PROJECTIVE GEOMETRY IN ENGINEERING[1]

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Dedicated to Professor Julius Strommer on the occasion of his 75th birthday

Abstract

After a historical introduction some applications of projective geometry are indicated by the modern calculus of linear algebra.

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1. Historical Approach

The verisimilar or – in up-to-date terminology – photorealistic representation of the phenomena and objects of the world was a primeval endeavour of the painters. From hundreds rather thousands of years the realism was an indispensable condition of the talent and success. The correctly applied perspective is an essential condition of verisimilitude. In the investigations of the laws of picturing of the eye LEONARDO DA VINCI (1452–1519) and ALBRECHT DÜRER (1471–1528) had important roles. As the laws we are referring to belong to geometry, the question arises: how long have these laws been examined by geometricians, and with what sort of results? Before enumerating names and dates, mention must be made that geometry originally meant '*measuring*', but neither the distances nor the angles remain stationary at central projection. That means central projection does not belong to geometry in its classical meaning. Nevertheless we can find theorems at the ancient Greek mathematicians: MENELAOS (about 100 AD), PAPPOS (about 320 AD) both from Alexandria, which can be inserted in the sequence of theorems of projective geometry developed later on.

Thinking of the usual perspective with horizontal ground plane and vertical picture plane, the horizon is a line of the picture plane, which does not have any '*original*', that means there is not any line of the ground plane, which would have an image as the horizon. The idea, to extend the

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ground plane with the '*line at infinity*', which is the geometrical set of the '*points at infinity*' or 'ideal points' comes naturally. The idea that the projective plane equals a usual (affine) plane plus an ideal line, (which can be transformed into general position by central projection) was the opinion of DESARGUES (1593–1662). A converse relation was suggested by PONCELET (1788–1867). He had elaborated in his work *Traité des propriétés projectives des figures* the system of projective geometry as an independent science in which the affine plane appears as a projective plane minus an arbitrary line. It is important to underline the more comprehensive character of this conception, because not only the affine but the hyperbolic and the elliptic spaces (planes) are particular cases of the projective space (plane) grasped in this meaning (see e.g. [2] and [5]).

The basic tools of the analytic projective geometry are the homogeneous coordinates and the linear transformations. The homogeneous coordinates we are discussing in the 2nd chapter were initiated by MÖBIUS (1790–1868) and PLÜCKER (1801–1868). For further historical remarks we refer to STROMMER [6] and [7].

2. Real Projective Plane $\wp^2(\Re)$

The two dimensional projective space over the field of real numbers is an analytical model for the projective plane. In this model we make realisations of the object '*point*', '*straight line*' and the relation '*incident*'. The proof that the model satisfies the usual set of axioms of the projective plane can be found e.g. in [1] and [2].

Definitions

point: $\mathbf{X}(\mathbf{x}(x^1, x^2, x^3)^t)$ triplet of real numbers (in column matrix), if there is at least one element different from zero. The triplets $\mathbf{x}(x^1, x^2, x^3)^t$ and $\mathbf{x}\lambda(x^1\lambda, x^2\lambda, x^3\lambda)^t$ $(\lambda \in \Re\backslash 0)$ represent the same point.

straight line: $u(\mathbf{u}[u_1, u_2, u_3])$ triplet of real numbers (in row matrix), if there is at least one element different from zero. Again the equivalence $\mathbf{u}[u_1, u_2, u_3] \sim \lambda\mathbf{u}[\lambda u_1, \lambda u_2, \lambda u_3]$ $(\lambda \in \Re\backslash 0)$ means that they represent the same line.

incidence: the point $\mathbf{X}$ lies on the line u, or in other words the line u passes through the point $\mathbf{X}$, if the equality

$$u_1 x^1 + u_2 x^2 + u_3 x^3 = 0 \tag{2.1}$$

satisfies. This can be written in the form $u_i x^i = 0$ or $\mathbf{ux} = 0$, too.

We do not make strict distinction between the analytic and the vector space model. A point can be given by coordinates in parentheses or by a column vector, a line by coordinates in brackets or by row vector, with superscript indices and subscript indices, respectively. It can easily be verified that any pair of points obtains a straight line, any pair of lines determines a point.

Non-Collinear Points, Projective Reference System

Let $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ triplet of points be not collinear:

$$\begin{vmatrix} p_1^1 & p_2^1 & p_3^1 \\ p_1^2 & p_2^2 & p_3^2 \\ p_1^3 & p_2^3 & p_3^3 \end{vmatrix} \neq 0 .$$

If the determinant above was zero, the points were collinear. The free proportionality factors can be normalized by a fourth point $\mathbf{P}(\mathbf{p})$, which is non-collinear with any two of $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$, so that $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. Then any point of the plane $\mathbf{Z}(\mathbf{z})$ can be written in the form of

$$\mathbf{z} = \mathbf{p}_1 z^1 + \mathbf{p}_2 z^2 + \mathbf{p}_3 z^3 ,$$

or in detail:

$$\mathbf{z} = \begin{pmatrix} p_1^1 z^1 + p_2^1 z^2 + p_3^1 z^3 \\ p_1^2 z^1 + p_2^2 z^2 + p_3^2 z^3 \\ p_1^3 z^1 + p_2^3 z^2 + p_3^3 z^3 \end{pmatrix} .$$

The triplet $(z^1, z^2, z^3)^t \sim (z^1\lambda, z^2\lambda, z^3\lambda)^t \nsim (0,0,0)^t$ can be considered as coordinates of the point with respect to the points $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ and $\mathbf{P}$. Now we can introduce a reference system in the projective plane. This frame can be visualized by the perspective view of the Cartesian mosaic as an infinite chess board (see *Fig. 2.1*).

The general projective reference system as the perspective image of the Cartesian mosaic is determined by the quadruple of points $\{\mathbf{E}_3, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}\}$. This frame is called *projective homogeneous reference system.*

Cartesian Homogeneous Rectangular Reference System

A particular case of the coordinate system represented in *Fig. 2.1* is the Cartesian homogeneous reference system shown in the *Fig. 2.2.*

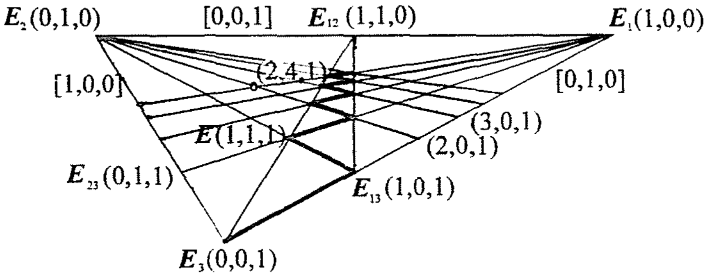


Fig. 2.1. Projective homogeneous coordinates

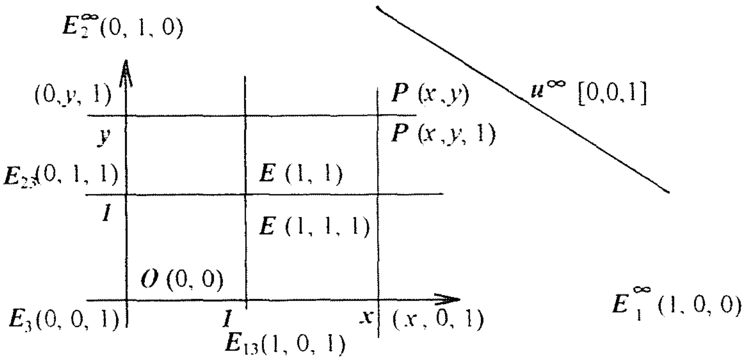


Fig. 2.2. Cartesian homogeneous rectangular reference system

The correspondence between the Cartesian homogeneous rectangular and the usual Cartesian coordinates of points can be read off the *Fig. 2.2.*

Historically, Cartesian homogeneous coordinates and the projective homogeneous ones were derived from the Cartesian rectangular coordinates. For usual points the procedure is

$$\mathbf{P}(x,y) \Rightarrow \mathbf{P}(x,y,1) \sim \mathbf{P}(x^1 = xc, x^2 = yc, x^3 = c) \quad \text{with} \quad c \in \Re \setminus 0$$

vice and

$$x^3 \neq 0, \quad x = \frac{x^1}{x^3}; \quad y = \frac{x^2}{x^3}, \quad P(x^1, x^2, x^3) \sim \mathbf{P}(x,y,1) \Rightarrow \mathbf{P}(x,y)$$

versa.

This procedure can be extended for any point at infinity determined by proportional direction vectors

$$\mathbf{V}^{\infty}(\mathbf{v} \sim \mathbf{v}\lambda)$$

or

$$\mathbf{V}^{\infty}(v,w) \Leftrightarrow \mathbf{V}^{\infty}(v,w,0) \sim V^{\infty}(x^1 = vc,\ x^2 = wc,\ x^3 = 0)\ .$$

All these arguments can be formulated that the projective plane is '*spanned*' in a real 3-dimensional vector space: points as 1-subspaces, lines as 2-subspaces, or equivalently, as proportionality classes of linear 1-forms which takes 0 on the corresponding 2-subspaces.

3. Real Projective Space $\wp^3(\Re)$

Following the strategy of the previous chapter we define the real projective space in any real 4-vector space and its dual '*form*' space.

Definitions

point: $\mathbf{X}(\mathbf{x}(x^1, x^2, x^3, x^4)^t)$ quadruple of real numbers written in column, if there is at least one element different from zero. The quadruples $\mathbf{x}(x^1, x^2, x^3, x^4)^t$ and $\mathbf{x}\lambda(x^1\lambda, x^2\lambda, x^3\lambda, x^4\lambda)^t$ $(\lambda \in \Re\backslash 0)$ represent the same point.

plane: $u(\mathbf{u}[u_1, u_2, u_3, u_4])$ quadruple of real numbers arranged in row, if there is at least one element different from zero. The equivalence $\mathbf{u}[u_1, u_2, u_3, u_4] \sim \lambda\mathbf{u}[\lambda u_1, \lambda u_2, \lambda u_3, \lambda u_4]$ $(\lambda \in \Re\backslash 0)$ characterizes the same plane.

incidence: the point $\mathbf{X}$ lies on the plane u, or in other words the plane u passes through the point $\mathbf{X}$, if the equality

$$u_1x^1 + u_2x^2 + u_3x^3 + u_4x^4 = 0\ ,$$

i.e. $\mathbf{ux} = 0$ satisfies.

A straight line can be determined for instance with the help of two planes. The points, whose coordinates satisfy two homogeneous equations of the given planes form a straight line as a 1-dimensional set of points, incident to both planes:

$$u_1^1x^1 + u_2^1x^2 + u_3^1x^3 + u_4^1x^4 = 0\ ,$$
$$u_1^2x^1 + u_2^2x^2 + u_3^2x^3 + u_4^2x^4 = 0\ .$$

The next modification of the previous sentence is an example for the duality principle: the planes, whose coordinates satisfy two homogeneous equations of the given points form a pencil as a 1-set of planes through the line determined by both points:

$$u_1x_1^1 + u_2x_1^2 + u_3x_1^3 + u_4x_1^4 = 0 \ ,$$
$$u_1x_2^1 + u_2x_2^2 + u_3x_2^3 + u_4x_2^4 = 0 \ .$$

The mutual positions of two straight lines are the following: complanar (or intersecting) if they have a plane (point) in common, otherwise skew.

Projective Homogeneous Reference System, Simplex Coordinates

Coordinatization in projective space $\wp^3(\Re)$ is analogous to the method followed in $\wp^2(\Re)$. Let the base tetrahedron (i.e. simplex) and the unit point be given as *Fig. 3.1* shows.

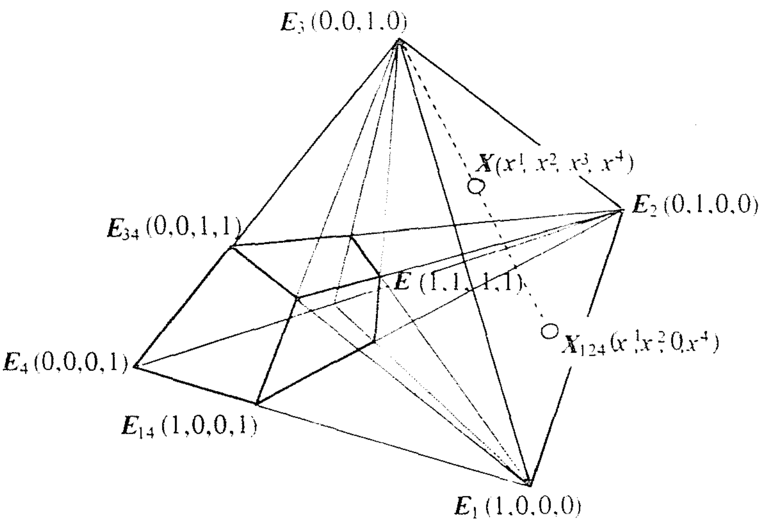


Fig. 3.1. Projective homogeneous reference system

The Cartesian homogeneous rectangular reference system can also be derived analogously. The point $\mathbf{X}_{124}(x^1, x^2, 0, x^4)$ is the projection of $\mathbf{X}(x^1, x^2, x^3, x^4)$ from $\mathbf{E}_3$ onto the plane $\{\mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_4\}$. The plane $[0, 0, 0, 1]$

which contains the points $(x^1, x^2, x^3, 0)^t$ corresponds to the '*plane at infinity*'. Except the points at infinity the others can be analogized with the points of the Cartesian rectangular coordinate system:

$$\mathbf{P}\left(x = \frac{x^1}{x^4}\ ,\quad y = \frac{x^2}{x^4}\ ,\quad z = \frac{x^3}{x^4}\right)\ ,\qquad x^4 \neq 0\ .$$

4. Projective Reference System in General, Transformation of Bases and Coordinates

We keep in our mind the reference system $\{\mathbf{E}_4,\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \mathbf{E}\}$ given by the four base vectors

$$\mathbf{E}_1 : \mathbf{e}_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\ ,\ \mathbf{E}_2 : \mathbf{e}_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\ ,\ \mathbf{E}_3 : \mathbf{e}_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\ ,$$

$$\mathbf{E}_4 : \mathbf{e}_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \text{ and } \mathbf{E} : \mathbf{e} = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

of a 4-vector space, which satisfy the equation $\mathbf{e} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4$.

Let us take any set of five points $\{\mathbf{E}_{4'},\ \mathbf{E}_{1'},\ \mathbf{E}_{2'},\ \mathbf{E}_{3'}, E'\}$ given by the vectors

$$\mathbf{E}_{1'} : \mathbf{e}_{1'} = \begin{pmatrix}e^1_{1'}\\e^2_{1'}\\e^3_{1'}\\e^4_{1'}\end{pmatrix}\ ;\ \mathbf{E}_{2'} : \mathbf{e}_{2'} = \begin{pmatrix}e^1_{2'}\\e^2_{2'}\\e^3_{2'}\\e^4_{2'}\end{pmatrix}\ ;\ \mathbf{E}_{3'} : \mathbf{e}_{3'} = \begin{pmatrix}e^1_{3'}\\e^2_{3'}\\e^3_{3'}\\e^4_{3'}\end{pmatrix}\ ;$$

$$\mathbf{E}_{4'} : \mathbf{e}_{4'} = \begin{pmatrix}e^1_{4'}\\e^2_{4'}\\e^3_{4'}\\e^4_{4'}\end{pmatrix}\ ;\ \mathbf{E}' : \mathbf{e}' = \begin{pmatrix}e^1\\e^2\\e^3\\e^4\end{pmatrix}$$

which satisfies two conditions

1. there are not any four of them lying on a plane,
2. $\mathbf{e}'\rho = \mathbf{e}_{1'} + \mathbf{e}_{2'} + \mathbf{e}_{3'} + \mathbf{e}_{4'}$ holds for appropriate $\rho \in \Re\backslash 0$.

An arbitrary point is supposed to be given by the coordinates $(x^1, x^2, x^3, x^4)^t$ and $(x^{1'}, x^{2'}, x^{3'}, x^{4'})^t$ with respect to $\{\mathbf{E}_4,\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3,\ \mathbf{E}\}$ and $\{\ \mathbf{E}_{4'},\ \mathbf{E}_{1'},\ \mathbf{E}_{2'},\ \mathbf{E}_{3'},\ \mathbf{E}'\}$, respectively. We use the Einstein summation convention for repeated lower and upper indices, furthermore we use Schouten's primes for indices. Thus we express

$$\mathbf{e}_{i'} = \mathbf{e}_j e^j_{i'} \quad \text{and} \quad \mathbf{x} = \mathbf{e}_j x^j = \mathbf{e}_{i'} x^{i'} = \mathbf{e}_j e^j_{i'} x^{i'} \ . \tag{4.1}$$

The connection of the bases above determines the coordinate transformation of any point $\mathbf{X}(\mathbf{x})$ with respect to both bases

$$x^j = e^j_{i'} x^{i'} \quad \text{or more precisely} \quad x^j \sim e^j_{i'} x^{i'} \ . \tag{4.2}$$

Now we add some remarks to the transformation of coordinates.

- Let $e^{i'}_j$ denote the inverse matrix of $e^j_{i'}$ (proportional to the transposed matrix of corresponding 3×3 sub-determinants, i.e. to the adjoint matrix). Thus

$$\mathbf{e}_j = \mathbf{e}_{i'} e^{i'}_j \ , \quad e^{i'}_j e^j_{k'} = \delta^{i'}_{k'} \quad \text{or} \quad e^k_{i'} e^{i'}_j = \delta^k_j \tag{4.3}$$

  hold with the Kronecker delta symbol. Then for any plane $u(\mathbf{u})$ the equations

$$0 = \mathbf{u}\mathbf{x} = u_j x^j = u_j e^j_{i'} x^{i'} = u_{i'} x^{i'} = u_{i'} e^{i'}_j x^j$$

  stand with respect to both reference systems, which shows us the coordinate change of a plane

$$u_j \sim u_{i'} e^{i'}_j \quad \text{or} \quad u_{i'} \sim u_j e^j_{i'}$$

  up to a real factor of proportionality, again.

- The projection $\mathbf{X} \mapsto \mathbf{X}_{124}$ in *Fig. 3.1* can be expressed in matrix form:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} x^1 \\ x^2 \\ 0 \\ x^4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \ . \tag{4.4}$$

  It is no more a coordinate transformation but a projective mapping (see later on).

– If the second condition for $\{\mathbf{E}_{4'},\ \mathbf{E}_{1'},\ \mathbf{E}_{2'},\ \mathbf{E}_{3'},\ \mathbf{E}'\}$ is not satisfied, we can reduce to the problem just discussed. Then the equation

$$\mathbf{e}\rho = \mathbf{e}_{1'}\lambda + \mathbf{e}_{2'}\mu + \mathbf{e}_{3'}\nu + \mathbf{e}_{4'}\sigma$$

is solvable for $\lambda, \mu, \nu, \sigma$ and the new basis vectors $\mathbf{e}_{1'}\lambda, \mathbf{e}_{2'}\mu, \mathbf{e}_{3'}\nu, \mathbf{e}_{4'}\sigma$ fulfil the second condition. The second condition is a kind of normalization of the basis vectors. Any set of five points, which satisfies the first condition can be considered as a basis of $\wp^3(\Re)$, but the coordinates must be normalized.

– In that case when both $\{\mathbf{E}_4, \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \mathbf{E}\}$ and $\{\mathbf{E}_{4'}, \mathbf{E}_{1'}, \mathbf{E}_{2'}, \mathbf{E}_{3'}, \mathbf{E}'\}$ are Cartesian homogeneous rectangular normed systems, the matrix $e^j_{i'}$ gets a special (affine) form

$$(e^j_{i'}) = \left(\begin{array}{ccc|c} e^1_{1'} & e^1_{2'} & e^1_{3'} & e^1_{4'} \\ e^2_{1'} & e^2_{2'} & e^2_{3'} & e^2_{4'} \\ e^3_{1'} & e^3_{2'} & e^3_{3'} & e^3_{4'} \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \quad \text{symbolized by} \quad \left(\begin{array}{c|c} \mathbf{R} & \mathbf{d} \\ \hline 0 & 1 \end{array}\right) .$$

where the $\mathbf{R}$ is a 3×3 orthogonal matrix and $\mathbf{d} = (e^1_{4'}, e^2_{4'}, e^3_{4'})^t$ describes the translation of the origin $\mathbf{E}_{4'}$ in the old system by (4.1). Conclusion: The old Cartesian coordinates can be expressed by the new ones according to (4.2) as follows

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ \dots \\ 1 \end{pmatrix} = \left(\begin{array}{c|c} \mathbf{R} & \mathbf{d} \\ \hline 0 & 1 \end{array}\right) \begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \\ \dots \\ 1 \end{pmatrix} .$$

– We sharply distinguish a projective transformation of the projective space $\wp^3 := \wp^3(\Re)$ from the coordinate transformation, although both will be described by linear mappings. As usual, a projective mapping α can be described by a linear mapping of the modelling 4-vector space

$$\alpha : \wp^3 \to \wp^3 , \quad \mathbf{X}(\mathbf{x}) \ \mapsto \ \mathbf{Y}(\mathbf{y}) , \quad \mathbf{x} \ \mapsto \ \mathbf{y} \sim \alpha\mathbf{x} .$$

In a fixed reference system
$\{\ \mathbf{E}_4, (\mathbf{e}_4),\ \mathbf{E}_1, (\mathbf{e}_1),\ \mathbf{E}_2, (\mathbf{e}_2),\ \mathbf{E}_3, (\mathbf{e}_3),\ \mathbf{E}(\mathbf{e} \sim \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4)\}$
the equations

$$\mathbf{y} = \mathbf{e}_i y^i \sim \alpha\mathbf{x} = \alpha(\mathbf{e}_j x^j) = (\alpha\mathbf{e}_j)x^j = \mathbf{a}_j x^j = \mathbf{e}_i a^i_j x^j \qquad (4.5)$$

describe how to obtain the coordinates of the image $\mathbf{Y}(\mathbf{y} = \mathbf{e}_i y^i)$ from the object $\mathbf{X}(\mathbf{x} = \mathbf{e}_i x^i)$:

$$y^i \sim a^i_j x^j \quad \text{by the matrix} \quad a^i_j \tag{4.6}$$

up to a real factor of proportionality, again.

We see that a^i_j is determined by the α - images of base points $\mathbf{E}_j$ and $\mathbf{E}$. Thus we can describe the motion of a coordinate system, too. Projective transformations will describe, e.g., proper motions in $\wp^3$ with regular α, i.e. $\det(a^i_j) \neq 0$, or projections, e.g. as (4.3) by degenerated linear mapping.

– Of course in another reference system $\{\mathbf{E}_{i'}(\mathbf{e}_{i'}),\ \mathbf{E}'(\mathbf{e}' \sim \mathbf{e}_{1'} + \mathbf{e}_{2'} + \mathbf{e}_{3'} + \mathbf{e}_{4'})\}$ our projective mapping α has another matrix $a^{i'}_{j'}$ defined by

$$\alpha \mathbf{e}_{j'} = \mathbf{a}_{j'} = \mathbf{e}_{i'} a^{i'}_{j'}. \tag{4.7}$$

The basis transformation (4.1) provides us

$$\alpha \mathbf{e}_{j'} = \alpha(\mathbf{e}_j e^j_{j'}) = \mathbf{e}_i a^i_j e^j_{j'} \quad \text{and} \quad \mathbf{e}_{i'} a^{i'}_{j'} = \mathbf{e}_i e^i_{i'} a^{i'}_{j'} \Rightarrow$$

$$a^i_j e^j_{j'} = e^i_{i'} a^{i'}_{j'} \quad \text{or} \quad e^{i'}_i a^i_j e^j_{j'} = a^{i'}_{j'} \tag{4.8}$$

the appropriate formula, where the inverse basis transformation (4.3) has also been applied. Our conventions simplify cumbersome calculations, they are independent of dimension. We remark that coordinate transformations and mappings may depend on time and we can combine them by appropriate multiplications.

5. Application 1: Fitting Conic Section

The problem arises in the plane: write the parametric equation system of a conic section inscribed in a triangle according to the *Fig. 5.1.*

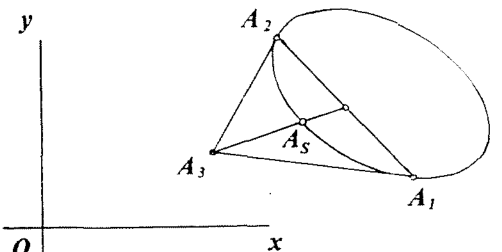


Fig. 5.1. Fitting conic section

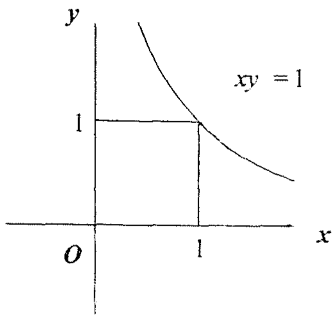
The points are given by their Cartesian coordinates: $\mathbf{A}_1(x1, y1)$, $\mathbf{A}_2(x2, y2)$, $\mathbf{A}_3(x3, y3)$, $\mathbf{A}_S(xs, ys)$. Before starting with the solution of the problem, we recall the equation of the unit hyperbola in Cartesian rectangular and Cartesian homogeneous reference systems.

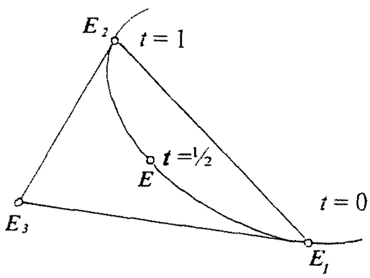
For this concrete problem we simplify our notation, do not use the prime convention, but proceed in that sense:

$$xy = 1 ,$$

$$x = \frac{z^1}{z^3} , \quad y = \frac{z^2}{z^3} \quad \Rightarrow \quad \frac{z^1}{z^3} \cdot \frac{z^2}{z^3} - 1 = 0 ,$$

$$z^1 z^2 - z^3 z^3 = 0 \quad \Rightarrow \quad \begin{aligned} z^1 &\sim (1-t)\cdot(1-t) , \\ z^2 &\sim t \cdot t , \\ z^3 &\sim t\cdot(1-t). \end{aligned}$$





Now the steps of the algorithm are as follows.

1. Change to the Cartesian homogeneous rectangular coordinates of the given points.

$$\begin{aligned} &\mathbf{A}_1(a_1^1 = x1 , \quad a_1^2 = y1 , \quad a_1^3 = 1) , \\ &\mathbf{A}_2(a_2^1 = x2 , \quad a_2^2 = y2 , \quad a_2^3 = 1) , \\ &\mathbf{A}_3(a_3^1 = x3 , \quad a_3^2 = y3 , \quad a_3^3 = 1) , \\ &\mathbf{A}_S(a_S^1 = xs , \quad a_S^2 = ys , \quad a_S^3 = 1) . \end{aligned}$$

2. $\{\mathbf{A}_3, \mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_S\}$ can be considered as a basis, but the coordinates must be normalized.
   That means to solve the equation

$$\mathbf{a}_S = \mathbf{a}_1 h^1 + \mathbf{a}_2 h^2 + \mathbf{a}_3 h^3$$

for h^i $i = 1, 2, 3$. Now take the basis given by the vectors and coordinates as follows:

$$\begin{array}{l} \mathbf{A}_1(\mathbf{a}_1 h^1 (a_1^1 h^1 \ , \ \ a_1^2 h^1 \ , \ \ a_1^3 h^1)^t) \ , \\ \mathbf{A}_2(\mathbf{a}_2 h^2 (a_2^1 h^2 \ , \ \ a_2^2 h^2 \ , \ \ a_2^3 h^2)^t) \ , \\ \mathbf{A}_3(\mathbf{a}_3 h^3 (a_3^1 h^3 \ , \ \ a_3^2 h^3 \ , \ \ a_3^3 h^3)^t) \ , \\ \mathbf{A}_S(\mathbf{a}_S (a_S^1 \ , \ \ a_S^2 \ , \ \ 1)^t) \ . \end{array}$$

3. The formula analogous to (4.2) leads back to the original Cartesian homogeneous system. The equation system $x^j = a_i^j z^i$ with the given data has the following form

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \sim \begin{pmatrix} a_1^1 h^1 & a_2^1 h^2 & a_3^1 h^3 \\ a_1^2 h^1 & a_2^2 h^2 & a_3^2 h^3 \\ h^1 & h^2 & h^3 \end{pmatrix} \begin{pmatrix} (1-t)\cdot(1-t) \\ t \cdot t \\ t\cdot(1-t) \end{pmatrix} .$$

4. As the last step let us return to the Cartesian rectangular frame by substituting

$$x^1/x^3 = x \ , \quad x^2/x^3 = y \ ,$$

$$a_1^1 = x1 \ , \ a_1^2 = y1 \ , \ a_2^1 = x2 \ , \ a_2^2 = y2 \ , \ a_3^1 = x3 \ , \ a_3^2 = y3 :$$

$$x = \frac{x1 \cdot h^1 \cdot (1-t)\cdot(1-t) + x2 \cdot h^2 \cdot t \cdot t + x^3 \cdot h^3 \cdot t \cdot (1-t)}{h^1 \cdot (1-t)\cdot(1-t) + h^2 \cdot t \cdot t + h^3 \cdot t \cdot (1-t)} ,$$

$$y = \frac{y1 \cdot h^1 \cdot (1-t)\cdot(1-t) + y2 \cdot h^2 \cdot t \cdot t + y^3 \cdot h^3 \cdot t \cdot (1-t)}{h^1 \cdot (1-t)\cdot(1-t) + h^2 \cdot t \cdot t + h^3 \cdot t \cdot (1-t)} .$$

This type of fitting conic section is in close connection with the NURBS theory. Some more details can be found in [3].

6. Application 2: Kinematics

The first course of robotics begins with kinematics of manipulator. The geometrical background of robot kinematics is concerned with the establishment of transformations in various moving frames and the relations among them. The type of frame used most frequently in robot geometry is the Cartesian homogeneous rectangular (normed) system, which we have discussed in chapter 4.

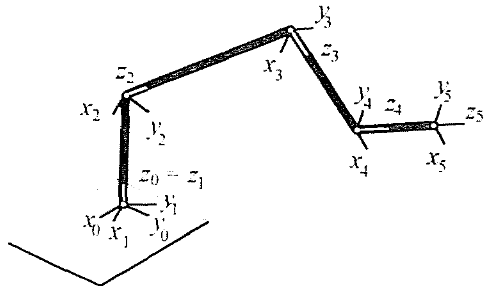


Fig. 6.1. Open kinematic chain

Kinematic chain

In kinematical point of view one can think of a robot as a set of rigid links connected by various joints [4]. In geometrical point of view it is a set of frames, fixed to the links, and matrices, which join together couples of consecutive frames. This set of links and joints is composed into an *open kinematic chain.* (*Fig. 6.1*).

The simple *revolute joint* like a hinge allows a relative rotation about a single axis, and a simple *prismatic joint* permits a linear motion along a single axis. Both have a single *degree-of-freedom* of motion as a parameter: the angle of rotation in the case of revolute joint and the amount of linear displacement in the case of prismatic joint. The restriction to the simple joints does not involve any real loss of generality, since joints such as socket joint (two degrees-of-freedom) or spherical wrist (three degrees-of-freedom) can always be thought as a succession of single degree-of-freedom joints with links of length zero in between. With the assumption that each joint has a single degree-of-freedom, the action of each joint can be described by a single real number; the angle of rotation in the case of revolute joint or the distance of displacement in the case of prismatic joint. The aim of this chapter is to determine the cumulative effect of the entire set of joint variables.

Suppose the links of the robot and the bases attached to them are numbered from 1 to n on the left as follows

$$\{{}^1\mathbf{e}_s\}, \{{}^2\mathbf{e}_s\}, \dots, \{{}^n\mathbf{e}_s\} \quad s = 1,2,3,4 \text{ indices of the basis vectors.} \tag{6.1}$$

The basis $\{{}^o\mathbf{e}_s\}$ of the robot is taken as link 0 (zero). The joints are also numbered from 1 to n (on the left) and the i-th joint connects the i-1-th and the i-th links by a matrix

$$({}^{i}\mathbf{A}) = ({}^{i}a_s^r), \text{ i.e. } ({}^{i-1}\mathbf{e}_r)({}^{i}a_s^r) = ({}^{i}\mathbf{e}_s) \tag{6.2}$$

as a particular projective mapping.

The matrix $({}^{i}\mathbf{A})$ depends on a single variable $({}^{i}t)$, on a rotation angle in the case of a revolute joint or a joint displacement in a prismatic joint. In (6.2) the summation convention is applied for the running index r (on the right only). Let the transformation between the i-th and j-th reference systems be denoted, in general, by

$$({}_{i}^{j}\mathbf{A}) = ({}_{i}^{j}a_s^r) \text{ , i.e. } ({}^{i}\mathbf{e}_r)({}_{i}^{j}a_s^r) = ({}^{j}\mathbf{e}_s) \ . \tag{6.3}$$

That means:

$$\begin{aligned} &({}_{i}^{i}\mathbf{A}) = \mathbf{I} \quad \text{the identity matrix, moreover,} \\ &({}_{i}^{j}\mathbf{A}) = ({}^{i+1}\mathbf{A})({}^{i+2}\mathbf{A})\dots({}^{j}\mathbf{A}) \quad \text{if} \quad i < j \ , \\ &({}_{i}^{j}\mathbf{A}) = ({}_{j}^{i}\mathbf{A})^{-1} \qquad\qquad\quad \text{if} \quad i > j \ . \end{aligned}$$

Since the 4×4 matrices come from Cartesian homogeneous orthonormed 3-bases, each of them is composed from a rotation sub-matrix and a transition vector, as we saw in chapter 4:

$$^{i}\mathbf{A} = \left(\begin{array}{c|c} {}^{i}\mathbf{R} & {}^{i}\mathbf{d} \\ \hline 0 & 1 \end{array} \right) .$$

Let $\mathbf{t} = [{}^{1}t, {}^{2}t, \dots, {}^{n}t]$ denote the n-tuple of joint variables. The matrix ${}_{0}^{n}\mathbf{A}$, which describes the moving object in the base frame, is a function of $\mathbf{t}$.

$${}_{0}^{n}\mathbf{A}(\mathbf{t}) = \left(\begin{array}{c|c} {}_{0}^{n}\mathbf{R}(\mathbf{t}) & {}_{0}^{n}\mathbf{d}(\mathbf{t}) \\ \hline 0 & 1 \end{array} \right)$$

7. Application 3: Analytic Description of Central Projection

CAD systems use many frames. There is a distinguished one among them, the *World-Coordinate System*. This is the frame, where the scene is furnished, even the system of projection. A system of projection is usually given by the *Camera Position*, the *Target Point* and the *Lens Length*. The *camera, target* and *lens length* (or *zoom*) metaphors help the user to imagine

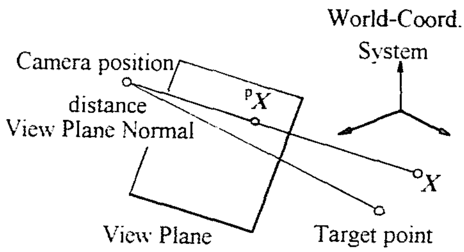


Fig. 7. 1.a. System of projection

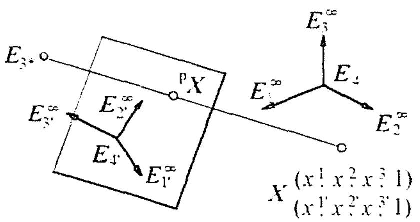


Fig. 7. 1.b. Reference systems

the viewing system. In geometrical point of view these data can be transformed into the common geometric data (*Fig. 7.1.a.*): *camera* $\leftrightarrow$ *Centre of Projection,* the line between *camera* and *target* $\leftrightarrow$ *View Plane Normal* and *lens length* $\leftrightarrow$ *View Plane Distance* (distance between the centre of projection and the picture plane).

The world-coordinate system is given by the basis $\{\mathbf{E}_4, \mathbf{E}_1^\infty, \mathbf{E}_2^\infty, \mathbf{E}_3^\infty, \mathbf{E}\}$ and the viewing-reference coordinate system attached to the picture plane is precisely given by the basis $\{\mathbf{E}_{4'}, \mathbf{E}_{1'}^\infty, \mathbf{E}_{2'}^\infty, \mathbf{E}_{3'}^\infty, \mathbf{E}'\}$ (*Fig. 7.1.b*) with respect to $\{\mathbf{E}_4, \mathbf{E}_1^\infty, \mathbf{E}_2^\infty, \mathbf{E}_3^\infty, \mathbf{E}\}$

$$\mathbf{E}_{1'} : \mathbf{e}_{1'} = \begin{pmatrix} e_{1'}^1 \\ e_{1'}^2 \\ e_{1'}^3 \\ 0 \end{pmatrix}, \mathbf{E}_{2'} : \mathbf{e}_{2'} = \begin{pmatrix} e_{2'}^1 \\ e_{2'}^2 \\ e_{2'}^3 \\ 0 \end{pmatrix}, \mathbf{E}_{3'} : \mathbf{e}_{3'} = \begin{pmatrix} e_{3'}^1 \\ e_{3'}^2 \\ e_{3'}^3 \\ 0 \end{pmatrix},$$

$$\mathbf{E}_{4'} : \mathbf{e}_{4'} = \begin{pmatrix} e_{4'}^1 \\ e_{4'}^2 \\ e_{4'}^3 \\ 0 \end{pmatrix}, \mathbf{E}' : \mathbf{e}' = \begin{pmatrix} e^1 \\ e^2 \\ e^3 \\ 0 \end{pmatrix}; \tag{7.1}$$

that means

$$\mathbf{e}_{i'} = \mathbf{e}_j e^j_{i'} \quad \text{and} \quad \mathbf{e}' \sim \mathbf{e}_{1'} + \mathbf{e}_{2'} + \mathbf{e}_{3'} + \mathbf{e}_{4'}.$$

The coordinates of the point **X** in the viewing-reference coordinate system are determined by $e^{i'}_j$ (the inverse matrix of $e^j_{i'}$) in the form of $x^{i'} \sim e^{i'}_j x^j$ (see 4.1 - 4.3).

Let us introduce a third reference system $\{\mathbf{E}_{4^*}(\mathbf{e}_{4^*}), \mathbf{E}_{1^*}(\mathbf{e}_{1^*}), \mathbf{E}_{2^*}(\mathbf{e}_{2^*}), \mathbf{E}_{3^*}(\mathbf{e}_{3^*}), \mathbf{E}^*(\mathbf{e}^* \sim \mathbf{e}_{1^*} + \mathbf{e}_{2^*} + \mathbf{e}_{3^*} + \mathbf{e}_{4^*})\}$ by $\mathbf{e}_{i^*} \sim \mathbf{e}_i e^{i'}_{i^*}$.

$$(\mathbf{e}_{1^*}, \mathbf{e}_{2^*}, \mathbf{e}_{3^*}, \mathbf{e}_{4^*}) \sim (\mathbf{e}_{1'}, \mathbf{e}_{2'}, \mathbf{e}_{3'}, \mathbf{e}_{4'}) \begin{pmatrix} 1 & 0 & e^{1'}_{3^*} & 0 \\ 0 & 1 & e^{2'}_{3^*} & 0 \\ 0 & 0 & e^{3'}_{3^*} & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad e^{3'}_{3^*} \neq 0 . \tag{7.2}$$

The third system just fits to the point $\mathbf{E}_{3^*}$ which will be the centre of projection. Think e.g. of $\mathbf{E}_{3^*}(e^{1'}_{3^*} = 0, e^{3'}_{3^*} = 0, e^{3'}_{3^*} = d, e^{4'}_{3^*} = 1)$, a point on the z' axis of distance $d \neq 0$ from the *View Plane.*

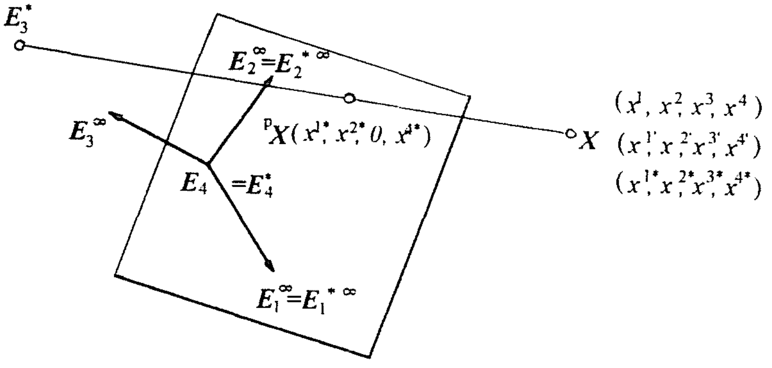


Fig. 7.2. Coordinates of the image

Since we want to get the coordinates of **X** with respect to the '*starred*' system

$$\mathbf{x} = \mathbf{e}_j x^j \sim \mathbf{e}_{k'} x^{k'} \sim \mathbf{e}_{i^*} x^{i^*} \sim \mathbf{e}_{k'} e^{k'}_{i^*} x^{i^*} \sim \mathbf{e}_j e^j_{k'} e^{k'}_{i^*} x^{i^*}, \tag{7.3}$$

we need the inverse matrix of $e^{k'}_{i^*}$, i.e. $e^{i^*}_{k'}$ to get $x^{i^*} \sim e^{i^*}_{k'} x^{k'}$ furthermore the inverse of $e^j_{k'}$, i.e. $e^{k'}_j$ to obtain $x^{k'} \sim e^{k'}_j x^j$ by the world coordinates x^j. Combining the coordinate transformations $\{\mathbf{E}_4, \mathbf{E}_1^\infty, \mathbf{E}_2^\infty, \mathbf{E}_3^\infty, \mathbf{E}\} \Rightarrow \{\mathbf{E}_{4'}, \mathbf{E}_{1'}^\infty, \mathbf{E}_{2'}^\infty, \mathbf{E}_{3'}^\infty, \mathbf{E}'\} \Rightarrow \{\mathbf{E}_{4^*}, \mathbf{E}_{1^*}^\infty, \mathbf{E}_{2^*}^\infty\ \mathbf{E}_{3^*}, \mathbf{E}^*\}$ we get

$$\mathbf{x} = \mathbf{e}_{i^*} x^{i^*} = \mathbf{e}_{i^*} e^{i^*}_{k'}, x^{k'} = \mathbf{e}_{i^*} e^{i^*}_{k'} e^{k'}_j x^j . \tag{7.4}$$

First in the base $\{\mathbf{e}_{i^*}\}$ we apply the projection Π by (4.4)

$$\mathbf{x} = \mathbf{e}_{i^*} x^{i^*} \mapsto {}^{p}\mathbf{x} := \mathbf{e}_{j^*}\, {}^{p}x^{j^*} \sim \Pi(\mathbf{e}_{i^*} x^{i^*}) := \mathbf{e}_{j^*} p^{j^*}_{i^*} x^{i^*} \tag{7.5}$$

as a linear mapping. Then holds

$${}^{p}x^{j^*} \sim p^{j^*}_{i^*} x^{i^*}\ , \quad \text{i.e.} \quad \begin{pmatrix} {}^{p}x^{1^*} \\ {}^{p}x^{2^*} \\ {}^{p}x^{3^*} \\ {}^{p}x^{4^*} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{1^*} \\ x^{2^*} \\ x^{3^*} \\ x^{4^*} \end{pmatrix} . \tag{7.6}$$

Second, we express this in the observer's frame $\{\mathbf{e}_{k'}\}$:

$${}^{p}\mathbf{x} = \mathbf{e}_{k'}\, {}^{p}x^{k'} \sim \mathbf{e}_{k'} e^{k'}_{j^*} p^{j^*}_{i^*} x^{i^*} = \mathbf{e}_{k'} (e^{k'}_{j^*} p^{j^*}_{i^*} e^{i^*}_{i'}) x^{i'} = \mathbf{e}_{k'} p^{k'}_{i'} x^{i'}\ ,$$

$$\begin{pmatrix} {}^{p}x^{1'} \\ {}^{p}x^{2'} \\ {}^{p}x^{3'} \\ {}^{p}x^{4'} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & e^{1'}_{3^*} & 0 \\ 0 & 1 & e^{2'}_{3^*} & 0 \\ 0 & 0 & e^{3'}_{3^*} & 0 \\ 0 & 0 & e^{4'}_{3^*} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & e^{1'}_{3^*} & 0 \\ 0 & 1 & e^{2'}_{3^*} & 0 \\ 0 & 0 & e^{3'}_{3^*} & 0 \\ 0 & 0 & e^{4'}_{3^*} & 1 \end{pmatrix}^{-1} \begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \\ x^{4'} \end{pmatrix} .$$

Simple computation yields the matrix $(p^{k'}_{i'})$ as follows:

$$\begin{pmatrix} {}^{p}x^{1'} \\ {}^{p}x^{2'} \\ {}^{p}x^{3'} \\ {}^{p}x^{4'} \end{pmatrix} \sim \frac{1}{e^{3'}_{3^*}} \begin{pmatrix} e^{3'}_{3^*} & 0 & -e^{1'}_{3^*} & 0 \\ 0 & e^{3'}_{3^*} & -e^{2'}_{3^*} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{4'}_{3^*} & e^{3'}_{3^*} \end{pmatrix} \begin{pmatrix} x^{1'} \\ x^{2'} \\ x^{3'} \\ x^{4'} \end{pmatrix} , \tag{7.7}$$

where the centre $\mathbf{E}_{3^*}$ is excluded from the mapping. We allow also $e^{4'}_{3^*} = 0$ where $\mathbf{E}_{3^*}$ is a point at infinity and we get the parallel projection.

Third, we consider (7.5) in the World-Coord. System $\{\mathbf{e}_j\}$. We get

$${}^{p}\mathbf{x} = \mathbf{e}_j\, {}^{p}x^j \sim \mathbf{e}_j e^j_{k'} e^{k'}_{j^*} p^{j^*}_{i^*} x^{i^*} \Rightarrow \mathbf{e}_j\, {}^{p}x^j \sim \mathbf{e}_j e^j_{k'} (e^{k'}_{j^*} p^{j^*}_{i^*} e^{i^*}_{i'}) e^{i'}_{i} x^{i} \tag{7.8}$$

just as before. If we are given the observer's frame (Camera Position) $\mathbf{e}_{k'} \sim \mathbf{e}_j e^j_{k'}$ by (7.1), then we obtain

$$\begin{pmatrix} {}^{p}x^{1} \\ {}^{p}x^{2} \\ {}^{p}x^{3} \\ {}^{p}x^{4} \end{pmatrix} \sim \begin{pmatrix} e^1_{1'} & e^1_{2'} & e^1_{3'} & e^1_{4'} \\ e^2_{1'} & e^2_{2'} & e^2_{3'} & e^2_{4'} \\ e^3_{1'} & e^3_{2'} & e^3_{3'} & e^3_{4'} \\ 0 & 0 & 0 & 1 \end{pmatrix} (p^{k'}_{i'})(e^j_{k'})^{-1} \begin{pmatrix} x^{1} \\ x^{2} \\ x^{3} \\ x^{4} \end{pmatrix} \tag{7.9}$$

a typical task to solve by computer. We can vary our problem. E.g. the projection in the observer's frame can be related to the world coordinates:

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \mapsto \begin{pmatrix} {}^px^{1'} \\ {}^px^{2'} \\ {}^px^{3'} \\ {}^px^{4'} \end{pmatrix} \sim \frac{1}{e_{3*}^{3'}} \begin{pmatrix} e_{3*}^{3'} & 0 & -e_{3*}^{1'} & 0 \\ 0 & e_{3*}^{3'} & -e_{3*}^{2'} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -e_{3*}^{4'} & e_{3*}^{3'} \end{pmatrix} (e_{k'}^{j})^{-1} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix}. \quad (7.10)$$

Finally, the Cartesian rectangular coordinates ${}^p\mathbf{X}({}^px', {}^py')$ of the image of the point $\mathbf{X}$ can be obtained from (7.10)

$${}^px' = \frac{{}^px^{1'}}{{}^px^{4'}}, \quad {}^py' = \frac{{}^px^{2'}}{{}^px^{4'}} \quad (7.11)$$

as linear fractions. Then some exceptional points are excluded from the mapping, namely: the points at infinity and every point whose image is at infinity. Our projective method does not make such a distinction.

Our results are obviously the same, as one can find in monographs (there is a large number of books of this kind), but the authors are convinced that the method based on projective geometry provides the straightforward solution. We emphasize the importance of distinguishing projective transformations and coordinate transformations as clearly as possible.

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