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ON DIRICHLET–VORONOI CELL
Part II. Diagrams[1]

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Abstract

This second part of my paper discusses the determination of the DIRICHLET–VORONOI cell of a lattice of dimension n. We give some concrete D–V cells and their automorphism groups, the cells of the lattices A_n, D_n and E_n, respectively. These lattices are root lattices to the corresponding finite root systems A_n, D_n, E_n. They have an important role in crystallography, physics and differential geometry. We illustrate the diagram method for visualizing very symmetric higher dimensional polytops.

Keywords: lattice of dimension n, DIRICHLET–VORONOI cell.

1. Automorphism Group of a Lattice

One of the most important tools in the deep investigations of discrete point system is the concept of its symmetry group. In general, a symmetry of a discrete point system of the Euclidean n-space is an orthogonal linear transformation which maps the system onto itself. In the case when the point system is a lattice, we define first the translations L which carry an origin 0 into the other lattice points. But it is more characteristic for L to consider a transformation fixing the origin which maps an integral vector base onto another integral base of the lattice. These transformations form a group with the composition as group operation, written multiplicatively. If L is a lattice with an integral base $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ then we can write its matrix M with respect to a standard orthonormed basis $\{\mathbf{e}_i\}$ of the Euclidean space. So we have $\mathbf{b}_i = \sum_{j=1}^{n} \mathbf{e}_j M_i^j$, where M_i^j is the j^{th}-element of the i^{th}-column of the matrix M. (Using the Einstein convention we shortly write that $\mathbf{b}_i = \mathbf{e}_j M_i^j$.) This matrix M is called a *generator matrix* of L. An orthogonal linear transformation (with matrix) $B : \mathbf{e}_i \longrightarrow B\mathbf{e}_i$ is a symmetry of L if and only if it carries the element $\mathbf{b}_i$ of $\mathcal{B}$ into another

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lattice vector $\mathbf{b}'_i = B\mathbf{b}_i = B \cdot (\mathbf{e}_j M_i^j) = (B\mathbf{e}_j)M_i^j = \mathbf{e}_k B_j^k M_i^j$ and these image vectors give another integral basis of the lattice. But $\mathbf{b}_i = \mathbf{e}_j M_i^j$ and there is an integral unimodular matrix U whose j^{th} column U_j gives the integral coordinates of the j^{th} element of the new basis $\{\mathbf{b}'_i, i = 1 \dots n\}$ with respect to the base $\mathcal{B}$, thus we have $\mathbf{b}'_i = \mathbf{b}_l U_i^l = \mathbf{e}_j M_l^j U_i^l$. This means that $\mathbf{e}_j M_l^j U_i^l = \mathbf{b}'_i = \mathbf{e}_k B_j^k M_i^j$ or $M_l^j U_i^l (M^{-1})_k^i = B_k^j$ which means that

$$MUM^{-1} = B.$$

The group of these transformations (for a fixed base $\{\mathbf{e}_i\}$) is the *automorphism group* (or symmetry group or point group or finite Weyl group) of the lattice. In the usual Euclidean space it can be proved that this group denoted by $Aut(L)$ is finite. On some occasions we consider the infinite group of all distance-preserving transformations of the underlying space that take the lattice to itself. This is obtained by adjoining the set L of translations, written in additive lattice vectors to the former $Aut(L)$. This is the *group of complete Euclidean affine automorphisms* (or infinite Weyl group) of the lattice (lattice means point lattice for brevity).

The automorphism groups of the lattices in dimensions up to 8 are especially interesting: they have subgroups of low index that are reflection groups. An orthogonal transformation A of the n-space E^n is a *general reflection* iff there is a k-dimensional subspace N of E^n such that $A|_N = I|_N$ and $A|_{N^\perp} = -I|_{N^\perp}$, where $N^\perp$ is the orthogonal complement of N and the transformation I is the identity, $-I$ is the inversion (point reflection in the origin) of the space. A usual *hyperplane reflection* (or briefly reflection) is a general reflection with $k = n-1$. We can specify each reflecting hyperplane by a minimal lattice vector orthogonal to it, and sometimes thus we get a generator system of the lattice. Then we say that the mentioned lattice vector is a *root vector*, the lattice is a *root lattice* and the subgroup of $Aut(L)$ generated by the reflections in the roots is the *reflection subgroup* of $Aut(L)$.

It is clear that the elements of the symmetry group leave invariant the D–V cells of the lattice L, which means that some of these transformations give the symmetries of a fixed one. Thus $Aut(L)$ can be determined as the group of symmetries of the D–V cell (see e.g. [11]).

For instance, if the lattice is the n-dimensional cubic lattice Z^n of the integer vectors in the standard orthonormed basis, the cell of the origin is the cube with 2^n vertices of all coordinates $\pm\frac{1}{2}$. The symmetry group of this cube can be described by all permutations and sign changes of these coordinates and the automorphism group of Z^n contains precisely these symmetries. As the walls of the D–V cell give reflecting hyperplanes of Z^n we have already n vectors corresponding to these hyperplanes which generate the lattice. Thus this lattice Z^n is a root lattice, the reflection subgroup (denoted by C_n) and the root system is generated by the roots:

$$(1,0,\cdots,0)^T,(1,-1,\cdots,0)^T,(0,1,-1,\cdots,0)^T,\ldots,(0,\cdots,1,-1)^T$$

which are called *fundamental roots*. The fundamental roots give an integral basis of the lattice. We remark that the reflections corresponding to the fundamental roots now generate the automorphism group of the lattice which means that $Aut(Z^n)$ is a so-called (finite) reflection group.

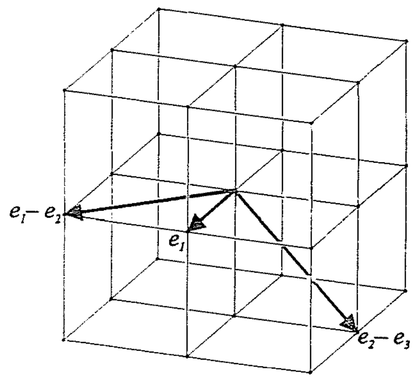

Fig. 1. The fundamental roots of Z^n

2. Coxeter Groups in Spherical and in Euclidean Spaces

As we saw in the previous paragraph the groups generated by reflections are very important for the automorphism group of a root lattice. Moreover, the infinite Euclidean reflection groups always have invariant lattices. In this section we characterize the finite reflection groups independently of the lattices. This problem has a complicated interesting history, see e.g. [7]. In [3] and [4] COXETER gave the Euclidean representatives of abstract groups defined by the relations

$$(R_iR_j)^{p_{ij}} = I, \quad p_{ij} \in \mathbf{Z}, \quad p_{ii} = 1.$$

His idea was the following: an element of such a group can be represented by a reflection in a hyperplane of an n-dimensional Euclidean space and the required assumption says that the product of two reflections is a rotation of finite order about the intersection $n-2$-subspace through twice the angle between the reflecting hyperplanes. This means that the angle between the mirror hyperplanes is commensurable with π. (Since the images of any point are distributed on a circle of the plane orthogonal to the $n-2$-dimensional point-wise fixed axis plane of the rotation.) It will be sufficient to consider submultiples of π so the dihedral angle between two hyperplanes is $\frac{\pi}{p_{ij}}$. Regarding the generating mirror hyperplanes and all their transforms, we get a partition of the space into a finite or infinite number of congruent convex regions; and the examined group is generated by reflections in the bounding hyperplanes of any one of the regions. (In our finite case each reflection is one in an $n-1$-subspace, the examined hyperplanes are concurrent, the common point is the origin, thus we have finite number of congruent copies of such infinite regions.)

To determine the number and the structure of generating hyperplanes we have to introduce the concept of irreducibility. If all the elements of the group leave invariant an m-dimensional subspace (through the origin) $(1 \leq m \leq n-1)$, they also leave invariant the completely orthogonal $(n-m)$-subspace, the group is then called *reducible* otherwise it is *irreducible*. In the first case the reflecting hyperplanes fall into two sets: some containing the m-space and the others containing the $(n-m)$-space in common. If these two sets are non-empty, the normal vectors of the generating reflections span the whole space. Thus the orbit of any point of the space is not parallel to a hyperplane so the group cannot be represented in a less dimensional space. Hence the reflection group is *non-degenerated* and the number p_{ij} is two whenever R_i is in one set and R_j in the other. This means that the group is the direct product of groups generated by the separate sets. (The irreducible reflection group does not have such a direct product decomposition, the other one must have.)

If the reflection group is non-degenerated the normal vectors of the generating hyperplanes span the whole space. Thus the number of generators is greater than or equal to n. Consider a fundamental region and a point O within it. Denote by $\mathbf{e}_i$ $i = 1 \dots m$, $m \geq n$ the set of unit normal vectors of the generating hyperplanes starting in directed inwards from the walls. The quadratic form

$$\sum_{i=1}^{m} \sum_{j=1}^{m} < \mathbf{e}_i \cdot \mathbf{e}_j > x^i x^j$$

is not negative because it expresses the length square of the variable vector $\sum x^i \mathbf{e}_i$. This means that it is a positive definite or semidefinite quadratic form in m variables. But each (symmetric) coefficient $e_{ij} :=< \mathbf{e}_i \cdot \mathbf{e}_j >= e_{ji}$ is

$$-\cos\left(\frac{\pi}{p_{ij}}\right) \leq 0 \quad i \neq j$$

since the angle between two $\mathbf{e}_i$'s is the supplement of the dihedral angle of the fundamental region. Examining the definiteness and rank of this quadratic form (see [5]) we can distinguish two cases. In the first one the quadratic form is semidefinite and has rank n and nullity 1, i.e. $m = n+1$. The fundamental domain is a Euclidean simplex where one normal vector is (negative) linear combination of the others. In the second case is $n = m$, the form is positive definite. The fundamental domain is a corner domain with n walls through a point and the group acts on every $n-1$-sphere with centre of the common point. Thus we have:

Theorem 1 ([5]) *The fundamental region of a finite group generated by reflections of the Euclidean n-space is a spherical simplex, and that of an irreducible infinite group generated by reflections is a Euclidean simplex. Furthermore, every group generated by reflections in a Euclidean space is a direct product of groups whose fundamental regions are Euclidean and spherical simplices.*

Now we deduce the types of the finite groups generated by reflections from the possible fundamental domains. Before enumerating the particular groups we remark that an infinite discrete reflection group arises from a finite reflection group. In fact, the reflecting hyperplanes determine a finite number of different directions because the angles of the hyperplanes cannot be arbitrarily small by discreteness. Thus the hyperplanes belong to a finite number of families each consisting of parallel hyperplanes. Representing every family by a single hyperplane through any fixed point O of the space and parallel to the element of the family, we have a finite group S generated by these reflections. The fundamental domain of S is a simplex of an $n-1$-dimensional spherical space or n-hedral angle so we have a generator system of n elements. A fundamental domain of the original infinite group G is bounded by the n-hyperplanes from those of the finite group S and by some others. (Thus the fundamental region of S occurs at one corner of the fundamental region of G.) This means any fundamental domain for G has at least one vertex which lies in one hyperplane of every family. Let us call this (by [3]) a *special* vertex of the fundamental region and S a *special* subgroup of G. (In general, this is the so-called *point group* of the considered space group which is a crystallographic subgroup of the isometries of E^n.) For example, for a group generated by reflections in the walls of a brick all vertices are special and the special subgroup of order 2^n generated by reflections in any n orthogonally concurrent walls. This is the largest finite subgroup of the considered infinite group. Another instance with $n = 2$ is the complete symmetry group of the regular triangular lattice generated by reflections in the sides of a ruler triangle. We have only one special vertex (where the angle $\frac{\pi}{6}$ occurs). The largest finite subgroup (the point group) has order 12.

Now, we have reduced the enumeration of discrete groups generated by reflections to that of spherical and Euclidean simplexes whose dihedral angles are submultiples of π. For any fundamental simplex we introduce a graph or diagram whose nodes represent the walls and whose p-marked edge (branch) indicates a pair of walls inclined in angle $\frac{\pi}{p}$, $p > 2$. Any perpendicular wall pair is represented by two nodes not joined by a branch. This means that the number of nodes is equal to n or $n+1$ and the graph is connected or disconnected according to that the group is irreducible or reducible. We remark that this diagram has been introduced first by Coxeter, we use now first a refined version in *Fig. 2* which contains a little bit more information on the simplex and the lattice related to it. This is the so-called *Coxeter–Dynkin* diagram introduced by Dynkin.

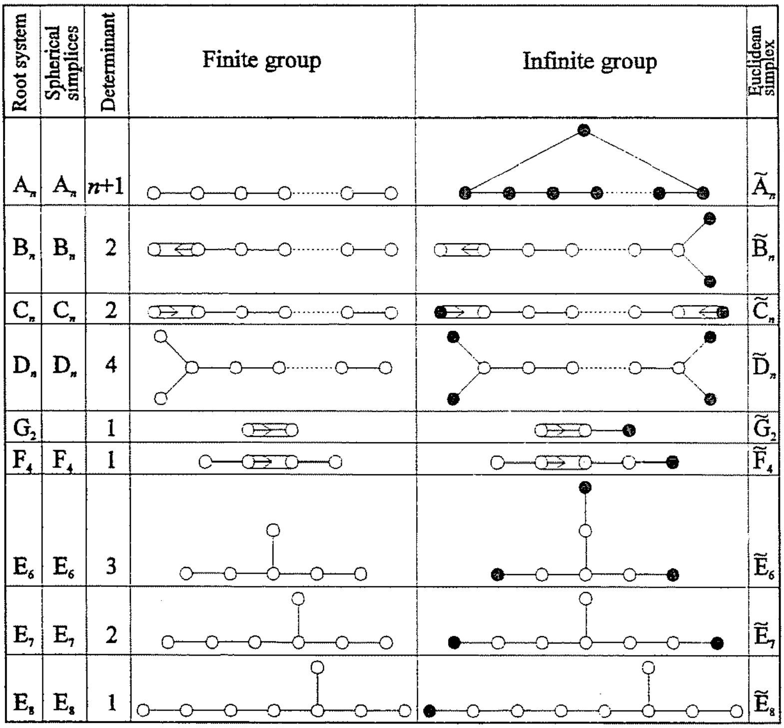

Fig. 2. Diagrams (graphs) for fundamental simplices of irreducible finite spherical and infinite Euclidean reflection groups

In this diagram we have the following conventions: the nodes representing the roots $\mathbf{r}$ and $\mathbf{s}$ are joined by a k-fold (directed) edge if they have the angle Θ where $4\cos^2\Theta = k$ and $\cos\Theta \leq 0$, the right-most roots in the fourth column have norm 2, and the lengths of the other roots are determined by the rule that a k-fold edge, whose arrowhead (if any) pointing from $\mathbf{r}$ to $\mathbf{s}$ indicates that $< \mathbf{r}, \mathbf{r} >= k < \mathbf{s}, \mathbf{s} >$.

Each Coxeter diagram of Euclidean reflection groups has two important properties:

1 The removal of any node (together with any branches which emanate from that node) leaves the graph for a spherical simplex, because $n-1$ of the n walls form the angular region at one vertex of the original simplex independently from the fact that this is Euclidean or spherical one. So by adding a fresh node to the graph of a Euclidean simplex we can never obtain a new admissible connected graph.

2 New Euclidean connected Coxeter diagram cannot be obtained by inserting a branch between two nodes already present, nor by increasing the mark on a branch, because such a positive semidefinite connected quadratic form becomes indefinite when any of its coefficients are decreasing (see e.g. [5]).

These principles help give the standard list of Euclidean simplices which can be found on the fifth column of the figure (*Fig. 2*). All these diagrams represent quadratic forms being semidefinite thus they represent Euclidean simplices. We note that in this list there are simplices which give the same finite group (e.g. B_n and C_n), however, the root lattices are different, and so are the infinite groups obtained by adjoining the translations to the finite group. From these Euclidean simplices we can derive spherical reflection simplices by drawing spheres round the vertices. In practice this means that we remove such a node from each graph which leaves the graph connected. (We consider the irreducible groups only and $n \geq 3$.) The total list of the spherical Coxeter simplices can be found also in *Fig. 2*. The fact that this list is complete follows from the above two properties of the admissible graphs.

3. The Fundamental Simplices of the Affine Weyl Groups $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$ and their Root Lattices

In this section we give the fundamental simplices of the root lattices (see e.g. [1]). First we recall an important theorem :

Theorem 2 ([1]) *For any root lattice L, the D–V cell around the origin is the union of the images of the fundamental simplex of $W_a(L)$ under the finite reflection group or Weyl group $W(L)$ of the lattice L.*

Thus the D–V cell of a root lattice L can be and will be described by the fundamental simplex of $W_a(L)$.

Sketch of the proof: In this case the fundamental domain of the infinite or affine Weyl group $W_a(L)$ is a simplex S. It has the property that the origin is the closest lattice point to any interior point of it. In fact, if a reflecting hyperplane separates an interior point $\mathbf{x}$ from the closest lattice point $\mathbf{u}$ then the lattice point $\mathbf{u}'$ which is the mirror image of $\mathbf{u}$ in this hyperplane is closer to $\mathbf{x}$ as $\mathbf{u}$. Thus for every element g of the group $W(L)$ the origin is a fixed point. Let now $\mathbf{x}$ be any point of the D–V cell around the origin. Then $\mathbf{x}$ is an element of an image $g(S)$ of the fundamental simplex S for some $g \in W_a(L)$ assuming that $\mathbf{x}$ is an interior point of $g(S)$ we get by the remark above that g is in $W(L)$ and the D–V cell covered by the images of the fundamental simplex under the elements of the finite Weyl group $W(L)$. The converse statement follows by reversing the steps and discussing the case when $\mathbf{x}$ is a boundary point of $g(S)$, the origin is one of the closest lattice points, too.

A simple consequence of this theorem is that the relevant vectors of the D–V cell of a root lattice are the minimum vectors of the lattice. In fact, the $|W(L)|$ copies of the fundamental simplex fill the cell, thus the images of the facet of the simplex do not contain the origin also filling the surface of the D–V cell. So the lengths of the relevants are equal, i.e. they are the minima of the lattice. (Each of the minimum vectors is a relevant.)

Using the diagram of the affine group we can choose the normal vectors of the facets of the fundamental simplex. We now search for the corresponding root lattices as the sublattices of the n-dimensional cubic lattice, so the scalar products of any two lattice vectors are integer. Since any of the root vectors are parallel to these normal vectors (by the above remark) the root vector with minimal norm corresponds to the special vertex of the diagram.

3.1 $\tilde{A}_n$ and the Corresponding Root Lattice A_n

For a unified method we consider an orthonormed basis $\{\mathbf{e}_0, \dots, \mathbf{e}_n\}$ of the Euclidean space E^{n+1}, and we will define A_n and $\tilde{A}_n$ in an n-dimensional subspace of E^{n+1} through the origin. From the general diagram of $\tilde{A}_n$ we can check that the normal vectors of its walls can be defined as

$$\mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i \text{ and } \mathbf{n}_{n+1} = \mathbf{e}_0 - \mathbf{e}_n = \mathbf{n}_1 + \cdots + \mathbf{n}_n ,$$

where $i = 1, \dots, n$, respectively.

These vectors generate the root lattice A_n and an integer basis of this lattice is $\{\mathbf{n}_1, \dots, \mathbf{n}_n\}$. This means that we can give the following description of this lattice:

$$A_n = \left\{ \sum_{i=0}^{n} x_i \mathbf{e}_i \mid \mathbf{x} = (x_0, \dots, x_n) \in \mathbf{Z}^{\,n+1} \qquad \sum_{i=0}^{n} x_i = 0 \right\}.$$

The walls of the fundamental simplex $\tilde{A}_n$ are

$$x_0 = x_1, \quad \dots, \quad x_{n-1} = x_n, \quad -x_0 + x_n = 1.$$

In the following we use point coordinates with subindices in accordance with our references. *Fig. 3* shows the concrete facet diagrams of $\tilde{A}_n$, $\tilde{D}_n$, and $\tilde{E}_n$, respectively.

Now we determine the vertices of this simplex as the intersection of the corresponding hyperplanes and the hyperplane $\sum_{i=0}^{n} x_i = 0$. On the *Fig. 4* we gave the vertex diagram of the fundamental simplices of the examined lattices. Each of the vertices of the simplex corresponds to that node of the facet diagram which means the opposite wall of it.

The order of the corresponding finite reflection group of A_n can be determined from the quotient of the volume of the fundamental simplex and the volume of the fundamental parallelepiped, i.e. that of the D–V cell of the lattice. Computing these values we have that:

$$|W(A_n)| = \frac{\sqrt{n+1}}{\frac{1}{n!}\frac{\sqrt{n+1}}{n+1}} = (n+1)!.$$

The elements of this group can be represented as the coordinate permutations of the points of the $n+1$-dimensional space. Since the reflection subgroup G_0 is a normal subgroup of the automorphism group thus if we know the factor group $G_1 = Aut(A_n)/G_0$ then we also know the group $Aut(A_n)$. But this group consists of the isometries of the fundamental simplex of the reflection subgroup of the lattice so this group is nothing else but the graph automorphism group of the Coxeter–Dynkin diagram of the finite reflection group. This second group is the cyclic group of order 2. (The non-trivial element of it can be represented in the space as the negation of all coordinates of a point.) So the order of $Aut(A_n)$ is $2(n+1)!$.

3.2 $\tilde{D}_n$ and the lattice D_n

In this case we choose an orthonormed basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, $n \geq 4$, of the n-dimensional space E^n. From the general diagram we can define a normal vector system of the walls of the fundamental simplex. It is

$$\mathbf{n}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i \text{ and } \mathbf{n}_{n+1} = \mathbf{e}_{n-1} + \mathbf{e}_n,$$

where $i = 2, \dots, n$, respectively. The facet diagram can be seen in *Fig. 3* and the fundamental roots are:

$$\mathbf{n}_1 = \mathbf{e}_1 + \mathbf{e}_2 \text{ and } \mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i,$$

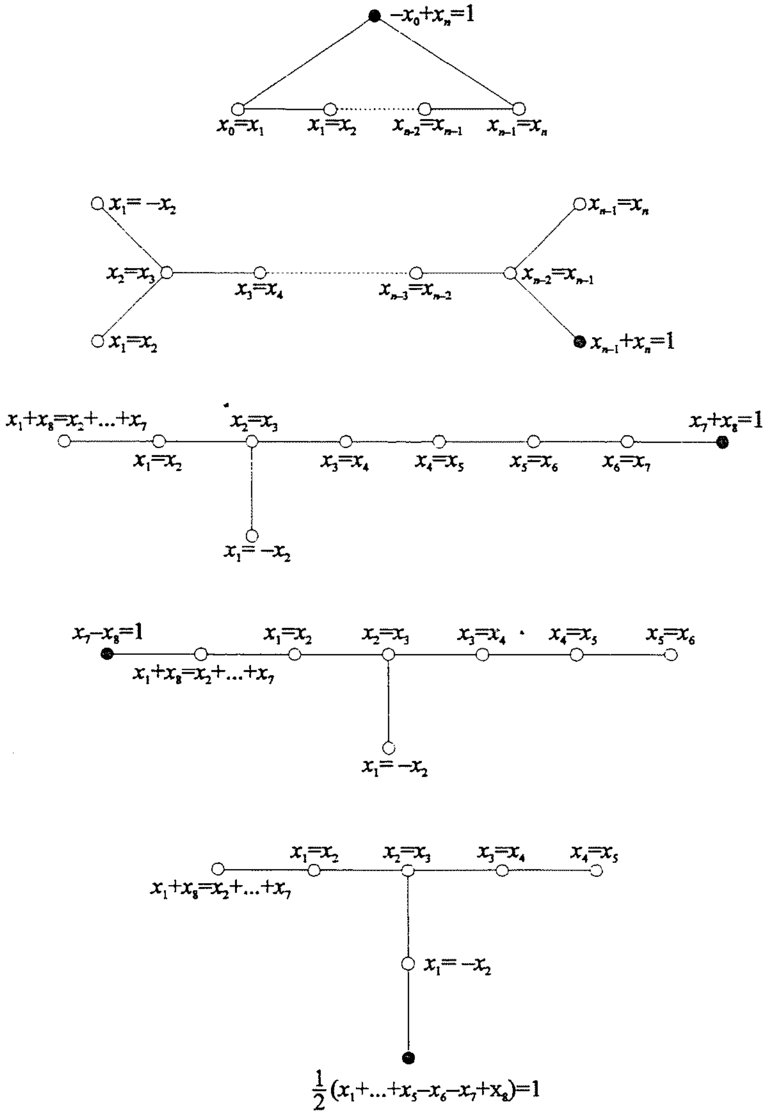

Fig. 3. Concrete facet diagrams for the fundamental simplices of the root lattices

where $i=2,\ldots,n$. The algebraic definition of the lattice is

$$D_n=\left\{\sum_{i=1}^{n} x_i\mathbf{e}_i \mid \mathbf{x}=(x_1,\ldots,x_n)\in\mathbf{Z}^n \qquad \sum_{i=1}^{n} x_i\equiv 0(2)\right\}.$$

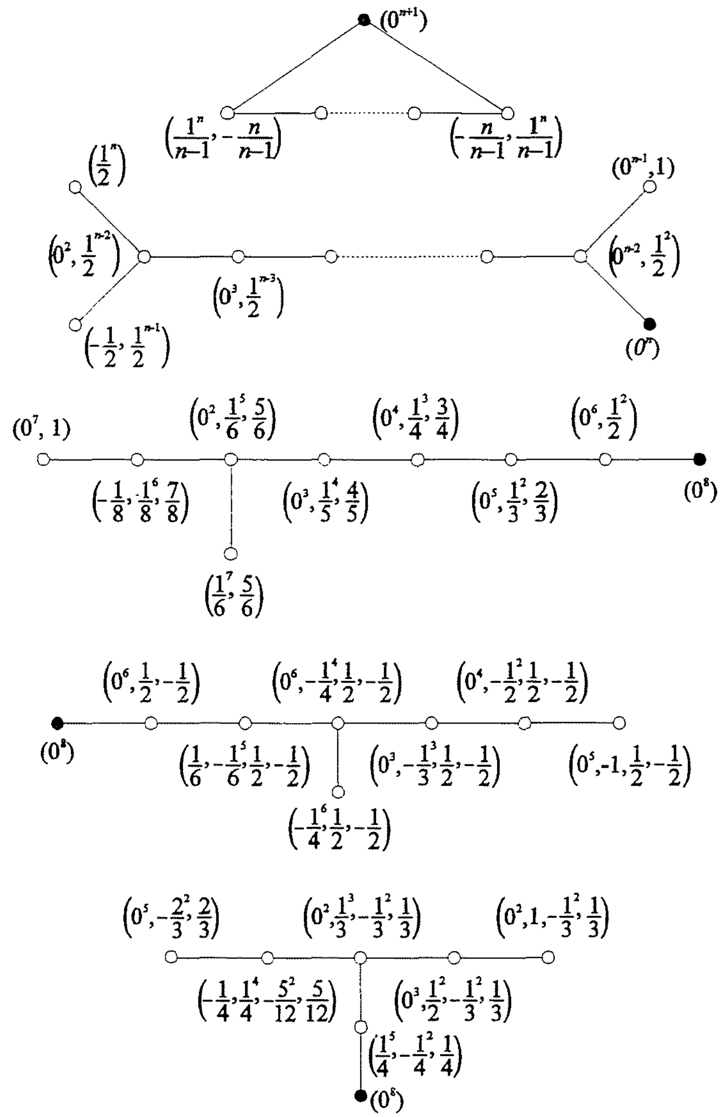

Fig. 4. Vertex diagrams for the fundamental simplices of $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_8$, $\tilde{E}_7$ and $\tilde{E}_6$, resp. E.g. $(0^5, -\frac{2}{3}^2, \frac{2}{3})$ means $(0,0,0,0,0,-\frac{2}{3},-\frac{2}{3},\frac{2}{3})$.

The computed vertices of the simplex $\tilde{D}_n$ can be found in *Fig. 4*. Because of the volume of the fundamental parallelepiped is

$$\det\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 \end{pmatrix} = 2,$$

and the volume of the simplex is

$$\frac{1}{n!}\det\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \dots & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{n!}(\frac{1}{2})^{n-2}.$$

The order of the reflection group is $2^{n-1} \cdot n!$. The automorphism group of the diagram D_n is a second order cyclic group if $n > 4$ and the permutation group of 3 symbols if $n = 4$. Thus the order of the automorphism group is $2^n n!$ if $n > 4$, and $2^7 \cdot 3^2$ if $n = 4$.

3.3 $\tilde{E}_8$ and the lattice E_8

Using the diagram of *Fig. 2* we know that we have only one special vertex (in the diagram) that corresponds to the hyperplane not containing the origin. First we fix the normal vector of this wall; it is the vector $(0,0,0,0,0,0,1,1)^T$. Since the walls are perpendicular to each other or have an angle $\frac{\pi}{3}$ between two of them, we can choose other 6 hyperplanes of the six elements chain $x_7 = x_6, x_6 = x_5, x_5 = x_4, \dots x_2 = x_1$, respectively. At the wall of the hyperplane $x_2 = x_3$, we have to search a further hyperplane perpendicular the others and has an angle $\frac{\pi}{3}$ between its and $x_2 = x_3$. The good hyperplane is $x_1 = -x_2$. The normal vector $\mathbf{n} = (n_1, \dots, n_8)^T$ of the last hyperplane now can be choosen by solving of the linear equation array:

$$\begin{array}{cccc} n_1 = -n_2 & n_2 = n_3 & n_3 = n_4 & n_4 = n_5 \\ n_5 = n_6 & n_6 = n_7 & n_7 = -n_8 & n_1 = 1. \end{array}$$

This means that the corresponding hyperplane is $x_1+x_8 = x_2+\cdots+x_7$ with normal vector $\mathbf{n}(1,-1,-1,-1,-1,-1,-1,1)^T$. The corresponding root is the vector $\frac{1}{2}\mathbf{n}$ and a basis of the root lattice E_8 is

$$\mathbf{n}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i \text{ and } \frac{1}{2}\mathbf{n},$$

where $i = 2, \dots, 7$. The algebraic definition of this lattice is

$$E_8 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \dots, x_8) \in \mathbf{Z}^8 \quad or \quad \mathbf{x} \in (\mathbf{Z} + \frac{1}{2})^8 \quad \sum_{i=1}^{8} x_i \equiv 0(2) \right\}.$$

The facet diagram of the fundamental simplex can be seen in *Fig. 3* and it is easy to determine the vertices. (See *Fig. 4*.) Computing the volumes of the fundamental parallelepiped of E_8 and the fundamental simplex $\tilde{E}_8$, respectively, we get the order of the reflection group:

$$v(E_8) = 1, v(\tilde{E}_8) = \frac{1}{2^{14} \cdot 3^5 \cdot 5^2 \cdot 7} \text{ and } |W(E_8)| = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

The automorphism group of this lattice E_8 is the reflection group because the automorphism group of its diagram is trivial. A nice theorem characterizes this reflection group, W.L. EDGE proved that

Theorem 3 ([8]) *A reflection group of order* $2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$ *is generated by all permutations of* 8 *letters, all even sign changes, and the matrix*:

$$H := diag\ \{H_4, H_4\} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}$$

This matrix corresponds to the general reflection in the 4-space orthogonal to the (lattice) subspace of dimension 4, spanned by the orthogonal minimum system:

$$\mathbf{m}_1 = [0,-1,1,0,0,0,0,0]^T \qquad \mathbf{m}_2 = [0,0,0,0,0,-1,1,0]^T$$
$$\mathbf{m}_3 = \left[-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right]^T \quad \mathbf{m}_4 = \left[-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}\right]^T.$$

3.4 $\tilde{E}_7$ and its Lattice E_7

In this case seven walls of the simplex are the same as those of the simplex $\tilde{E}_8$. This means that the normal vector system

$$\mathbf{n}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i, \frac{1}{2}\mathbf{n}_7(1,-1,-1,-1,-1,-1,-1,1)^T,$$

where $i = 2, \dots, 6$, can be chosen first in the space E^8 and the coordinates of the last vector $\mathbf{n}(n_1, \dots, n_8)$ satisfy the following equation array:

$$n_1 = \ldots = n_6,\ n_1 = -n_2,\ -n_7 + n_8 = \sqrt{2(n_7^2 + n_8^2)}.$$

So the last normal vector is $(0^6, 1, -1)^T$ where 0^6 means that the first six coordinates are zero. We see that the fundamental simplex and the lattice lie in the hyperplane $x_7 + x_8 = 0$ and change the standard basis to the new orthonormed system $\{\mathbf{e}_1 \cdots \mathbf{e}_6, \mathbf{e}_7^* = \frac{1}{\sqrt{2}}(\mathbf{e}_7 - \mathbf{e}_8)\}$. Then we get the full-dimensional description of $\tilde{E}_7$, i.e. in the 7-dimensional Euclidean space E^7. (See in *Fig. 5*)

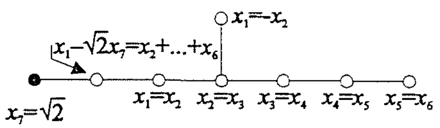

Fig. 5. The full-dimensional Coxeter–Dynkin diagram of the fundamental simplex of $\tilde{E}_7$

The algebraic description of the root lattice generated by the fundamental roots:

$$E_7 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \dots, x_8) \in E_8 \ \ \sum_{i=1}^{8} x_i = 0 \right\}.$$

The fundamental volume of this lattice is $\sqrt{2}$ while the volume of the fundamental simplex of $\tilde{E}_7$ can be computed from the vertex diagram (see in *Fig. 4*) it is $\frac{\sqrt{2}}{2^{10}\cdot 3^4 \cdot 5 \cdot 7}$. The order of the reflection group is $2^{10}\cdot 3^4 \cdot 5 \cdot 7$. The full automorphism group is the reflection group because the automorphism group of the diagram is trivial. We note that from our first geometric presentation we also can verify this result. In fact the automorphism group $Aut(E_7)$ of E_7 is a subgroup of $Aut(E_8)$. This follows from the fact that the lattice E_7 is just the sublattice of E_8 which lies in the plane $x_7 + x_8 =$

0 (parallel to the wall $x_7 + x_8 = 1$ of $\tilde{E}_8$). Hence any automorphism of $Aut(E_8)$ which leaves invariant the hyperplane $x_7 + x_8 = 0$, by fixing its normal vector $\mathbf{e}_7 + \mathbf{e}_8$ is just the corresponding automorphism of E_7, too. These automorphisms obviously form a subgroup of $Aut(E_8)$. (This subgroup isn't normal because for a $\beta \in Aut(E_8)$ fixing the vector $\mathbf{e}_7 + \mathbf{e}_8$, and for an $\alpha \in Aut(E_8)$, the automorphism $\alpha^{-1}\beta\alpha$ in general does not fix the vector $\mathbf{e}_7 + \mathbf{e}_8$.) The index of $Aut(E_7)$ in $Aut(E_8)$ is 240 which is the number of facets of the examined D–V cell of E_8.

3.5 $\tilde{E}_6$ and its Lattice E_6

In this case we fix two hyperplanes $x_7 + x_8 = 0$ and $x_6 - x_7 = 0$ in the lattice E_8. These hyperplanes will contain the fundamental simplex of $\tilde{E}_6$. The normal vectors of the simplex of E_6 are

$$\mathbf{n}_1 = \mathbf{e}_1 + \mathbf{e}_2, \mathbf{n}_i = \mathbf{e}_{i-1} - \mathbf{e}_i, \frac{1}{2}\mathbf{n}_7(1,-1,-1,-1,-1,-1,-1,1)^T,$$

where $i = 2, \dots, 5$. A normal vector $\mathbf{n}(n_1, \dots, n_8)$ of the last hyperplane satisfies the conditions:

$$n_1 = \dots = n_6,\ -n_1 - n_8 + \sum_{i=2}^{7} n_i = 0,\ n_7 + n_8 = 0 \text{ and } n_1 + n_2 = \frac{|\mathbf{n}|}{\sqrt{2}}.$$

The normal vector is $\mathbf{n}(1^5, -1^2, 1)^T$ and we take $\frac{1}{2}\mathbf{n}$. With respect to the basis $\{\mathbf{e}_1 \cdots \mathbf{e}_5, \mathbf{e}_6^* = \frac{1}{\sqrt{3}}(-\mathbf{e}_6 - \mathbf{e}_7 + \mathbf{e}_8)\}$ these vectors have the form:

$$(1,-1,0^4)^T, \dots (0^4,1,-1)^T, (1,1,0^4)^T, ((\frac{1}{2})^5, \frac{\sqrt{3}}{2})^T.$$

The volume of the fundamental parallelepiped is $\sqrt{3}$. The facet diagram can be seen in *Fig. 3*, the vertex diagram in *Fig. 4*. The order of the corresponding reflection group is

$$\frac{v(\text{parallelepiped})}{v(\text{simplex})} = 2^7 \cdot 3^4 \cdot 5.$$

The automorphism group has order $2^8 \cdot 3^4 \cdot 5$ because of the graph automorphism group of E_6.

The algebraic description of the corresponding root-lattice is

$$E_6 = \left\{ \sum_{i=1}^{8} x_i \mathbf{e}_i \mid \mathbf{x} = (x_1, \dots, x_8) \in E_8\ x_1 + x_8 = \sum_{i=2}^{7} x_i = 0 \right\}.$$

4. An Algorithmic Setting up of the D–V Cell of E_8

In this paragraph we illustrate mainly on the base of [5] how to imagine and construct higher dimensional polyhedra, i.e. polytopes by diagram methods. So we obtain a tool for visualizing very symmetric polytopes related to very symmetric lattices. The question now is the following:

How can we describe the combinatorial and geometric structure of a facet of the D–V cell D of the lattice E_8?

In Theorem 2 we have seen the general strategy: We have to construct the simplex of $\tilde{E}_8$ and form the union of its images under the finite reflection group $W(E_8)$ of the root lattice E_8. We know that the facet, meeting the midpoint of its normal vector $\mathbf{e}_7+\mathbf{e}_8$ from the origin, is invariant under the subgroup $Aut_{\mathbf{e}_7+\mathbf{e}_8}(E_8)$ hence those 7-dimensional wall S of the fundamental simplex $\tilde{E}_8$ which lies on the hyperplane $x_7+x_8=1$ is a fundamental simplex S of this facet with respect to the group $Aut_{\mathbf{e}_7+\mathbf{e}_8}(E_8)$. Translating the facet and its fundamental simplex S into the origin we get congruent copies of these polyhedra. Hence we can regard this translated facet as the non-overlapping union of the images of the simplex $S'=S-\frac{1}{2}(\mathbf{e}_7+\mathbf{e}_8)$ under the complete automorphism group $Aut(E_7)$. In the following we give the vertex and facet diagrams of the simplex S'. The vertices of S' are:

$$\begin{array}{cccc} (0^6,-\frac{1}{2},\frac{1}{2})^T & (-\frac{1}{8},\frac{1}{8}^5,-\frac{3}{8},\frac{3}{8})^T & (\frac{1}{6}^6,-\frac{1}{3},\frac{1}{3})^T & (0^2,\frac{1}{6}^4,-\frac{1}{3},\frac{1}{3})^T \\ (0^3,\frac{1}{5}^3,-\frac{3}{10},\frac{3}{10})^T & (0^4,\frac{1}{4}^2,-\frac{1}{4},\frac{1}{4})^T & (0^5,\frac{1}{3},-\frac{1}{6},\frac{1}{6})^T & (0^8)^T. \end{array}$$

These points are lying on the hyperplane $x_7+x_8=0$. Observe that the non-zero position vectors pairwise parallel to the corresponding position vectors of the fundamental simplex $\tilde{E}_7$ so the hyperplanes containing those walls of S' which contain the origin are the same as of the simplex $\tilde{E}_7$. Hence, we have to determine only such a hyperplane which intersects the hyperplane $x_7+x_8=0$, i.e. we search for the six-dimensional affine subspace containing the wall of S' opposite to the origin. From this hyperplane we have seven non-zero points and know that does not contain the origin. The normal vector of it is orthogonal to the six difference vectors:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \\ n_6 \\ n_7 \\ n_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which means that $n_1 = n_2 = n_3 = n_4 = n_5 = 0$ and $n_6 + n_7 - n_8 = 0$. Let now be $n_6 = n_7 = 1$ so $n_8 = 2$. Then the equation of this hyperplane is $x_6 + x_7 + 2x_8 = \frac{1}{2}$ and the walls of the simplex S' are on the 6-planes:

$$\begin{array}{ll} \{x_7 + x_8 = 0 \cap x_2 = x_3\} & \{x_7 + x_8 = 0 \cap x_1 + x_8 = x_2 + \ldots + x_7\} \\ \{x_7 + x_8 = 0 \cap x_3 = x_4\} & \{x_7 + x_8 = 0 \cap x_4 = x_5\} \\ \{x_7 + x_8 = 0 \cap x_5 = x_6\} & \{x_7 + x_8 = 0 \cap x_1 = -x_2\} \\ \{x_7 + x_8 = 0 \cap x_1 = x_2\} & \{x_7 + x_8 = 0 \cap x_6 + x_7 + 2x_8 = \frac{1}{2}\} \end{array}$$

Now we can draw the facet diagram of this simplex, see in *Fig. 6*. The branch denoted by double dotted line means that the angle of the corresponding facets is $\varphi = \arccos\frac{1}{2\sqrt{3}}$.

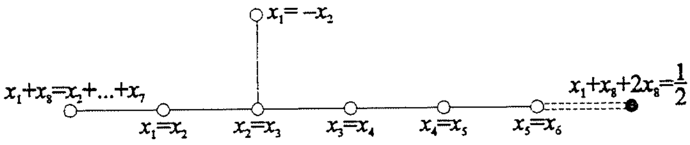

Fig. 6. The Coxeter–Dynkin diagram of the simplex S'

The full-dimensional variation of this diagram is better for determining the geometric properties of the polyhedra built up by this simplex. First we have to change the coordinate system. If $\{\mathbf{e}_1 \cdots \mathbf{e}_8\}$ was the original orthonormed base we consider the new orthonormed system $\{\mathbf{e}_1 \cdots \mathbf{e}_6, \mathbf{e}_7^* = \frac{1}{\sqrt{2}}(\mathbf{e}_7 - \mathbf{e}_8)\}$. Seven normal vector are in the hyperplane spanned by this system but the last vector $(0^5, 1, 1, 2)^T$ is not here. We have to take the orthogonal projection of the vector $(0^5, 1, 1, 2)^T$ onto the hyperplane $x_7 + x_8 = 0$. This vector $(0^5, 1, -\frac{1}{2}, \frac{1}{2})^T$ has to be expressed in the new base. Thus the normal vectors of the examined intersection six-planes with respect to this base have the form:

$$\begin{array}{cccc} (1,-1,0^5)^T & (0,1,-1,0^4)^T & (0^2,1,-1,0^3)^T & (0^3,1,-1,0^2)^T \\ (0^4,1,-1,0)^T & (\frac{1}{2},-\frac{1}{2}^5,\frac{\sqrt{2}}{2})^T & (1,1,0^5)^T & (0^5,1,-\frac{1}{\sqrt{2}})^T \end{array}$$

The corresponding diagrams can be seen in *Fig. 7*. The branch denoted by dotted line means that the angle of corresponding facets is $\varphi = \arccos\frac{1}{\sqrt{3}}$.

Taking the images of S' under the automorphism group of E_7 we get a polyhedron $P = \bigcup\{\alpha(S') : \alpha \in Aut(E_7)\}$ which is congruent to the facets of the D–V cell of the lattice E_8. Translate this polyhedron with the vector

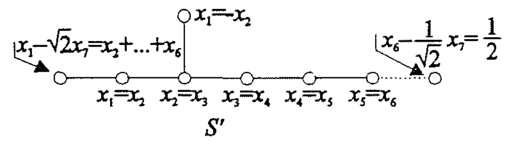

Fig. 7. The full-dimensional facet diagram of S'.

$\frac{1}{2}(\mathbf{e}_7+\mathbf{e}_8)$, and transform with the coset representatives of $Aut_{\mathbf{e}_7+\mathbf{e}_8}(E_8)$ in $Aut(E_8)$ then we get the surface of the D–V cell of the lattice E_8. As a summary we have the following theorem:

Theorem 4 *The D–V cell D of the lattice E_8 can be built up by the following algorithmic way:*

$$D=\bigcup\left\{\beta\left(\frac{1}{2}(\mathbf{e}_7+\mathbf{e}_8)+\cup\{\alpha(S')\ \alpha\in Aut_{\mathbf{e}_7+\mathbf{e}_8}(E_8)\}\right)\ \beta\in H\right\}$$

where S' is the simplex in Figs. 6 and 7, and H is a coset representative system of $Aut_{\mathbf{e}_7+\mathbf{e}_8}(E_8)$ in $Aut(E_8)$.

From the full-dimensional diagrams we can read the difference between the seven-cell of $\tilde{E}_7$ and the facet of the eigth-cell D of the lattice E_8. The fundamental seven-simplex S' of the latter is such an affine image of the simplex $\tilde{E}_7$ which changes only two dihedral angles between the facet of the special vertex and the other walls.

We note that there is a more geometric diagram description of the D–V cells of the lattices E_n. GOSSET investigated some semi-regular polyhedra in [10], and obtained three important polyhedra in dimensions $n=6,7$ and 8, respectively. His essay started with the systematic investigation of the problem to find the n-dimensional semi-regular polytopes. His method, as Burnside wrote in a letter to Glaisher, is 'a sort of geometrical intuition' and his idea of regarding an $(n-1)$-dimensional honeycomb as a degenerated n-dimensional polytope seemed 'fanciful'. Later his results were rediscovered by ELTE in [9] and COXETER in [6]. The polytopes defined in a similar way as the original polyhedra made by Gosset is called Gosset polytopes. The D–V cells of the lattices E_n are the reciprocal or polar to the Gosset polytopes $4_{21}, 2_{31}, 1_{22}$ of the same dimension, resp. The notation k_{ij} means that the vertex diagram of the fundamental simplex has unmarked branches which are connected to each other on the way of *Fig. 8*, where the black node means that vertex of the fundamental simplex (which is also vertex of the honeycomb) which all transforms give the vertices of the polyhedron. This diagram was introduced by WYTHOFF in [14] and [15]. The *reciprocal* (or polar body) of a polyhedron with respect to a point of its interior as the origin, collects those points of the space whose scalar product with any point of the body is not greater than 1. The above observations are consequences of the following theorem:

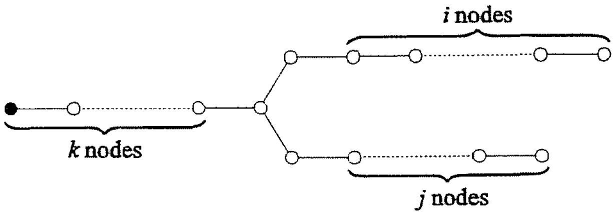

Fig. 8. What does the symbol k_{ij} mean?

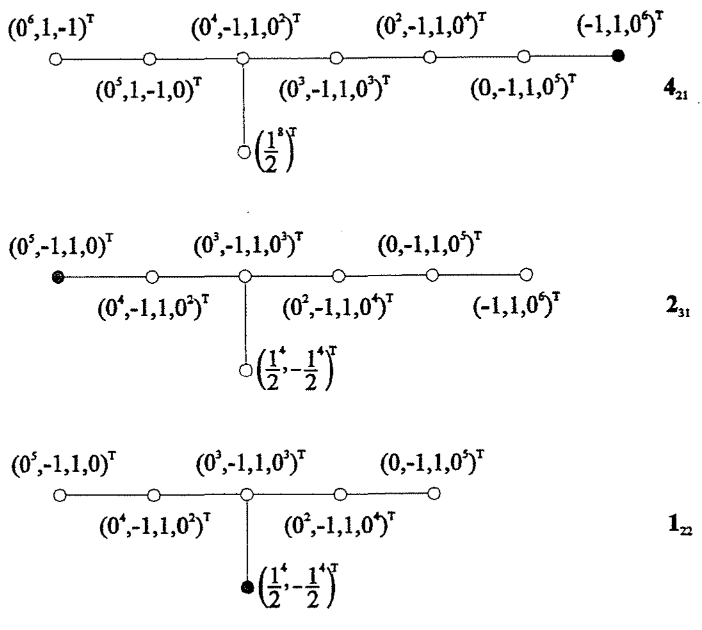

Fig. 9. Wythoff diagrams of the vertex figures of E_n

Theorem 5([1]) *If the relevant vectors of a lattice are precisely the minimal vectors, then the D–V cell of the lattice is similar to the reciprocal of the polytope whose vertices are the minimal vectors of the lattice.*

Imagine the cubic lattice, whose 6 relevants point to the 6 vertices of a regular octahedron whose reciprocal is just the cube, i.e. the D–V cell.

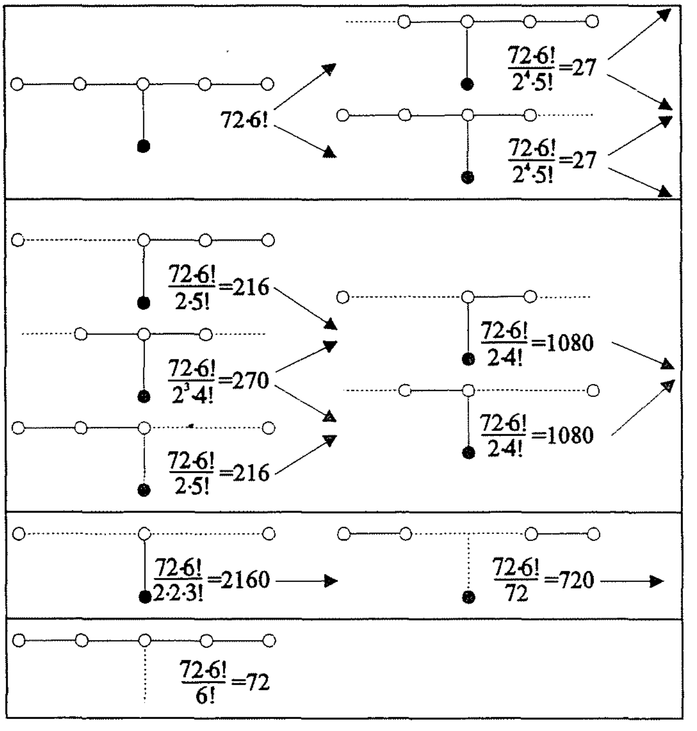

Fig. 10. The number of faces of the polytope $\mathbf{1}_{22}$

Finally we collect the main properties of the D–V cells of the lattices E_n.

Theorem 6 ([1], [5]) *Let us denote N_i the number of i-dimensional faces of the D–V cell, the radius of its circumscribed ball is R and the radius of its inscribed ball is r. Then these parameters for the D–V cells of the root lattices E_n $n = 6, 7, 8$ can be found in the following tables:*

	N_0	N_1	N_2	N_3	N_4	N_5	N_6	N_7
E_8	19440	207360	483840	483840	241920	60480	6720	240
E_7	632	4788	14112	20160	10080	2016	126	–
E_6	54	702	2160	2160	720	72	–	–

	R	r	det
E_8	1	$\frac{1}{\sqrt{2}}$	1
E_7	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	2
E_6	$\frac{2}{\sqrt{3}}$	$\frac{1}{\sqrt{2}}$	3

Sketch of the proof: By Theorem 5 in the case of the lattices E_n we have to determine the number of faces of dimension i of the polytope whose

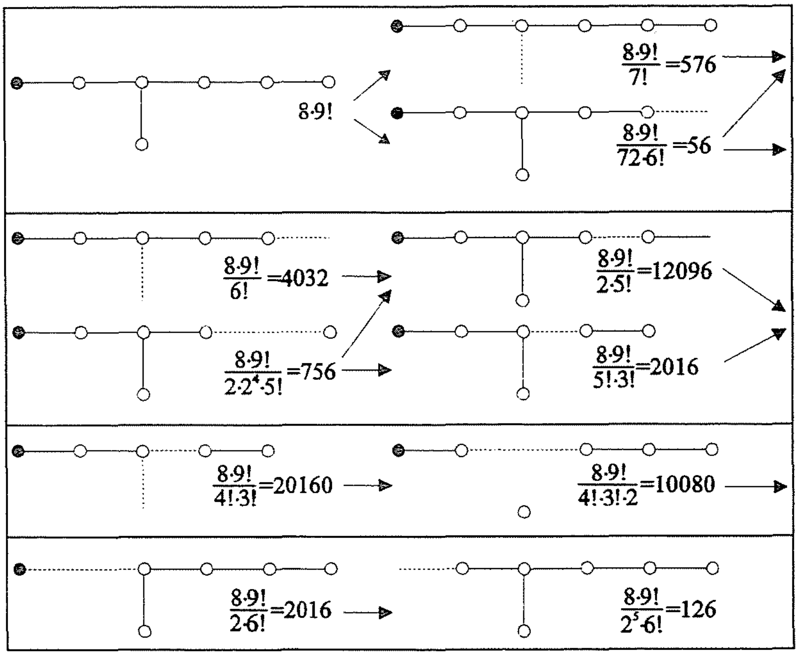

Fig. 11. The number of faces of the polytope 2_{31}

vertices are the endpoints of the minimal vectors of the lattice. Since the automorphism group of the lattice is E_n these polytopes can be built up from the vertex diagram of (suitable) fundamental simplices by drawing a ring round one of their nodes. These ringed (black) nodes have to be a midpoint of a minimal norm lattice vector so in *Fig. 9* we can state the Wythoff diagram of these lattices from the given diagrams of them. These are $1_{22}, 2_{31}$ and 4_{21}, respectively. In the case of $n = 8$ the numbers N_i can be found in [5], the numbers r, R in all cases in [1]. As a new illustration of the method, we determine the number of faces when $n = 6$ or $n = 7$.

First, we take the diagram 1_{22}. Removal of an unringed (white) node (and its branches) from the diagram yields that of a 5-face of the polytope iff the graph remains connected. Since this diagram is a tree we have only two possibilities for it, we can remove only a free end of the graph which are unringed nodes belonging to only one branch. The nodes of the remaining diagram represent hyperplanes of symmetry of this 5-face. These, regardless of the ring, represent the fundamental simplex for the subgroup leaving invariant this 5-face. The congruent image facets correspond to the cosets by this subgroup. This means that the number of congruent 5-faces is equal to the index of this subgroup in the original group. From the diagram we see that the 5-cells of the examined polytope fall into two congruent classes, both have the diagram D_5. Thus the cardinality $M_5 = N_0$ by [5] is

$$N_0 = 2 \cdot \frac{|E_6|}{|D_5|} = 2 \cdot \frac{72 \cdot 6!}{2^4 \cdot 5!} = 54$$

by the polarity.

It is clear that if now we remove a free end of the diagram of each above 5-face we get the diagram of any 4-face. The diagram of a 4-face will be a ringed piece of the full diagram. Dropping the neighbouring branches and nodes of this piece from the full diagram, we have a new diagram which has two parts. (The new diagram is not connected.) If we consider a node which belongs to the ringed piece as a hyperplane of symmetry of the polytope it is also a symmetry of the 4-face of it. Simultaneously, the symmetry hyperplanes, corresponding to the nodes of the other piece of this unconnected diagram, mean hyperplanes are containing the regarded 4-face. Thus the graph for this 4-face, regardless of the ring, represents the fundamental region for the subgroup leaving this 4-face invariant. This means that the image copies of this face correspond to the cosets of the above subgroup, so the number of this image 4-faces is the index of this subgroup in the full symmetry group.

Continuing this process, we can determine from the diagram the number of the faces of dimension k for every possible k. If we denote by M_k this number we have that $N_k = M_{5-k}$ by the reciprocity. In *Fig. 10* we can see the steps of the calculation.

Similar arguments show the case of the reciprocal to the polytope 2_{31} for the lattice E_7 (see *Fig. 11*).

Finally we remark that the analogous data of the other root lattices can be calculated in a similar way. The most detailed description of the D–V cells of the lattices A_n, D_n and their reciprocals A_n^* and D_n^* can be found in the works [1] and [2].

Acknowledgement

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References

1. CONWAY, J. M. – SLOANE, N. J. A.: Sphere Packings, Lattices and Groups. Springer-Verlag, 1988.
2. CONWAY, J. M.: On the Voronoi Regions of Certain lattices. *SIAM Journal on Algebraic and Discrete Methods.* Vol. 5 (1984), pp. 294–305.

3. COXETER, H. S. M.: Groups whose Fundamental Regions are Simplices. *J. of the London Math. Soc.* Vol. 6 (1931), pp. 132–136.
4. COXETER, H. S. M.: Discrete Groups Generated by Reflections. *Annals of Mathematics* Vol. 35 (1934), pp. 588–621.
5. COXETER, H. S. M.: Regular Polytopes. Methuen and Co. Ltd., London, 1948.
6. COXETER, H. S. M. : The Pure Archimedean Polytopes in Six and Seven Dimensions. *Proceedings of the Cambridge Philosophical Society*, Vol. 24 (1928) pp. 1–9.
7. DAVIS, C. – GRÜNBAUM, B. – SHERK, F. A.: (eds): The Geometric Vein: The Coxeter Festschrift, Springer-Verlag 1981 [B].
8. EDGE, W. L.: An Orthogonal Group of Order $2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$. *Annali di Mat.* Vol. 61 (1963), pp. 1–95.
9. ELTE, E. L.: The Semiregular Polytopes of the Hyperspaces, Groningen, 1912.
10. GOSSET, T.: On the Regular and Semi-Regular Figures in Space of n Dimensions. *Messenger Math.* Vol. 29 (1900), pp. 43–48.
11. G. HORVÁTH, Á.: On the Dirichlet-Voronoi Cell, Part I. Classical Problems. *Periodica Poly. Math. Ser. Mech. Eng.* Vol. 39, No. 1, (1995) pp. 25–42.
12. GRUBER, P. M. – LEKKERKERKER, C. G.: Geometry of Numbers. North-Holland Amsterdam-New York-Oxford-Tokyo, 1987.
13. WEYL, H.: Theorie der Darstellung kontinuierlicher halb-einfacher Gruppen durch lineare Transformationen (II), *Mathematische Zeitschrift*, Vol. 24 (1926), pp. 328–376.
14. WYTHOFF, W. A.: The Rule of Neper in the Four Dimensional Space. *Koninklijke Akademie van Wetenschappen te Amsterdam, Proceedings of the Section of Sciences*, Vol. 9 (1907), pp. 529–534.
15. WYTHOFF W. A.: A Relation between the Polytopes of the C_{600}-family. *Ibid.* Vol. 20 (1918), pp. 966–970.