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STABILITY OF PERIODICALLY OPERATING DYNAMIC SYSTEMS

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Abstract

This paper focuses on parametric vibrations excited on mechanical systems, mechanisms and machinery. It derives the stability conditions of free and forced vibration of periodically operating systems, and investigates the distributions of instability domains for low variation excitation. The extensive and abstract theory will be discussed briefly, by using a specific interpretation. Possibilities of applications of the results are also described. The main purpose of the paper is to facilitate deepening in the comprehensive theory and, at the same time, to cover the fundamental knowledge needed in spreading mechanical and vibration applications.

Keywords: dynamic stability, parametric excitation, resonance, self-induced vibration.

1. Introduction

A wide range of mechanical systems may be considered, roughly, as a system of rigid bodies, or alternatively, mechanisms and machinery. They can be considered to be *dynamic* in that, after having eliminated the structural constraints, their motion can be described by ordinary (second-order) system of differential equations. In general, a discrete model of finite degrees of freedom can be applied even if, due to elasticity of components, local motions and vibrations are superimposed to the rigid-body motion. The equation concerned, that is motion equation, can be replaced by a power series related to rigid-body motion, and thus, the local motion can be investigated independently of the global motion. In the simplest case, the approximate differential equation is linear, and its coefficients, i.e. its *parameters* vary – either directly or indirectly – as a time function.

Irrespective of transients, e.g. starting or stopping conditions, the alteration of parameters is mostly *periodical*. For example, let us consider the cranking and control mechanism of a piston engine operating at a nearly constant speed [1, 2, 3, 4], a gear drive [5, 6, 7], a belt and chain drive

[7, 8] or a robot performing a fixed working cycle [10]. In the cases given before and also in other technical applications (e. g. [11, ..., 17]) all, or some, of the parameters, namely the overall masses, damping and stiffness vary periodically as a function of time. Also the purely time-dependent force or amplitude excitation is periodical, thus, its history is similar to the parametric excitation. In the following, fundamental results for this type of system will be considered. Primarily, we focus on the analysis of the stability conditions, since the concerned systems are prone to *self-induced, parametric resonance*. In practice, it is very important to know the response of these systems from the point of the view of motion stability.

The required mathematical tools, based on the theorems by FLOQUET, LYAPUNOV, POINCARÉ, are of course available in the relevant literature, e. g. in monograph [19] and in part, in various other papers on mathematics [20, 21, 22] or on mechanics [13, ..., 18], or elsewhere. In spite of the proliferating number of applications, the extensive mathematical theory is tedious to understand, and its use often reflects unawareness. As a consequence of the facts given above, this paper summarises the most essential theorems. Of course, the discussion of the theorems cannot be exhaustive, since it would extend the frame of this paper. Motion stability of periodically operating systems is discussed for both free and forced vibrations, and distribution of resonance spots for small parameter changes will be presented. Some thoughts will also be presented on physical, technical and vibration aspects of the theories. Hopefully, this approach facilitates acquiring the needed theoretical and practice oriented knowledge.

2. Formulation of the Dynamic Model

a. In view to the aims given in the introduction, consider a motion equation of a system of N degrees of freedom:

$$\mathbf{Q}(\ddot{\mathbf{q}}, \dot{\mathbf{q}}, \mathbf{q}, t) = 0 \ ,$$

where $\mathbf{q}$ represents the vector of general coordinates, $\dot{\mathbf{q}}$, $\ddot{\mathbf{q}}$ give (t) time derivatives of $\mathbf{q}$, i.e. the speed and acceleration. Reorganise the function $\mathbf{Q}$ into a first order *Taylorian series* for the $\mathbf{q}_0(T) = \mathbf{q}_0(t+T)$ global movement of periodicity T:

$$\mathbf{Q}\Big|_{\mathbf{q}_0} + \frac{\partial \mathbf{Q}}{\partial \ddot{\mathbf{q}}}\Big|_{\mathbf{q}_0}(\ddot{\mathbf{q}} - \ddot{\mathbf{q}}_0) + \frac{\partial \mathbf{Q}}{\partial \dot{\mathbf{q}}}\Big|_{\mathbf{q}_0} + \frac{\partial \mathbf{Q}}{\partial \mathbf{q}}\Big|_{\mathbf{q}_0}(\mathbf{q} - \mathbf{q}_0) = 0 \ .$$

Together with $\mathbf{q}_0(t)$, time derivatives $\dot{\mathbf{q}}_0, \ddot{\mathbf{q}}_0$ are also known, hence $\mathbf{Q}$ and its partial derivatives are specified time functions. Thus, for local motion, that is for vibration $\mathbf{x} = \mathbf{q} - \mathbf{q}_0$, the following system of approximate linear differential equations:

$$\mathbf{M}(t)\ddot{\mathbf{x}} + \mathbf{D}(t)\dot{\mathbf{x}} + \mathbf{S}(t)\mathbf{x} = \mathbf{f}(t)$$

is valid. In the equation given above, $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{S}$, respectively, represent the first order (general and reduced) members of the matrix of tensor of mass, damping, and stiffness. At the same time, the force vector $\mathbf{f}$ represents the zero order member. Furthermore, valid is the periodicity condition:

$$\mathbf{M}(t+T) = \mathbf{M}(t)\ , \quad \mathbf{D}(t+T) = \mathbf{D}(t)\ , \quad \mathbf{S}(t+T) = \mathbf{S}(t)\ , \quad \mathbf{f}(t+T) = \mathbf{f}(t)\ .$$

b. In order to simplify computations, let us transform the second-order equation to a first-order (*Cauchy's*) normal form. Multiplied by $\mathbf{M}^{-1}$, because $\mathbf{M}$ is regular according to our assumption, the equation becomes

$$\begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix}^{\cdot} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{S} & -\mathbf{M}^{-1}\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix} ,$$

or, by using symbol $\mathbf{x}$ for the state variable, the equation can be expressed in an even shorter form:

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{a}(t)\ , \quad \mathbf{A}(t+T) = \mathbf{A}(t)\ , \quad \mathbf{a}(t+T) = \mathbf{a}(t)\ .$$

This differential equation is a linear, homogeneous one, it has a periodic coefficient, therefore, in the general terms, it is *Hill* type.

c. It is known that the solution of linear differential equations can be composed from the solution $\mathbf{x}_h(t)$, meeting the initial condition $t_0 = 0$, $\mathbf{x}_0 = \mathbf{x}(t_0)$ of the *homogeneous part* ($\mathbf{a} = \mathbf{0}$), and from the *particular solution* $\mathbf{x}_p(t)$ of the inhomogeneous equation belonging to the zero initial condition, such as:

$$\mathbf{x}(t) = \mathbf{x}_h(t) + \mathbf{x}_p(t)\ , \quad \mathbf{x}_h = \mathbf{X}(t)\mathbf{x}_0, \quad \mathbf{x}_p(t) = \mathbf{X}(t) \int_{\tau=0}^{t} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau\ .$$

The solution fundamental matrix $\mathbf{X}(t)$ is to be built up of the independent vector solution of the linear differential equation in such a way that $\mathbf{X}(t = 0) = \mathbf{I}$ be a unit matrix. If, by chance, coefficient matrix $\mathbf{A}$ is constant, then the solution matrix is an exponential function

$$\mathbf{X}(t) = e^{\mathbf{A}t} = \exp(\mathbf{A}t)\ .$$

We will see later that the *Lyapunov's stability* of the movement decisively depends on the history of the free vibration. Consequently, first investigate the characteristics of the homogenous solution x_h, that is, the *self or free* (transient) vibration.

3. Stability of the Free Vibration

a. If matrix $\mathbf{A}$ is specifically constant for time domains $T_1, T_2, \cdot T_m$, as it is the case with Meissner's equations, the solution matrix is the product of the part solutions of the form given before:

$$\mathbf{X}(t) = \mathbf{X}_i(\tau)\mathbf{X}_{i-1}(T_{i-1})\dots\mathbf{X}_1(T_1) \ , \quad t = \tau + \sum_{j=1}^{j-1} T_j \ , \quad 0 \le \tau \le T_i \ ,$$

$$i, j = 1, 2, \dots \ .$$

Since $\mathbf{A}(t)$ is a function of time period T, the product defines a constant matrix $\mathbf{C}$ that is called *monodromy* operator (or period mapping, or fundamental matrix) as follows:

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{C} \ , \quad \mathbf{C} = \mathbf{X}(T) = \prod_{i=1}^{m} \mathbf{X}(T_i) \ , \quad T = \sum_{i=1}^{m} T_i \ ,$$

$$i = 1, 2, \dots, m \ .$$

For the existence of $\mathbf{C}$, it is indifferent whether $\mathbf{A}(t)$ is constant in the domain, or varies continuously (just like the well-known Mathieu equation). The point is that, if $\mathbf{C}$ is known, the solution can simply be obtained for each T:

$$\mathbf{X}(nT) = \mathbf{X}^n(T) = \mathbf{C}^n \ , \quad n = 0, 1, 2, \dots$$

b. Should the state of the system converge from any initial position $\mathbf{x}_0$ to an equilibrium $\mathbf{x} = 0$, it is necessary to satisfy:

$$\lim_{n\to\infty} \mathbf{C}^n = 0 \ .$$

At the same time, this is the sufficient condition. In order to understand that, in the intermittent positions, the state $\mathbf{x}$ can become zero, divide the periodic mapping into two arbitrary parts $\mathbf{X}_1$ and $\mathbf{X}_2$. The following can be derived by identical mappings:

$$\mathbf{C}\mathbf{x}_0 = \mathbf{X}_2\mathbf{X}_1\mathbf{x}_0 = \mathbf{C}_{21}\mathbf{x}_0 \ , \quad (\mathbf{X}_1\mathbf{X}_2)(\mathbf{X}_1\mathbf{x}_0) = \mathbf{C}_{12}\mathbf{x}_1 \ .$$

Thus, monodromy for initial time $t = T_1$, rather than for $t = 0$, can be obtained from $\mathbf{C} = \mathbf{C}_{12}$ by similarity transformation:

$$\mathbf{C}_{12} = \mathbf{X}_1\mathbf{C}_{21}\mathbf{X}_1^{-1} \ .$$

So, whatever reference starting time is chosen, satisfying the requirement for $\mathbf{C}$ implies to meet

$$\lim_{n\to\infty} \mathbf{C}_{12}^n = \lim_{n\to\infty} (\mathbf{X}_1 \mathbf{C}_{21} \mathbf{X}_1^{-1})^n = \lim_{n\to\infty} \mathbf{X}_1 \mathbf{C}_{21}^n \mathbf{X}_1^{-1} = 0 \ .$$

noindent Accordingly, if during a period T reckoned from time t the solution $\mathbf{x}_h = \mathbf{X}(t)\mathbf{x}_0$ is decreasing, then after a time the free vibration necessarily decays, i.e. it becomes *asymptotically stable.*

c. In the investigation of physical systems, it does not mean a significant restriction to consider, for the sake of simplicity, the eigenvalues of $\mathbf{C}$ to be pair-wise different ($\lambda_1 \neq \lambda_j$). Considering that in this case representation of $\mathbf{C}$ can be arranged diagonally, for the eigenvalues invariant to a similarity transformation, the following equation can be derived:

$$\lim_{n\to\infty} \mathbf{C}^n = 0 \ , \quad \lim_{n\to\infty} \lambda_i^n = 0 \ , \quad |\lambda_i| < 1 \ , \quad i = 1, 2, \ldots, 2N \ .$$

Accordingly, *the natural vibration is asymptotically stable if any eigenvalue of the monodromy is less than 1 in absolute value* (Lyapunov's theorem). The motion is *stable* if, among the eigenvalues of less than one, there is at least one of value 1. The same is true, again, if $\lambda = 1$ also occurs with single multiplicity (theorem of Andronov and Witt). Else, the motion may be unstable, and a *parametric resonance* may develop, with an exponentially increasing magnitude.

Note that a general and mathematically more exact discussion is longer and more cumbersome. Those interested may study, first of all, works by PONTRYAGIN and ARNOLD, [19, 20, 21]. The LYAPUNOV and ANDRONOV-WITT theorems formulate the sufficient conditions not only for the stability of solutions of linear equations but also for the periodical solutions of the original equation. (This is the outcome of the Lyapunov's stability theorem.)

d. If $\mathbf{C}$ is known, the stability of homogeneous system with periodical coefficients can be reasoned out similarly to the time invariable systems with constant coefficient. This is not incidental at all. For periodic equations a relevant equation of constant coefficient of $\mathbf{B} = \ \ln\,(\mathbf{C})/T$ can always be determined, such that the two equations take the same states by period T (or $2T$, in the real number space). The relationship

$$\mathbf{C} = \mathbf{Y}(T) = \exp(\mathbf{B}T) \ , \quad \text{that is} \quad \dot{\mathbf{Y}} = \mathbf{B}\mathbf{Y}$$

is unambiguous, irrespective of similarity transformations. Note that it can be proved that a non-singular $\mathbf{C}$ always has an unambiguous logarithm in complex. The proof is easy for single eigenvalues, but for multiple eigenvalues it is far from being elementary, as it can be seen e. g. in [20]. In the real case, difficulties may occur due to the fact that an arbitrary $\mathbf{C}$ may also contain reflections. It is not certain that $\mathbf{B}$ is real, too. In complex, reflections can be considered as realilty rotation, while in reality, normally $\mathbf{C}^2 = \mathbf{X}(2T)$ involves real rotation only. Here 'involves' refers to the theorem of polar decomposition.

e. The consequence of the relationship given below is referred more frequently in the literature. This is the Floquet's theorem. On the basis of this theorem, the solution $\mathbf{X}(t)$ is a product of a periodic function $\mathbf{P}(t)$ and $\exp(\mathbf{B}t)$:

$$\mathbf{X}(T) = \mathbf{P}(T)\exp(\mathbf{B}t) \ . \quad \mathbf{P}(t+T) = \mathbf{P}(t) \ .$$

If the monodromy for the equation above is produced based on the facts mentioned in the previous paragraph, we get:

$$\mathbf{X}(t+T) = \mathbf{P}(t+T)\exp(\mathbf{B}t)\exp(\mathbf{B}T) = \mathbf{P}(t)\exp(\mathbf{B}t)\exp\mathbf{B}T) = \mathbf{X}(t)\mathbf{C}$$

which, in turn, includes the proof of Floquet's theorem.

Thereafter a particular solution $\mathbf{x}_p$ of the motion equation independent of the initial condition, the forced vibration, will be investigated. To this end, theorems d, e referring to equivalent time invariant systems will be neglected, only obvious outcomes of theorems a, b and c will be used.

4. Stability and Periodicity of Forced Vibrations

Although conventional and parametric excitations are mostly concomitant, in general, only the parametric excitation related to the homogeneous system is investigated. Thus, discussion here will be restricted to the case where both conventional, that is zero order excitation $\mathbf{a}(t)$ and the parametric one of first order $\mathbf{A}(t)$, are functions with period T. Obviously, this condition is also satisfied if the primary periods are not the same, but they can be rated through rational numbers. (The simplest case is when function $\mathbf{a}(t)$ is constant.) This constraint is irrelevant in most of practical problems since it is automatically met.

a. Let's determine amplitude of *forced vibration* $\mathbf{x}_p$ similarly as for natural vibration, for finite periods T. Let's start out of the condition that the value of constant vector

$$\mathbf{x}_p(T) = \mathbf{X}(T)\int_{t=0}^{T}\mathbf{X}^{-1}(t)\mathbf{a}(t)dt = \mathbf{c}$$

is known for primary period T. Repeating integration from period to period yields:

$$\mathbf{x}_p(2T) = \mathbf{C}^2(\mathbf{I} + \mathbf{C}^{-1})\mathbf{c} = (\mathbf{C}^2 + \mathbf{C})\mathbf{c},$$
$$\mathbf{x}_p(3T) = \mathbf{C}^3(\mathbf{I} + \mathbf{C}^{-1} + \mathbf{C}^{-2})\mathbf{c} = (\mathbf{C}^3 + \mathbf{C}^2 + \mathbf{C})\mathbf{c}, \dots$$

Factor of $\mathbf{c}$ is a geometric series:

$$\mathbf{X}_p(nT) = \mathbf{C} + \mathbf{C}^2 + \dots + \mathbf{C}^n = \mathbf{S}_n \ .$$

Since the expression for sums of series is also true for matrices,

$$\mathbf{S}_{n+1} = \mathbf{S}_n + \mathbf{C}^{n+1} = \mathbf{S}_n\mathbf{C} + \mathbf{C} \ ,$$

hence, due to the interchangeability of factors, we receive:

$$\mathbf{S}_n = (\mathbf{I} - \mathbf{C}^n)\mathbf{C}(\mathbf{I} - \mathbf{C})^{-1} = (\mathbf{I} - \mathbf{C}^n)(\mathbf{I} - \mathbf{C})^{-1}\mathbf{C} \ .$$

Hence, the particular solution sought for:

$$\mathbf{x}_p(nT) = \mathbf{X}_p(nT)\mathbf{c} \ , \quad \mathbf{X}_p(nT) = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{X}_p(T) - \mathbf{C}^n(\mathbf{I} - \mathbf{C})^{-1}\mathbf{X}_p(T) \ .$$

If the natural vibration is asymptotically stable, then the second term of forced vibration should decay. The limit value is:

$$\lim_{n\to\infty} \mathbf{x}_p(nT) = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{c} = (\mathbf{I} - \mathbf{C})^{-1}\mathbf{x}_p(T) = \mathbf{x}_{p\infty} \ .$$

Summing up everything, condition of stability of forced vibration $\mathbf{x}_p$ is the regularity of matrix $\mathbf{I}$–$\mathbf{C}$. However, the functions that are periodical according to T are also periodical according to nT, hence:

$$\det(\mathbf{C}^n - \mathbf{I}) \neq 0 \ , \quad \text{that is} \quad |\lambda_i^n - 1| \neq 0 \ , \quad \text{i.e.} \quad |\lambda_i| \neq 1 \ .$$

It is a less strict condition than the earlier one. *The asymptotic stability of natural vibrations implies stability of forced vibrations.* It can be said that resonances of the concerned systems are due to variation of coefficient $\mathbf{A}(t)$, i.e. the parametric excitation, rather than the restricted periodical force, or amplitude excitation $\mathbf{a}(t)$. Note that the result is very close to the Andronov–Witt theorem. Namely, if the inhomogeneous equation is considered to be a homogeneous equation of degrees of freedom $2N - 1$, then in the spectrum of $\mathbf{C}$, the eigenvalue of unit value necessarily appears. If it is of single multiplicity, the motion is certainly stable.

b. The formula obtained for the particular solution is essentially self-explanatory. It means that for a t large enough the stable system returns to the same condition $\mathbf{x}_{p\infty}$ at each period T:

$$\mathbf{x}_{p\infty} = \mathbf{C}\mathbf{x}_{p\infty} + \mathbf{x}_p(T) = \mathbf{X}(T)\left(\mathbf{x}_{p\infty} + \int_{t=0}^{T} \mathbf{X}^{-1}(t)\mathbf{a}(t)dt\right) .$$

Thus, our conjection can be that the subsistent component of forced vibration is a periodic function. In fact:

$$\mathbf{X}(t)\left(\mathbf{x}_{p\infty} + \int_{\tau=0}^{t} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau\right) = \mathbf{X}(t+T)\left(\mathbf{x}_{p\infty} + \int_{\tau=0}^{t+T} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau\right) ;$$

that is,

$$\mathbf{x}_{p\infty} + \int_{\tau=0}^{t} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau =$$

$$= \mathbf{C}\left(\mathbf{x}_{p\infty} + \int_{\tau=0}^{T} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau + \int_{\tau=0}^{t} \mathbf{X}^{-1}(\tau+T)\mathbf{a}(\tau+T)d\tau\right) .$$

Since here $\mathbf{X}(\tau+T) = \mathbf{C}\mathbf{X}(\tau)$, and $\mathbf{a}(\tau+T) = \mathbf{a}(\tau)$, the right-hand side becomes:

$$\mathbf{C}\left(\mathbf{x}_{p\infty} + \int_{\tau=0}^{T} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau + \mathbf{C}^{-1}\int_{\tau=0} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau\right) .$$

This equation yields the formula related to theorem a:

$$\mathbf{x}_{p\infty} = (\mathbf{I}-\mathbf{C})^{-1}\mathbf{C}\int_{\tau=0}^{0} \mathbf{X}^{-1}(\tau)\mathbf{a}(\tau)d\tau = (\mathbf{I}-\mathbf{C})^{-1}\mathbf{x}_p(T) .$$

Thus, stability and periodicity conditions of the forced vibration are mutually *equivalent*. The history of the asymptotic, permanent vibration is not affected by the fact that the initial conditions are mostly uncertainly known. Now, the initial condition may be specified as:

$$\mathbf{x}_0 = \mathbf{x}_{p\infty}(\mathbf{I}-\mathbf{C})^{-1}\mathbf{x}_p(T) .$$

Significance of these results is in the fact that it is sufficient to investigate to a single period of motion comprehensively on the global behaviour of the system. The components of motion can be reasoned out, they are similar from period to period.

c. The amplitude of periodic vibration is limited if $|\lambda_j| \neq 1$, and $\mathbf{x}_p$ is finite. This is true even if the other two components of the general solution are divergent, resonant, that is unstable. Although motion instability is not due to the inhomogeneous term, the persistence of the stationary vibration requires certain excitation $\mathbf{a}(t)$ ($\mathbf{a}$ can also be constant). Namely, in the considered linear systems the natural vibration either decays or is intensified, its persistence has a slight (practically zero) probability because of the damping that occurs regularly. This marginal, rather unstable than stable case is theoretically much more significant for it makes a separation between asymptotically stable and unstable motions. It is an exciting problem to reveal the position of domains, related to parameters, furthermore, stability diagram and resonance map of the system. It is relatively easy to be answered for specifically low vibration of parameter values.

5. Distribution of Resonance Spots

a. Let us consider homogeneous equation

$$\dot{\mathbf{x}} = (\mathbf{A}_0 + \mathbf{P}(t))\mathbf{x}\,, \quad \mathbf{P}(t+T) = \mathbf{P}(t)\,, \quad |\mathbf{P}| \ll |\mathbf{A}_0|\,,$$

where $\mathbf{P}$ is of a small value by an appropriate norm, and is a purely periodical function (disturbance, perturbance). Let's find solution of the equation like in Chapter 2. about solution $\mathbf{x}_0$ of the system of constant coefficient $\mathbf{A}_0$ from $\mathbf{x}(t) = \mathbf{x}_0(t) + \mathbf{d}$

$$\dot{\mathbf{x}}_0(t) + \dot{\mathbf{d}} = (\mathbf{A}_0 + \mathbf{P}(t))(\mathbf{x}_0(t) + \mathbf{d})\,.$$

Neglecting secondary small term $\mathbf{Pd}$, deviation $\mathbf{d}$: obeys the following equation:

$$\dot{\mathbf{d}} = \mathbf{A}_0\mathbf{d} + \mathbf{P}(t)\mathbf{x}_0(t)\,.$$

Thus, solution of the inhomogeneous equation for initial conditions $t_0 = 0$, $\mathbf{x}_0(t_0) = \mathbf{x}_0$, $\mathbf{d}_0 = \mathbf{0}$ according to item **2.c**:

$$\mathbf{d}(t) = \mathbf{e}^{\mathbf{A}_0 t} \int_{\tau=0}^{t} \mathbf{e}^{-\mathbf{A}_0\tau}\mathbf{P}(t)e^{\mathbf{A}_0\tau}d\tau\mathbf{x}_0\,.$$

b. Factors of integrand are known. Exp $(\mathbf{A}_0 t)$ can be expressed in the form:

$$e^{\mathbf{A}_0 t} = \sum_{j=1}^{2n} \mathbf{A}_j e^{(\delta_j + i\omega_j)t} , \quad j = 1, 2, \dots , 2N,$$

if, by chance, eigenvalues $\mu_j = \delta_j + i\omega_j$ of the system with constant parameter differ from each other. And of course,

$$e^{-\mathbf{A}_0 t} = \sum_{k=1}^{2n} \mathbf{A}_k e^{(-\delta_k + i\omega_k)t} , \quad k = 1, 2, \dots , 2N .$$

where δ_j are damping ($\delta_j < 0$), ω_i are natural frequencies, i is the imaginary unit. Denote the primary circular frequency of the function $\mathbb{P}(t)$ of periodicity T by $\Omega = 2\pi/T$. Produce $\mathbb{P}(t)$ by its complex Fourier series as given below:

$$\mathbb{P}(t) = \sum_{m=-\infty}^{+\infty} \mathbb{P}_m e^{im\Omega t} , \quad \mathbb{P}_0 = 0 , \quad m = \pm 1, \pm 2, \dots .$$

For a real $\mathbb{P}$, constant coefficients $\mathbb{P}_{-m}$ and $\mathbb{P}_m$ are complex conjugates of each other. Recall the correspondence between real and complex series. Scalar and matrix series are formally identical.

c. Let's perform integration in the final formula of item a:

$$\int_{\tau=0}^{t} \sum_{j,k,m} \mathbf{A}_j \mathbb{P}_m \mathbf{A}_k e^{((\delta_k - \delta_j) + i(\omega_k - \omega_j + m\Omega))\tau} d\tau =$$

$$\sum_{j,k,m} \mathbf{A}_j \mathbb{P}_m \mathbf{A}_k \frac{e^{((\delta_k - \delta_j) + i(\omega_k - \omega_j + m\Omega))\tau}}{(\delta_k - \delta_j) + i(\omega_k - \omega_j + m\Omega)} .$$

d. becomes doubtless high in some combination of indices:

$$\delta_k - \delta_j = 0 , \quad \text{and} \quad \omega_k - \omega_j + m\Omega = 0 .$$

If damping δ_j is negligible, or there exist two of them nearly balanced, then instability may occur at any of combined frequencies

$$\frac{\omega_j - \omega_k}{\pm m} = \Omega , \quad j, k = 1, 2, \dots , N , \quad m = 1, 2 \dots .$$

For a real $\mathbf{A}_0$, this case is almost regular. Namely, eigenvalues μ_j are pairwise conjugate, or are real numbers in themselves, such as e. g.:

$$\delta_{j+N} = \delta_j \ , \quad \text{and} \quad \omega_{j+N} = -\omega_j \ , \quad j = 1, 2, \dots, N \ .$$

Accordingly, parametric resonance may take place even if damping occurs in the system, that is when the integer multiple of the exciting frequency Ω equals twice either of the natural frequencies ω_j:

$$\frac{2\omega_j}{\pm m} = \Omega \ , \quad j = 1, 2, \dots, N, \quad m = 1, 2, \dots \ .$$

It can be seen that instability may take place not only above the twice of the maximum ω_j, but below.

d. Refining the approximation above, it may be demonstrated that parametric resonance spots, as a function of exciting frequency $1/T = \Omega/2\pi$ *cannot be isolated.* Thus, intervals belonging to, a combination of frequencies are finite as a rule. In this respect, systems with constant or variable parameters are rather different. Widths of instability domains and *bands*, respectively, may be estimated, and it can be demonstrated that with increasing the order number m the band width, and in turn, the resonance intensity decreases. *Damping* makes resonance bands narrower, and eliminates those that have less intensive divergences above certain m. Thus, in the case of occurrence of damping, there is a *minimum frequency limit*, such that at lower frequencies no parametric resonance develops. This limit affects, of course, the parametric excitation of a specific intensity. By increasing the excitation amplitude, the frequency limit is lowered.

Stochastic disturbance in the system shadows sharp limits, and at low frequencies the random alteration of parameters has an effect similar to damping. Nevertheless, in practical cases, parametric resonance occurs only for small m values. It is interesting to mention, that even a weak parametric excitation may multiply the numbers of critical domains of the system. Additionally, the resonance limits are also excited. Relative to that of the systems of constant parameters, the extension of the dangerous domain of frequencies is increasing depending on the strength of damping. The extension in the upper domain is approximately two-fold, and, in general, it decreases in the lower domain to its half or third.

With increasing changes of parameters, resonance bands grow wider, then result in extended, often overlapping intervals, called *instability domains.* The resonance map often appears to be rather intricate (sometimes it is fractal-like). These maps can only be produced by numeric computations. Of course, for simple cases, stability diagrams can be found in the relevant literature (e. g. [12, ..., 17]). Even if the appropriate diagram for a specific problem is found, it is better to get acquainted with the theoretical backgrounds.

Auxiliary motions in technical systems are in most cases definitely harmful. They cause a detriment of the accuracy of functioning, generate vibration and noise, therefore, reduce the duration, usability, etc. of the machine. For the machinery, the worst is operation at the resonance frequencies. The vibration of the machinery sooner or later will exceed the *linear* (e.g. knocking) *limits*. The amplitude is going to be large, but it still remains limited. In the worst case, the amplitude will result in a permanent change of the system, and may demolish it. In the proximity of certain resonance bands, mainly of those with even order numbers ($m = 2, 4, 6, \dots$), stationary vibration may be important in itself, and non-linear vibrations may develop earlier than the resonances.

In order to reduce the risk of resonance, changes of parameters have to be '*smoothed*', and the system should be '*tuned up*'. There are several possibilities to constrain parametric excitation. Most known ones are balancing by mass, increasing damping artificially, changing stiffness or mass of the elements. or, besides these, certain active compensations may be applied. Design parameters can possibly be selected so that the operation frequency be above the highest. or below the lowest critical values. or between two adjacent resonance bands of possible low order numbers. If the machinery is intended to generate excitation, our aim is just the opposite. If design parameters are randomly varied, without a stability map, then, instead of the expected reduction of vibration, we often get an increase. and vice versa. Thus, it is expedient to expose the expectable behaviour of systems with the parametric excitation earlier in the design phase. We have to be also conscious of fact that with increasing loads and operating velocities, parametric resonances may occur in mechanisms that have been believed to be stable.

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