions, because the two tendons 'work against each other'. The internal force will be minimum, if:

$$\kappa_1 = \max(0, -\tau_1) . \tag{6}$$

There are other manipulators, which can be described also with the above equations. For example, *Fig. 2.b* shows a spatial representation of the same structure.

Identification of structure matrices can be achieved relatively easily according to the following composition rule.

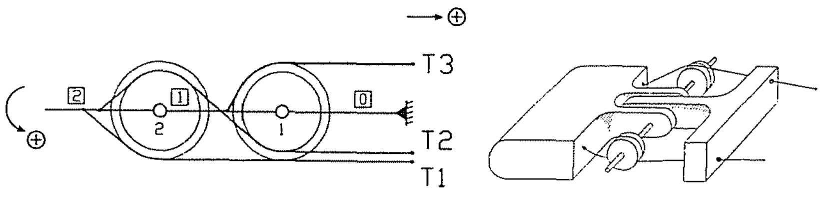

Fig. 3. Manipulator of 2×3 type a) planar b) spatial

ALGORITHM 1 Assume, given a planar manipulator, and its pulleys are all of unit or zero radii. In a structure matrix ($\mathbf{A}$), column j of the matrix refers to tendon j, while row i refers to joint i (joints numbered from proximal end in ascending order, starting by 1). For the element in row i and column j of $\mathbf{A}$ the following rules apply:

- the element is 1, if positive force on tendon j results in positive torque on link i;
- the element is -1, if positive force on tendon j results in negative torque on link i;
- the element is 0, if positive force on tendon j results in no torque on link i, because tendon crosses the joint axes or it is fixed to a segment proximal to joint i.

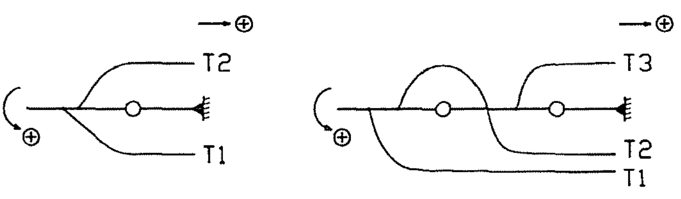

Fig. 4. Schematic drawings of manipulators a) 1×2 type b) 2×3 type

EXAMPLE 2 Let us now investigate a 2 DOF mechanism. For the 2×3 type of manipulator shown in *Fig. 3*. Algorithm 1 yields the following structure matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}^T . \tag{7}$$

For the 2×4 type of manipulator in *Fig. 5*, the following structure matrix applies:

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}^T . \tag{8}$$

Notice, the relation between number of degrees-of-freedom and tendons. The above manipulators were of $n \times (n+1)$ and $n \times 2n$ types, respectively.

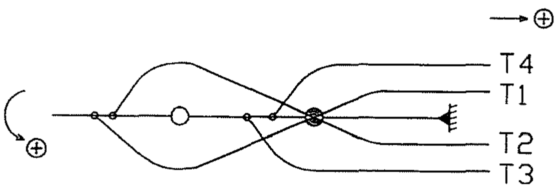

Fig. 5. Manipulator of 2×4 type

3. Classification of Manipulators

Since the concrete objective of the work presented was to elaborate the theoretical background of transmission design for dextrous robot hands, only a limited area of open-ended tendon applications will be discussed in details. In below, the word (general) *manipulator* ($\mathcal{M}$) refers to a system of the following components: (1) open-loop, serial *mechanism* with R-pairs, (2) *transmission*, which consists of open-ended tendons and conducting pulleys, that rotate freely around manipulator axes (distal end of a tendon is fixed to a segment, proximal end to a motor), (3) *actuator*, which is a number of rotating motors. Set $\mathcal{M}$ is infinite at a given number of degrees-of-freedom (DOF) and a given number of tendons.

Because structural analysis deals rather with the way how a manipulator is operated and not with the calculation of kinematic variables, thus exact values of geometrical parameters have no importance, and simplified models can be taken.

A *normalized manipulator* ($\mathcal{M}_n \subset \mathcal{M}$) is a manipulator, where the radius of pulleys is unit or zero at joints, and unit at motors. At given values of n and m, set $\mathcal{M}_n$ is finite with respect to possible structure matrices. On the other hand, there are infinite numbers of normalized manipulators having the same structure matrix, but differing in their D–H parameters (DENAVIT and HARTENBERG, 1955) from each other.

A *schematic manipulator* ($\mathcal{M}_s \subset \mathcal{M}_n$) is a planar, normalized manipulator, where all links are of unit length. Using the D–H parameter assignment: $\vartheta_i = 0, \alpha_i = 0, a_i = 1$ and $d_i = 0$. According to LEE and TSAI (1991), manipulators in plane can be represented with schematic drawings, see e.g. *Fig. 4*. Set $\mathcal{M}_s$ is finite at given values of n and m. The main advantage in using the schematic description is due to the one-to-one correspondence between normalized structure matrices and schematic manipulators.

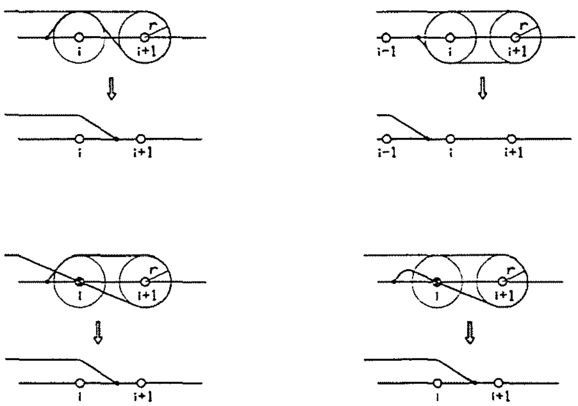

Fig. 6. Replacement of reversed tendons

Although it is not a necessary feature of transmissions, in practice, tendons are usually conducted only in forward direction, from the basement to the direction of the distal end. Nevertheless, there is a possibility for *reversed tendon routing* with one ore more changes in the direction of tendon conducting. *Fig. 6* displays basic cases and gives suggestions how to replace reversed tendons with straightforward ones. Tendons are *parallel*, if they are conducted in an identical way. In structural analysis, they can be replaced with only one tendon. It will be assumed in below, that schematic manipulators have neither reversed nor parallel tendons. For the sake of completeness it must be also imposed that if there was a tendon crossing only other joint axes beyond a certain joint, it is assumed to be fastened to the segment proximal to that joint. There is no tendon furthermore, which has no effect on the manipulator (no $\mathbf{0}^T$ row in $\mathbf{A}$).

4. Kinematic Description

Kinematic description of a manipulator relates finding connection between variables of *motor* and *joint space*, which are motor (ϕ) and joint position

($\mathbf{q}$), motor and joint velocity, motor ($\mathbf{m}$) and joint torque ($\boldsymbol{\tau}$).

Assume rigid tendons (with zero spring constant), zero mass and no friction! Conversion of kinematic variables at an $n \times m$ type of manipulator can be described with a structure matrix ($\mathbf{A}$) of $m \times n$ size. Values in the matrix depend on the geometrical parameters of the transmission (diameters of pulleys), and in a general case: $A_{i,j} \in \Re$. Assume pulley radii are of zero or unit value, which is the case at normalized manipulators, then: $A_{i,j} \in \{-1, 0, +1\}$.

Assume zero motor positions result zero joint positions, and in that simple case, the following equation applies:

$$\phi = \mathbf{A}\mathbf{q} \ . \tag{9}$$

Otherwise, there is an additional bias addend (ϕ_0) too in the equation:

$$\phi = \mathbf{A}\mathbf{q} + \phi_0 \ . \tag{10}$$

It follows from the law of energy conservation, that transformation matrices used in velocity (or displacement) and torque type of equations are transposed of each other:

$$\dot{\phi} = \mathbf{A}\dot{\mathbf{q}} \ , \tag{11}$$

$$\boldsymbol{\tau} = \mathbf{A}^T \mathbf{m} \ . \tag{12}$$

The structure matrix of a feasible manipulator is non-quadratic (see Lemma 8), that complicates the solution of the inverse tasks. Inverse equation on angular positions can always be expressed algebraically by reordering *Eq.* (9):

$$\mathbf{q}_{n\times 1} = \mathbf{A}^{-1}_{n\times m} \phi_{m\times 1} \ . \tag{13}$$

where $\mathbf{A}^{-1}$ denotes pseudo inverse of the structure matrix. Physically it is clear that not all the motor positions can be set arbitrary values, because tendons are rigid, thus certain combinations of motor positions could cause loosening or tearing of some tendons. Mathematically that is described with $m - n$ anholonomic-scleronomic constraints:

$$f_k = (\phi_1, \phi_2, \ldots, \phi_m) = 0 \ , \tag{14}$$

where $k = 1, \ldots, m - n$, which assure unambiguity of the task. For the inverse equation of angular velocity similar rules can be applied. The solution of the inverse torque equation is always ambiguous, since number of unknowns is higher, than number of equations. Physically, internal force(s) act in the system, which yield no joint torque, and their value(s) can be optimized. Formally, the inverse torque task can be described as a system of parametrical equations:

$$\mathbf{m} = \{\mathbf{f}(\boldsymbol{\tau}) : \boldsymbol{\tau} = \mathbf{A}^T\mathbf{m}, \quad \mathbf{m} \geq \mathbf{0}, \ \ c(\mathbf{m}) \to \min\} \ , \tag{15}$$

where $c(\mathbf{m})$ is a criterion function. Solution method depends mainly on $c(\mathbf{m})$, and cannot often be obtained in explicit form – see Section 6.

5. Structural Analysis

This section deals with main operational features of manipulators based on the analysis of structure matrices with the aim to provide theoretical basis for effective algorithms for constructional purposes.

5.1. Feasible Tendon Structures

Taking an arbitrary structure matrix, there is no guarantee that it describes a fully controllable manipulator.

DEFINITION 1 The open-ended tendon transmission of a manipulator is *feasible*, if the variables of any joint can be set to any arbitrary value, not depending on the variables of other joints.

The above definition is fulfilled, if the torque equation at an arbitrary $\boldsymbol{\tau} \in \Re$ constant vector:

$$\mathbf{A}^T \mathbf{m} = \boldsymbol{\tau} \tag{16}$$

has a solution, which fulfills the *condition of unidirectionality*:

$$\mathbf{m} \geq \mathbf{0} \, . \tag{17}$$

For simplicity, torque equation was taken only, but similar rules could be obtained for velocity. According to the linear algebra, a necessary condition of feasibility can be obtained directly, considering (16) system of linear inhomogeneous equations.

LEMMA 1 *If a manipulator is feasible, then its structure matrix must have a rank equal to the number of degrees-of-freedom:*

$$rank\,(\mathbf{A}) = n \, . \tag{18}$$

A manipulator can be equipped with passive and active tendons. A tendon is *passive*, if without that tendon the manipulator remains feasible (i.e. the passive tendon is unnecessary). If not, the tendon is *active*. It must be noted that active and passive tendons of a manipulator can often be selected in various combinations.

LEMMA 2 *If a manipulator is feasible, there is a homogeneous solution of Eq. (16), which has only non-zero, positive elements.*

Proof 1 Assume given a feasible manipulator without passive tendons.

Solutions of *Eq.* (16) can be composed from a particular solution ($\mathbf{m}^P$) and from the homogeneous solution ($\mathbf{m}^H$):

$$\mathbf{m} = \mathbf{m}^P + \mathbf{m}^H = \mathbf{m}^P + \sum_{k=1}^{m-n} \kappa_k \mathbf{m}_k^H , \tag{19}$$

where $\kappa_k \in \Re$. The second summand is an internal force, which has no effect on net joint torque. Because all tendons are active, thus signs of elements of $\mathbf{m}^P$ can be either positive or negative at an arbitrary joint torque. Consider the case that all elements are negative. It can be compensated by the homogeneous part, if there exists a set of κ_k coefficients, that:

$$\boldsymbol{\beta} = \sum_{k=1}^{m-n} \kappa_k \mathbf{m}_k^H > \mathbf{0} , \tag{20}$$

then one possible (not necessarily optimum) solution is:

$$\mathbf{m} = \mathbf{m}^P - \min_{j=1,\ldots,m} \left(\frac{m_j^P}{\beta_j} \right) \boldsymbol{\beta} . \tag{21}$$

Eq. (20) expresses the same condition, as in the statement.

Assume now that the manipulator has one passive tendon, too. Although, at the first sight it seems that the motor torque acting on the passive tendon could be zero, too, it can be proved that condition (20) is automatically fulfilled. Since the manipulator is feasible even without the passive tendon, $m - n - 1$ homogeneous basis vector can be generated having zero elements in the row related to the tendon. Assume that the last tendon is the passive one, then *Eq.* (16) can be decomposed:

$$\boldsymbol{\tau} = \left[\mathbf{A}_{\text{active}}^T \mathbf{a}_{\text{passive}} \right] \begin{bmatrix} \mathbf{m}_{\text{active}} \\ m_{\text{passive}} \end{bmatrix} = \mathbf{A}_{\text{active}}^T \mathbf{m}_{\text{active}} + \mathbf{a}_{\text{passive}} m_{\text{passive}} . \tag{22}$$

Obviously, the active part of the equation can be solved for $\boldsymbol{\tau} = -\mathbf{a}_{\text{passive}}$ constant vector, let us denote the solution with $\mathbf{m}_p$:

$$-\mathbf{a}_{\text{passive}} = \mathbf{A}_{\text{active}}^T \mathbf{m}_p . \tag{23}$$

The $\mathbf{m} = [\mathbf{m}_p^T \ 1]^T$ motor torque vector will result zero joint torque, furthermore, it is linearly independent from the $m - n - 1$ basis vectors since the element in its last row is not zero, consequently it is an $(m - n)$-th homogeneous basis vector. Because its last element is positive and its other elements can be compensated by the other basis vectors, condition (20) can automatically be fulfilled. If there were more than one passive tendons, the above procedure could be applied for each of them, one-by-one.

Proof 2 A necessary and sufficient condition of feasibility follows directly from Lemma 1 and 2 according to *Eqs.* (16) and (17).

LEMMA 3 *A manipulator is feasible if and only if Lemma 1 and 2 are fulfilled.*

Although Lemma 3 provides an analytical tool on checking feasibility of an arbitrary manipulator, it is often complicated to generate vector β. The following lemma can be better applied in computerized analysis.

LEMMA 4 *Let* $\mathbf{t}_i \in \Re^n$ *be a unit vector, which is parallel with the i-th joint axis, and points to the positive direction. A manipulator is feasible if and only if the following equation can be solved at any* $i \in \{1, \ldots, n\}$ *indices:*

$$\mathbf{A}^T\mathbf{m} = \mathbf{t}_i\,, \tag{24}$$

$$\mathbf{A}^T\mathbf{m} = -\mathbf{t}_i\,, \tag{25}$$

where (17) condition of unidirectionality must be also fulfilled.

Proof 3 Necessity is trivial, because $\mathbf{t}_i$ and $-\mathbf{t}_i$ are some of the arbitrary $\boldsymbol{\tau}$ vectors. Sufficiency can be proved, if we consider that any $\boldsymbol{\tau}$ vectors can be provided as a sum of the $\mathbf{t}_i$ and $-\mathbf{t}_i$ vectors, respectively, multiplied with $|\tau_i|$. According to the statement, all addends of that sum can be provided with positive motor torque vectors. Consequently, the sum itself (the $\boldsymbol{\tau}$ vector) can also be provided with the resultant of positive motor torque vectors.

ALGORITHM 2 Feasibility of an arbitrary manipulator can be checked according to Lemma 4. *Eqs.* (24) and (25) can be solved with tools of linear programming, using the 2-phased simplex method (GÁSPÁR and TEMESI, 1990). Although optimization vector is not defined here, it is not a problem, because only the existence of the solution must be checked with the 1st phase of the method. The algorithm necessitates to solve $2n$ linear programming tasks in total at a given structure matrix.

Another approach to feasibility of manipulators is based on mechanical considerations. If a manipulator is not feasible, it occurs because it has synergetic or limited joints. Joint $i \in \{1, \ldots, n\}$ and the group of joints $p_1, \ldots, p_s \in \{1, \ldots, n\}$ of a manipulator are *synergetic*, if there exist $\lambda_{p_1}, \ldots, \lambda_{p_s} \in \Re$ coefficients, that:

$$\boldsymbol{\tau}_i = \lambda_{p_1}\boldsymbol{\tau}_{p_1} + \lambda_{p_2}\boldsymbol{\tau}_{p_2} + \ldots + \lambda_{p_s}\boldsymbol{\tau}_{p_s}\,. \tag{26}$$

where at least one coefficient is not zero. It occurs, if column vectors of a structure matrix are not linearly independent, i.e. Lemma 1 is not fulfilled. Joint $i \in \{1, \ldots, n\}$ and the group of joints $p_1, \ldots, p_s \in \{1, \ldots, n\}$ of a manipulator are *limited*, if there exist $\lambda_s \in \{+1, -1\}$ and $\lambda_{p_1}, \ldots, \lambda_{p_s} \in \Re$ coefficients, that:

$$\lambda_i \boldsymbol{\tau}_i \leq \lambda_{p_1} \boldsymbol{\tau}_{p_1} + \lambda_{p_2} \boldsymbol{\tau}_{p_2} + \ldots + \lambda_{p_s} \boldsymbol{\tau}_{p_s} . \tag{27}$$

Possibility of limitation follows from the unidirectional character of the transmission. It occurs, if there exist $\lambda_1, \ldots, \lambda_n \in \Re$ coefficients and $\boldsymbol{\alpha} \in \Re^m$, $\boldsymbol{\alpha} \geq \mathbf{0}$, $|\boldsymbol{\alpha}| > 0$ vector, that:

$$\mathbf{0} = \lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_n \mathbf{a}_n + \boldsymbol{\alpha} . \tag{28}$$

If *Eq.* (28) can be fulfilled so, that all λ_i coefficients are zero, then joint i cannot be operated in one or both directions (see Lemma 5). If *Eq.* (28) could be satisfied with $|\boldsymbol{\alpha}| = 0$, that would be the case of synergy.

Although Algorithm 2 can be programmed relatively easily, its application is time consuming. If a huge number of arbitrary manipulators should be checked, it is reasonable to filter out previously the obviously unfeasible structures with simpler rules, e.g. with Lemma 5, which is a direct consequence of unidirectionality.

LEMMA 5 *If a manipulator is feasible, its rows contain both positive and negative elements.*

5.2. Isomorphism

Manipulators, which differ in tendon routing, can still behave in similar manner, because of similarities in their structures. This section deals with that isomorphism.

Assume that given a planar, schematic manipulator, where external moments and forces acting on the manipulator are balanced with a joint torque vector of $\boldsymbol{\tau} = [\tau_1 \tau_2 \ldots \tau_n]^T$. Take segment i of the manipulator, let M_i be a moment, which is obtained by transforming the external forces and moments acting on the segment and the effect of the distal segment $(i+1)$ to a frame, fixed to the centre point of the proximal axis of the segment. Executing such reduction at each joint, an $\mathbf{M} = [M_1 M_2 \ldots M_n]^T$ system of moments will be obtained. Joint torque vector equals to the vector of moments in that simple, planar case. Let us define a diagonal transformation matrix $\mathbf{R}$ with $n \times n$ size and $R_{k,k} \in \{+1, -1\}$. It executes mirroring of moments through the $x - y$ plane.

DEFINITION 2 Schematic manipulators A and B are isomorphic, if there is a constant transformation $(\mathbf{R})$, that the following equation applies to the joint torque vectors $(\mathbf{m}_A, \mathbf{m}_B)$:

$$|\mathbf{m}_A(\mathbf{M})| = |\mathbf{m}_B(\mathbf{RM})| , \tag{29}$$

in case of arbitrary system of moments ($\mathbf{M}$).

The above definition describes two important features of isomorphic manipulators. Defining the positive direction to opposite at a joint (adding $\pm 180°$ to a twist angle) and/or renumbering the tendons (exchanging two tendons) yield isomorphic structures. Consequently, the following lemmas apply.

LEMMA 6 *If two rows of a structure matrix are exchanged, then the original and the new structure matrices are isomorphic.*

LEMMA 7 *If signs of the elements in a certain row of a structure matrix are exchanged, then the original and the new structure matrices are isomorphic.*

In practice, isomorphism of general manipulators can be checked by checking their schematic representations based on the above definitions. Notwithstanding, the definition of isomorphism could be extended. In that case, system of moments would be a vector with $3n$ elements $\mathbf{M} = [\mathbf{m}_1^T \mathbf{m}_2^T \ldots \mathbf{m}_n^T]^T$, and matrix $\mathbf{R}$ would represent a more general transformation in the $3 \times n$ dimensional space. For example, at normalized manipulators, where $\alpha_i \in \{0°, 90°, 180°, 270°\}$, $a_i = 1$ and $d_i = 0$, matrix $\mathbf{R}$ would be a stripe matrix, having rotational transformation matrices of 3×3 size in its diagonal stripe.

There are infinite number of general manipulators, and finite number of schematic manipulators, which are all isomorphic to each other. In order to represent all these isomorphic manipulators by one, it is useful to define *canonical manipulators* ($\mathcal{M}_c \subset \mathcal{M}_s$). A canonical manipulator can be unambiguously obtained from a group of isomorphic manipulators. Set $\mathcal{M}_c$ is finite at given values of n and m. The following algorithm describes a simple procedure to transform an arbitrary normalized structure matrix to its canonical form.

ALGORITHM 3 Assume, given a normalized structure matrix!

Step 1: Generate matrix $\mathbf{C}$ according to the following rule:

$$\begin{array}{llllll} \text{if} & A_{j,i} & = & 0 & \text{then} & C_{i,j} = 0 , \\ \text{if} & A_{j,i} & = & -1 & \text{then} & C_{i,j} = 1 , \\ \text{if} & A_{j,i} & = & 1 & \text{then} & C_{i,j} = 2 , \end{array} \tag{30}$$

where $i = 1, \ldots, n$; $j = 1, \ldots, m$. One row of matrix $\mathbf{C}$ can be considered as a trinomial number (c_i):

$$c_i = |C_{i,1}|C_{i,2}| \ldots |C_{i,m}|_3 , \tag{31}$$

and the whole matrix can be represented with a serial of such numbers.

Step 2: Maximize the trinomial numbers from top to down (according to rows of $\mathbf{C}$) applying the following rules:

- certain digits in a trinomial number can be changed with each other, but in the same time the same digits must be changed in all other trinomial numbers;
- in a trinomial number it is allowed to change all '1' to '2', and vice versa.

When a certain trinomial number (c_k) is under maximization, the previous trinomial numbers ($c_l, l < k$) cannot be decreased.

Step 3: The same as step 1, but in reversed direction.

EXAMPLE 3 A simple example for the application of Algorithm 3 is presented below:

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{matrix} 20103 \\ 22113 \\ 12123 \end{matrix} \rightarrow \begin{matrix} 21003 \\ 21213 \\ 11223 \end{matrix} \rightarrow \begin{matrix} 21003 \\ 21213 \\ 22113 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{bmatrix} . \quad (32)$$

The introduction of canonical manipulators is useful mainly in constructional synthesis, because the search for optimum tendon transmission of a mechanism with given D–H parameters can be cut to two fundamental steps: (1) search for optimum structure in the canonical manipulator subset; (2) search for optimum parameters in the manipulator set.

5.3. Number Analysis

The theoretically possible numbers of tendons and pulleys play an important role in constructional synthesis.

LEMMA 8 *The minimum number of tendons of a feasible manipulator is:*

$$m_{\min} = n + 1 \ . \quad (33)$$

Proof 4 Number of tendons must be more than the number of degrees-of-freedom, otherwise *Eq.* (16) would not have homogeneous solution in contrast with Lemma 2. If m is equal to $n+1$, a feasible solution always exists, for example, in accordance with the following rule:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & . & 0 & 0 \\ 1 & 1 & -1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & -1 & 0 \\ 1 & 1 & 1 & . & 1 & -1 \end{bmatrix}^T . \quad (34)$$

The above matrix is feasible, because it has a rank of n, its homogeneous solution is:

$$\mathbf{m}^H = \kappa[1 \ 2 \ \ldots \ 2^{n-1}]^T , \quad (35)$$

thus $\mathbf{m}^H$ can compensate any negative elements of the particular solution, if coefficient $\kappa \in \Re$ is big enough.

A tendon can be routed in three different ways around a joint: above, below or through an axis. Accordingly maximum number of tendons of normalized manipulator:

$$m_{\max} = 3^n - 1 . \tag{36}$$

Consider a feasible manipulator! It follows from Lemma 8, that the most distal joint should be wired around at least by two tendons, otherwise it would not be operable. The next joint must be wired around at least one more tendons, otherwise Lemma 8 would be injured for the sub-manipulator constituted by the two distal segments. Similarly, further proximal joints necessitate always one further tendon. In this way all tendons are as 'short' as possible, thus number of pulleys (t) will be minimum:

$$t_{\min} = 2n + (n-1) + (n-2) + \ldots + 1 = \frac{n^2 + 3n}{2} . \tag{37}$$

The manipulator obtained is feasible according to *Eq.* (34).

If degenerated pulleys, where a tendon crosses a joint axis, are not counted, the minimum number of tendons can be obtained from matrix (34) by replacing so much '1' elements with '0' elements as possible. That results the following feasible structure:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & . & 0 & 0 \\ 1 & 0 & -1 & . & 0 & 0 \\ . & . & . & . & . & . \\ 1 & 0 & 0 & . & -1 & 0 \\ 1 & 0 & 0 & . & 0 & -1 \end{bmatrix}^T , \tag{38}$$

accordingly the minimum number of tendons:

$$t_{\min} = 2n . \tag{39}$$

The number of pulleys is maximum, if number of tendons is also maximum and all tendons are conducted to the most distal segment. That would yield $t_{\max} = nm = n3^n - n$ tendons. On the other hand, it is not reasonable to count unnecessary pulleys, i.e. the degenerated distal pulleys, which belong to distal part of a tendon, that cross only joint axis beyond a certain joint. Consequently, the real figure becomes smaller:

$$t_{\max} = \frac{(2n-1)3^n + 1}{2} . \tag{40}$$

If degenerated pulleys are not counted at all:

$$t_{\max} = 2 \cdot 3^{n-1} \cdot n . \tag{41}$$

6. Solution of Inverse Torque Task

The inverse torque equation is ambiguous, optimum solution depends on the optimization criteria according to *Eq.* (15). Although, optimization should minimize internal forces, which increase frictional effects and necessitate higher motor torque, that purpose can be satisfied in many ways.

The min-sum criterion can be considered as an approximate energy criterion, which aims to minimize the energy consumption of the manipulator:

$$[-1 \; -1 \ldots -1]\mathbf{m} \rightarrow \min \; . \tag{42}$$

Since $c(\mathbf{m})$ is a linear function, the optimum solution of a given inverse torque task can be obtained with the methods of linear programming. Elements of the capacity vector can take negative values too, thus *Eq.* (15) can be solved as a general task with the 2-phased simplex method (GÁSPÁR and TEMESI, 1990).

The non-linear min-max criterion can be considered as a kind of mass criterion during manipulator design because of approximate proportionality between maximum current and mass of DC motors:

$$\max(m_1, m_2, \ldots, m_m) \rightarrow \min \; . \tag{43}$$

Reordering the (12) system of direct torque equations, n components of motor torque vector can be expressed as sums of linear combinations of the remainder $m-n$ components of motor torque (f_i) and linear combinations of the n components of joint torque (g_i):

$$m_{p_i} = f_i(m_{r_1}, \ldots, m_{r_{m-n}}) + g_i(\tau_1, \ldots, \tau_n) \; , \tag{44}$$

where $i = 1, \ldots, n$. The unidirectional character of the transmission can be represented with a system of inequalities:

$$f_i(m_{r_1}, \ldots, m_{r_{m-n}}) \geq -g_i(\tau_1, \ldots, \tau_n) \; , \tag{45}$$

$$m_{r_k} \geq 0 \; , \tag{46}$$

where $k = 1, \ldots, n$. *Eqs.* (45) and (46) define an intersection of m closed half-spaces limited by hyperplanes in the $(m-n)$-dimensional space, and the solution can be obtained by searching for the extreme value according to criteria (43) fulfilling *Eq.* (44). The analytical solution is often difficult, thus numerical methods must be applied.

Let us examine some special, but typical cases in more details!

EXAMPLE 4 *Symmetrical tendons* are a pair of tendons, which are wired around the joints of a manipulator symmetrically, always on the opposite side of a pulley or in case of degenerated pulley, both tendons cross joint axis. Assume an $n \times 2n$ type of normalized manipulator, which has only symmetrical tendons according to the following composition rule:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 & -1 & . & 1 & -1 \\ 0 & 0 & 1 & -1 & . & 1 & -1 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 1 & -1 \end{bmatrix}^T . \tag{47}$$

According to *Eq.* (19), the solution of inverse torque equation is a sum of particular and homogenous parts:

$$\mathbf{m} = \begin{bmatrix} 1 & -1 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 1 & -1 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 0 & 1 \\ 0 & 0 & 0 & . & 0 & 0 \end{bmatrix} \boldsymbol{\tau} + \kappa_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ . \\ 0 \\ 0 \end{bmatrix} + \kappa_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ . \\ 0 \\ 0 \end{bmatrix} + \ldots + \kappa_n \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ 1 \\ 1 \end{bmatrix} , \tag{48}$$

where the $m \times n$ type of matrix $(\mathbf{A}^{-T})$ on the right hand side is pseudo inverse of $\mathbf{A}^T$. Optimum solution is obtained based on either the min-sum or the min-max criteria, if:

$$\kappa_n = \max(0, -\tau_n) , \tag{49}$$

$$\kappa_j = \max(0, -(\tau_j - \tau_{j+1})) , \tag{50}$$

where $j \in \{1, 2, \ldots, n-1\}$. In some steps, the solution of the inverse position task can also be expressed with the above pseudo inverse:

$$\phi = \mathbf{A}^{-1}\mathbf{q} . \tag{51}$$

Similar equation can be applied for velocity.

EXAMPLE 5 Given a $n \times (m+1)$ type of manipulator. Let us extend the structure matrix with vector $\mathbf{c}$, which contains only constant $c > 0$, $c \in \Re$ elements. For the direct torque equation it yields:

$$\begin{bmatrix} \tau \\ \tau_p \end{bmatrix} = \begin{bmatrix} \mathbf{A}^T \\ \mathbf{c}^T \end{bmatrix} \mathbf{m} , \tag{52}$$

where $\boldsymbol{\tau}_p$ is a pseudo torque, used for calculation purposes. In a brief form:

$$\boldsymbol{\tau}^* = \mathbf{B}^T \mathbf{m} . \tag{53}$$

where $\boldsymbol{\tau}^*$ is the modified joint torque vector and $\mathbf{B}$ is the modified structure matrix.

Assume that row m of matrix $\mathbf{B}^T$ can be expressed as a linear combination of the other rows:

$$\lambda_1 \mathbf{a}_1 + \lambda_2 \mathbf{a}_2 + \ldots + \lambda_n \mathbf{a}_n = \mathbf{c} , \tag{54}$$

where $\lambda_1, \ldots, \lambda_n \in \Re$, and $\lambda_1 \neq 0$. After rearrangement it yields:

$$\left(-\frac{\lambda_2}{\lambda_1}\right) \mathbf{a}_2 + \ldots + \left(-\frac{\lambda_n}{\lambda_1}\right) \mathbf{a}_n + \mathbf{c} = \mathbf{a}_1 . \tag{55}$$

Taking *Eq.* (27) also into account, τ_1 could not be smaller than a certain linear combination of other motor torque components, which is impossible at a feasible manipulator. It follows that vector $\mathbf{c}$ is independent from the other rows of $\mathbf{B}^T$, thus the modified structure is regular.

Consequently, the inverse torque equation can be obtained with matrix inversion:

$$\mathbf{m} = \mathbf{B}^{-T} \boldsymbol{\tau}^* , \tag{56}$$

which can be decomposed to a particular and a homogeneous part:

$$\mathbf{m} = \mathbf{B}_{\text{sub}}^{-T} \boldsymbol{\tau} + \tau_p \mathbf{b} , \tag{57}$$

where matrix $\mathbf{B}_{\text{sub}}^{-T}$ contains the first n, and vector $\mathbf{b}$ the last column of matrix $\mathbf{B}^{-T}$. Because of unidirectionality, any rows of *Eq.* (57) must be non-negative. That yields m conditions, which can be fulfilled with a minimum value of τ_p, if:

$$\tau_p = \max_{j=1,\ldots,m} \left(-\frac{\sum_{i=1}^{n} B_{j,i}^{-T} \tau_i}{B_{j,m}^{-T}} \right) . \tag{58}$$

It must be noted that any component of the motor torque vector is minimum, if the pseudo torque is minimum, thus both the min-max and the min-sum criteria supply the same result in the case of $m \times (n+1)$ type of manipulators.

In algebraic way or with the law of energy conservation it can be proved that the inverse position equation can be expressed in the following form:

$$\mathbf{q} = \mathbf{B}_{\text{sub}}^{-1} \boldsymbol{\phi} , \tag{59}$$

where index 'sub' indicates, that the last row of matrix $\mathbf{B}^{-1}$ is eliminated. The given value of vector $\boldsymbol{\phi}$ must be feasible with respect to *Eq.* (14). Similar inverse equation applies for velocity.

7. Synthesis of Open-Ended Tendon Structures

It is a typical design problem that a transmission system with optimum features must be synthesized for a given mechanism. A solution is presented in below.

ALGORITHM 4 Assume that there is a mechanism, and its HD parameters are given.
Step 1: Generation of all possible normalized structure matrices at given values of m and n.
Step 2: Selection of feasible structures according to Algorithm 2. In order to reduce computations, previous filtering can be done based on Lemmas 5 and 1.
Step 3: Selection of structures, which are not isomorphic with each other according to Algorithm 3.
Step 4: Previous selection of admissible structures based on heuristic criteria.
Step 5: Optimization of geometrical parameters for admissible structures, selection of the best of them.
Step 6: Construction of the optimum manipulator.

Heuristic criteria in Step 4 can be for example the following:

- cost constraint: number of pulleys should not be too many,
- symmetry constraint: value of maximum motor torque should not depend too much on the direction of the force acting on the manipulator.
- proportionality constraint: number of pulleys at a joint (t_i) should decrease to distal direction (e.g. $t_i \leq 2(n - i)$).

The optimum structure can be found with a more detailed analysis in Step 5. For example, behaviour of the admissible structures can be analyzed with simulation in order to select the structure, which is optimum by the following criteria:

- minimum mass criterion: selection of structures which necessitates the lowest maximum motor torque, thus mass of the motors can be minimum,
- modularity constraint: selection of structures which necessitates about the same order of maximum torque at each motor,
- production constraint: diameters of pulleys should fit a certain selection,
- cost constraint: number of different pulley diameters should not be too many.

The above algorithm was verified during the design of the transmission system of the TUB-PC dextrous hand. In fact, only the three distal rotational joints of a finger were respected ($n = 3$), because the proximal translational joint is operated in a different manner (manually). In Step 1, $m \in \{4, 5, 6\}$ was taken, which resulted eight 3×6 type, four 3×5 type and one 3×4 type of admissible schematic structures in Step 4. According to Step 5, the best of them was a 3×4 type, selected with static simulation based on traditional robotic methods (LANTOS, 1997). The construction of a real finger is presented in *Fig. 7*.

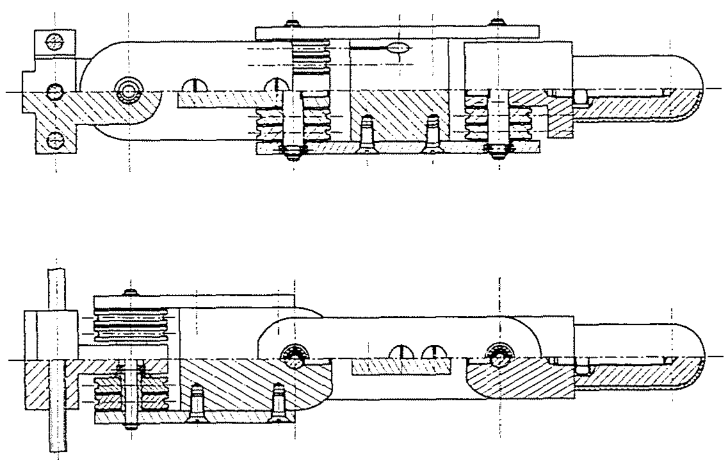

Fig. 7. The construction of a finger of the TUB-PC dextrous robot hand

8. Description of Real Manipulators

So far it was assumed that tendons are rigid, which is not true in practice. If elastic tendons are considered, spring constants of the tendons (k_j) must also be respected with a stiffness matrix: $\mathbf{K} = \mathbf{Diag}(k_j)$. For example, at the TUB-PC finger, the following equations can be obtained:

$$\phi = \mathbf{A}\mathbf{q} + \frac{1}{R_m^2}\mathbf{K}^{-1}\mathbf{B}^{-T}\tau^* , \tag{60}$$

$$\mathbf{q} = \mathbf{B}_{\mathrm{sub}}^{-1}\phi - \frac{1}{R_m^2}\mathbf{B}_{\mathrm{sub}}^{-1}\mathbf{K}^{-1}\mathbf{m} , \tag{61}$$

where R_m denotes radius of motor pulleys. Torque equations are the same as *Eqs.* (12) and (56). To make the model even more exact, effect of mass and of friction could be taken also into account.

9. Summary

The field of open-ended tendons is interesting, because the theory can be directly applied in the design of various robotic equipment. The article aimed to generalize known results with a few restriction only, since the primary goal of the work was to elaborate an efficient algorithm for the design synthesis of dextrous hand transmissions. One challenging problem for future researches is to generalize the description of real manipulators with respect not only to tendon elasticity, but also to mass, friction and even the non-linear features of the transmission to enable dynamic analysis of open-ended tendon transmissions.

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