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THE EQUATION OF MOTION OF MECHANICAL SYSTEMS BASED ON D'ALEMBERT–LAGRANGE'S EQUATION

Dedicated to Professor Franz Ziegler
on his 60-th birthday

Gyula BÉDA

Department of Technical Mechanics
Technical University of Budapest
H–1521 Budapest, Hungary

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Abstract

The equations of motion of mechanical (discrete or continuous) systems can be deduced from d'Alembert–Langrange's equation. The equation of motion of micropolar body is obtained on the continuous bodies. More conclusions and questions are given from the presented arithmetic.

Keywords: equation of motion, discrete and continuous system, micropolar body.

1. Introduction

A lot of forms of equation of motion are known in the mechanics, for example Newton–Euler's, Lagrange's, Appell's, Cauchy's equation, etc.

We will use the generalized d'Alembert–Langrange's equation [1]. This equation is valid on both the whole and a part of the material system.

Consider a material system (*Fig. 1*) [2]. This system consists of a lot of macro-mass elements dm which contain many micro-mass elements dm'. These elements cover the whole material system. This system has got mass m it is placed in domain V.

The generalized d'Alembert–Langrange equation [1, 3] in case of an arbitrary micro-mass element is

$$\mathbf{C'} \otimes (d\mathbf{F}' - \dot{\mathbf{v}}' dm') = 0, \tag{1}$$

where $\mathbf{C'}$ is a tensor of arbitrary rank, $d\mathbf{F}'$ is the force, $\mathbf{v}'$ is velocity and $\dot{\mathbf{v}}'$ is acceleration of a point of the micro element. $\mathbf{C'}$, dF' and $\dot{\mathbf{v}}'$ depend on the position vector $\mathbf{r}'$ and the time t. The notation $\otimes$ means an arbitrary multiplication.

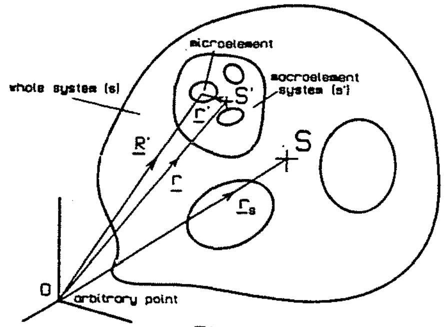

Fig. 1.

S' is the centre of mass of micro-mass elements (s') in macro-mass element (*Fig. 1*). That is

$$\sum_{dm} \mathbf{r}' dm' = 0 \quad \text{or} \quad \int_{dm} \mathbf{r}' dm' = 0$$

and

$$\sum_{dm} dm' = dm \quad \text{or} \quad \int_{dm} dm' = dm,$$

in case of discrete or continuous micro-mass elements. The first moment of a macro-mass element dm on an arbitrary point 0 is

$$\int_{dm} (\mathbf{r} + \mathbf{r}') dm' = \int_{dm} \mathbf{r} dm' + \int_{dm} \mathbf{r}' dm' = \int_{dm} \mathbf{r} dm' = \mathbf{r} dm.$$

The integral is Stieltjes integral now and in the following [3]. The first moment of the whole material system is (*Fig. 1*)

$$\int_m \int_{dm} (\mathbf{r} + \mathbf{r}') dm' = \int_m \mathbf{r} dm.$$

Eq. (1), in case of a macro-mass element, is

$$\int_{dm} \mathbf{C}' \otimes (d\mathbf{F}' - \dot{\mathbf{v}}' dm') = 0$$

and *the generalized d'Alembert–Lagrange equation* on the whole material system is

$$\int_m \int_{dm} \mathbf{C}' \otimes (d\mathbf{F}' - \dot{\mathbf{v}}' dm') = 0 \tag{2a}$$

or by integrating it with respect to time from t_1 to t_2

$$\int_{t_1}^{t_2}\int_{m}\int_{dm} \mathbf{C}' \otimes (d\mathbf{F}' - \dot{\mathbf{v}}' dm' dt) = 0. \tag{2b}$$

2. Discrete Material System

The micro element is a material point which has got mass m' with force $\mathbf{F}'$. Using *Eq.* (2a) we obtain

$$\sum_{(s)}\sum_{(s')}[\mathbf{C}' \otimes (d\mathbf{F}' - \dot{\mathbf{v}}' m')] = 0, \tag{3}$$

where (s) is the full material point system and its part is (s'). Parts (s') are disjunctive. **C'** is now equal to $\Phi'\mathbf{I} + \Psi'\mathbf{R}'\mathbf{I}$ and $\otimes$ means tensorial (or dyadic) multiplication which is marked by two side by side written tensors or vectors. **I** is the unit tensor, Φ' and Ψ' are arbitrary scalar and vector functions. The Ψ' depends on time t. **R'** is the position vector (*Fig. 1*). Now *Eq.* (3) is

$$\sum_{(s)}\sum_{(s')}(\Phi'\mathbf{I} + \Psi' \cdot \mathbf{R}'\mathbf{I})(\mathbf{F}' - \dot{\mathbf{v}}' m') = 0$$

that is,

$$\mathbf{I}\sum_{(s)}\sum_{(s')} \Phi'(\mathbf{F}' - \dot{\mathbf{v}}' m') + \Psi' \sum_{(s)}\sum_{(s')}(\mathbf{R}'\mathbf{F}' - \mathbf{R}'\dot{\mathbf{v}}' m') = 0$$

but $\mathbf{R}' = \mathbf{r} + \mathbf{r}'$, and Φ' and Ψ' are arbitrary functions. We obtain

$$\sum_{(s)}\sum_{(s')}(\mathbf{F}' - \dot{\mathbf{v}}' m') = 0,$$

that is,

$$\mathbf{F} - m\dot{\mathbf{v}}' = 0 \tag{4a}$$

and

$$\sum_{(s)}[\mathbf{r}\sum_{(s')}(\mathbf{F}' - \dot{\mathbf{v}}' m') + \sum_{(s')}(\mathbf{r}'\mathbf{F}' - \mathbf{r}'\dot{\mathbf{v}}' m') = 0,$$

that is,

$$\mathbf{r}_s\mathbf{F} - \mathbf{r}_s\dot{\mathbf{v}}_s m + \mathbf{M}_s - \mathbf{D}_s = 0, \tag{4b}$$

where S is the centre of mass of the full system and

$$\mathbf{r}_s\mathbf{F} \equiv \sum_{(s)}\mathbf{r}\sum_{(s')}\mathbf{F}', \quad \mathbf{r}_s\dot{\mathbf{v}}'m' \equiv \sum_{(s)}\mathbf{r}\sum_{(s')}\dot{\mathbf{v}}m, \quad \mathbf{M}_s \equiv \sum_{(s)}\sum_{(s')}\mathbf{r}'\mathbf{F}' \text{ and}$$

$$\mathbf{D}_s \equiv \sum_{(s)}\sum_{(s')}\mathbf{r}'\dot{\mathbf{v}}'m'.$$

We write a vector product instead of tensorial multiplication and by using that $\mathbf{M}_s + \mathbf{r}_s \times \mathbf{F}$ is equal to the moment of forces $\mathbf{M}_0$ on point 0 and similarly $\mathbf{D}_s + \mathbf{r}_s \times \dot{\mathbf{v}}_s m$ is equal to the moment of kinetic vector $\mathbf{D}_0$ on point 0. Finally we obtain

$$\mathbf{D}_0 = \mathbf{M}_0. \tag{4c}$$

Eqs. (4a) and (4c) are the *Newton–Euler equations of motion.*

The system is a free one. It does not contain any constraints. A system often contains constraints. Generally the properties of the constraint forces are unknown. This is an important problem in mechanics.

A group of the constraint forces satisfies the principle of virtual work, that is, $\sum_{(s)}\sum_{(s')}\mathbf{K}\cdot\delta\mathbf{r}' = 0$. $\mathbf{K}'$ is the constraint force and $\delta\mathbf{r}'$ is the virtual displacement. Using *Eq.* (2b) we obtain

$$\int_{t_1}^{t_2}\sum_{(s)}\sum_{(s')}\delta\mathbf{r}'(\mathbf{F}' - \dot{\mathbf{v}}'m')dt = 0. \tag{5}$$

Tensor $\mathbf{C}'$ is the virtual displacement and $\otimes$ is the scalar product between two vectors which is denoted by point. $\mathbf{F}'$ does not contain the constraint forces because zero is their virtual work. The Lagrange's second equation follows from *Eq.* (5).

Another group of constraint forces satisfies Gauss' principle, that is, $\sum_{(s)}\sum_{(s')}\mathbf{K}'\cdot\delta\dot{\mathbf{v}}' = 0$, $\delta\dot{\mathbf{v}}$ is the virtual acceleration. By using *Eq.* (3) similarly we obtain Appell's equation

$$\frac{\partial S}{\partial q} = Q_k \quad (k = 1,\dots n),$$

where S is the Appell's function, q_k are the generalized coordinates and Q_k are the generalized forces.

These equations can comprise the rigid body, too.

3. Continuous Body

Let us see the whole system (*Fig. 1*) as a continuum. Using *Eq.* (2a) we can obtain equations of motion of continuum when $d\mathbf{F}'$ is equal to $\frac{1}{\rho'}\sigma'.\nabla$, the divergence of stress tensor σ [4] and $\mathbf{C}'$ is equal to $\Phi'\mathbf{I} + \Psi'\mathbf{R}'\mathbf{I}$ and $\otimes$ means tensorial multiplication as previously. When $dm' = \rho' dV'$ and $dm = \rho dV$ *Eq.* (2a) will be [2]

$$\int_V \Big[\int_{dV} \Phi'(\sigma' \cdot \nabla' + q' - \rho'\dot{\mathbf{v}}')dV' \Big] dV'\mathbf{I} +$$

$$+\Psi' \cdot \int_V \int_{dV} \Big[\mathbf{R}'(\sigma' \cdot \nabla) + \mathbf{R}'q' - \mathbf{R}'\dot{\mathbf{v}}'\rho' \Big] dV'\mathbf{I} = 0, \tag{6}$$

where ρ is mass density, V is volume of continuum and q' is body force.

Transforming the first term of the first integral we obtain,

$$\int_{dV} \Phi'(\sigma' \cdot \nabla')dV' = \int_{dV} \Big[(\Phi'\sigma')\nabla' - \sigma' \cdot (\nabla'\Phi') \Big] dV' =$$

$$\int_{dA} \Phi'\sigma' \cdot dA' - \int_{dV} \sigma' \cdot (\nabla'\Phi')dV',$$

dA is the surface of a macro element.

The $\Phi', \Psi', \mathbf{R}'$ are arbitrary. The first integral of *Eq.* (6) is

$$\int_A \int_{dA} \Phi'\sigma' dA' + \int_V \Big[\int_{dV} -\sigma' \cdot (\nabla'\Phi') + q' - \rho'\dot{\mathbf{v}}' \Big] dV' = 0$$

when Φ' is equal to constant. This equation will be

$$\int_V [\sigma \cdot \nabla + q - \rho\dot{\mathbf{v}}]dV = 0, \quad \text{that is} \quad \sigma \cdot \nabla + q = \rho\dot{\mathbf{v}}, \tag{7a}$$

where

$$\int_{dA} \sigma' \cdot dA' \equiv \sigma \cdot dA, \int_{dV} q' dV' \equiv q dV,$$

$$\int_{dV} \rho'\dot{\mathbf{v}}' dV' \equiv \rho\dot{\mathbf{v}}' dV [2].$$

The second integral of *Eq.* (6) is transformed similarly as the first one's. The second integral is equal to zero, that is,

$$\int_{dV} [\mathbf{r}(\sigma \cdot \nabla + q - \rho\dot{\mathbf{v}}) + \sigma - \mathbf{S} + \lambda \cdot \nabla + \ell - \rho\dot{\Pi}]dV = 0$$

from this and *Eq.* (7a)

$$\sigma - \mathbf{S} + \lambda \cdot \nabla + \ell - \rho\dot{\Pi} = 0 \tag{7b}$$

where the notations are used [2]

$$\int\limits_{dV} \sigma' dV' \equiv \mathbf{S} dV, \int\limits_{dA} \mathbf{r}'\sigma' \cdot dA' \equiv \lambda dA,$$

$$\int\limits_{dV} \mathbf{r}' q' dV' \equiv \ell dV, \text{and} \int\limits_{dV} \rho' \mathbf{r}' \dot{\mathbf{v}} dV \equiv \rho\dot{\Pi} dV.$$

σ, $\mathbf{S} = \mathbf{S}^T$, λ and Π functions are unknown in *Eqs.* (7a) and (7b).

4. Conclusions

The equation of motion of discrete and continuous systems can be determined from the generalized d'Alembert–Lagrange's equation.

The surface force of the micro-mass element has to be expressed as density of body force.

Basic equations cannot be written in cases of discrete and continuous systems thus further equations are needed for example principle of virtual work or Gauss' principle or Hooke-law or generally a constitutive equation.

• The equation of motion is given for the continuum as the equation of motion of micropolar body.

5. Questions

We wondered if the stress tensor could characterize the micro-element only?
Why does force–couple system break?
How could we keep this force–couple system?

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