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STABILIZING AN INVERTED PENDULUM — ALTERNATIVES AND LIMITATIONS

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Abstract

In this work we discuss two basic alternatives of stabilizing an inverted pendulum — by applying a horizontal force to its base or by use of a speed controller which implements a kinematic constraint into the mechanical system.

Keywords: inverted pendulum, nonholonomic systems.

1. Description of the Mechanical System

The mechanical system consists of a pendulum hinged to a mobile cart (see *Fig. 1*). The cart and the pendulum are constrained to move in the same vertical plane. The cart consists of an electric motor with a gearbox connected to a pair of wheels. As can be seen from *Fig. 1*, our system has two degrees of freedom and its motion can be described by two generalized coordinates: the angle θ of the pendulum and the horizontal position q of the cart.

2. Mathematical Model

To obtain the equations of motion, we give the kinetic energy of the system in the form

$$T = \frac{1}{2}M_c\dot{q}^2 + \frac{1}{2}J_w\left(\frac{\dot{q}}{r}\right)^2 + \frac{1}{2}M_1\left[(\dot{q}+s\dot{\theta}\cos\theta)^2\right] + (s\dot{\theta}\sin\theta)^2] + \frac{1}{2}J_p\dot{\theta}^2, \quad (1)$$

where the following notation is used:

M_c the mass of the cart with the wheels;
J_w moment of inertia of the wheels, gearbox and rotor altogether;
r radius of the wheels;
M_1 mass of the pendulum;

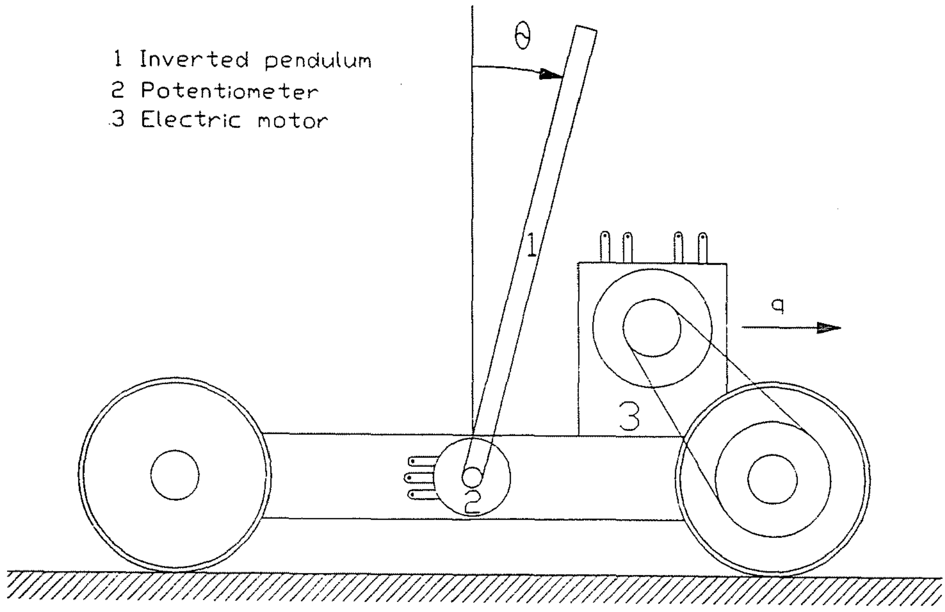

Fig. 1. Experimental device

s distance between the hinge and the mass centre of the pendulum;
J_p moment of inertia about the mass centre of the pendulum.

This kinetic energy will be used in forming the mathematical model of the system in cases of two different control strategies.

2.1 Balancing with a Horizontal Force

By means of the electric motor, we apply a horizontal force F to the cart, which is determined by a linear feedback of the state variables q, $\dot{q}$, θ and $\dot{\theta}$ of the system. We use the well known Lagrange's equations of motion of the second kind for holonomic systems in the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q_q, \tag{2a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta, \tag{2b}$$

where Q_q and Q_θ are the generalized forces obtained from the virtual power of the active forces:

$$\delta P = Q_q \delta \dot{q} + Q_\theta \delta \dot{\theta} = (F - S)\delta \dot{q} + (M_1 g s \sin\theta - C)\delta\dot{\theta}. \tag{3}$$

S and C stand for the dissipative horizontal force and the dissipative torque at the pivot axis respectively and g stands for the gravitational acceleration. Because of the use of ball bearings, the magnitude of C is negligible. The force S though, has a significant role in the system due to the DC motor connected to the wheels. We assume that it has a viscous nature, i.e. S is proportional to the velocity: $S = D\dot{q}$, which is a reasonable approximation in case of direct current motors. After substituting (1), and also Q_q, Q_θ from (3) into Eqs. (2) we obtain the equations of motion

$$(M_0 + M_1)\ddot{q} + M_1 s\ddot{\theta}\cos\theta = M_1 s\dot{\theta}^2 \sin\theta - D\dot{q} + F, \tag{4a}$$

$$M_1 s\ddot{q}\cos\theta + J_1\ddot{\theta} = M_1 g s\sin\theta, \tag{4b}$$

where $M_0 = M_c + J_w/r^2$ and $J_1 = M_1 s^2 + J_p$. If the state vector $\mathbf{x}^T = (\theta, \dot{\theta}, q, \dot{q})$ is introduced, the linearized equations of motion become:

$$\dot{\mathbf{x}} = \mathbf{A}\cdot\mathbf{x}, \tag{5}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{M_1 s((M_0+M_1)g-k_1)}{(M_0J_1+M_1J_p)} & \frac{-M_1 sk_2}{(M_0J_1+M_1J_p)} & \frac{-M_1 sk_3}{(M_0J_1+M_1J_p)} & \frac{M_1 s(D-k_4)}{(M_0J_1+M_1J_p)} \\ 0 & 0 & 0 & 1 \\ \frac{-M_1{}^2 s^2 g+J_1 k_1}{(M_0J_1+M_1J_p)} & \frac{J_1 k_2}{(M_0J_1+M_1J_p)} & \frac{J_1 k_3}{(M_0J_1+M_1J_p)} & \frac{J_1(k_4-D)}{(M_0J_1+M_1J_p)} \end{pmatrix},$$

where the horizontal force F is chosen to be a liner feedback of the four state variables:

$$F = k_1\theta + k_2\dot{\theta} + k_3 q + k_4\dot{q}. \tag{6}$$

By applying appropriate gains k_j, $j = 1, 2, 3, 4$, we would like to insure the asymptotic stability of the trivial solution $\mathbf{x}_0^T = (0,0,0,0)$. This could be achieved if the real parts of the eigenvalues of $\mathbf{A}$ are definitely negative [2]. The characteristic polynomial of $\mathbf{A}$ assumes the form:

$$\lambda^4 + \lambda^3\left(\frac{J_1(D-k_4) + M_1 sk_2}{M_0J_1 + M_1J_p}\right) + \lambda^2\left(\frac{M_1 s(k_1 - (M_0+M_1)g) - J_1 k_3}{M_0J_1 + M_1J_p}\right) + \lambda\left(\frac{M_1 s(k_4 - D)g}{M_0J_1 + M_1J_p}\right) + \frac{gM_1 sk_3}{M_0J_1 + M_1J_p} = 0. \tag{7}$$

The stability region of (7) in the parameter space of k_1, k_2, k_3 and k_4 is determined from the well known Routh – Hurwitz criterion, which is expressed by the following four inequalities:

$$(i)\quad M_1 s k_2 > J_1(k_4 - D);$$

$$(ii)\quad \Big(M_1 s k_2 - J_1(k_4 - D)\Big)\Big(M_1 s(k_1 - (M_0 + M_1)g) - J_1 k_3\Big) > \Big(M_0 J_1 + M_1 J_p\Big)\Big(M_1 s(k_4 - D)g\Big);$$

$$(iii)\quad (k_4 - D)\Big((M_1 s(k_1 - (M_0 + M_1)g) - J_1 k_3)(M_1 s k_2 - J_1(k_4 - D)) - (M_0 J_1 + M_1 J_p)(M_1 s g(k_4 - D))\Big) > \Big(M_1 s k_2 - J_1(k_4 - D)\Big)^2 k_3;$$

$$(iv)\quad k_3 > 0.$$

From (i)–(iv) it is evident, that all coefficients k_i $(i = 1, \dots, 4)$ should be positive, in other words, it is necessary to use all the state variables in the feedback.

If one does not require to position the cart at $q = 0$, it is not necessary to measure q. This is easily checked if we realize, that in this case the coordinate q does not appear in equations (4). This allows us to use the state vector $\mathbf{y}^T = (\theta, \dot{\theta}, \dot{q})$ and to repeat the above procedure. We obtain conditions (i)–(iii) with $k_3 = 0$ substituted in them.

Further, if $k_4 = D$ is also true, then the variable $\dot{q}$ also disappears from Eqs. (4) and thus we can rewrite them in terms of $(\theta, \dot{\theta})$ only. The linear stability in the phase plane $(\theta, \dot{\theta})$ then is defined by (i)–(ii) only, so when $t \to \infty$ the cart may travel with some constant speed $\dot{q}_0$. Note, that because of the presence of the frictional force $S = D\dot{q}$ it is still necessary to use $\dot{q}$ in the feedback.

Here we have neglected the natural time delay, which is always present in the system. The stability region described by (i)–(iv) in the parameter space is smaller in the case of a significant time delay: there is an additional upper bound for the parameters k_i. An exact calculation of the stability chart for the simplest case ($k_3 = 0$, $k_4 = D = 0$) has been carried out by G. STÉPÁN in [1].

2.2 Balancing with Speed Controller

As it can be seen from Section 2.1, a naturally unstable mechanical system could be stabilized with additional forces introduced in it. Although these forces may be of various kind, most frequently they are provided by actuators — in our case, by an electric motor, which are sold together with controllers. The best solution for our problem would be to use an I×R-compensation, because this controller type guaranties the desired force, proportional to the input signal. Sometimes we do not possess the best type of controller, but it is still worth trying to use it. In this section we show, that a speed controller used to control the horizontal velocity of the cart is able to balance the pendulum in a similar way, as it was done in Section 2.1. This solution however, involves an integration in the feedback, which complicates the controlling algorithm and increases the time delay in the system.

If we wish to stabilize an unstable system, a standard approach to the problem would be to use a feedback of its state variables. Because the input of the system in the case of a speed controller is a voltage proportional to the desired velocity of the cart, it seems to be reasonable to apply the kinematic constraint of the form:

$$\dot{q} = \eta_1\theta + \eta_2\dot{\theta} + \eta_3 q. \tag{8}$$

We deal with a nonholonomic mechanical system and therefore the equations of Routh – Voss [2] should be used:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} = Q_q + \nu A_{11}, \tag{9a}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta + \nu A_{12}, \tag{9b}$$

$$A_{11}\dot{q} + A_{12}\dot{\theta} + A_1 + A_2 = 0, \tag{9c}$$

where $A_{11} = 1$; $A_{12} = -\eta_2$; $A_1 = -\eta_1\theta$; $A_2 = -\eta_3 q$. Taking into account (1) and (3), *Eqs.* (9) become

$$(M_0 + M_1)\ddot{q} + M_1 s\cos(\theta)\ddot{\theta} = M_1 s\sin(\theta)\dot{\theta}^2 - D\dot{q} + \nu, \tag{10a}$$

$$M_1 s\cos(\theta)\ddot{q} + J_1\ddot{\theta} = M_1 gs\sin(\theta) - \eta_2\nu, \tag{10b}$$

$$\dot{q} = \eta_1\theta + \eta_2\dot{\theta} + \eta_3 q. \tag{10c}$$

In the *Eqs.* (10), two new terms appeared (ν and $\eta_2\nu$), which correspond to an additional force and torque, applied to the cart and to the pendulum, respectively. Unfortunately, the pendulum swings freely about its pivot and no torque could be transferred from the cart to the pendulum. This means, that the coefficient η_2 cannot be set to a non-zero value, otherwise the solution of equations (10) does not describe the motion of the system.

Now, from *Eqs.* (10) with $\eta_2 = 0$, the linearized equation of motion can be simplified in the form:

$$J_1 \frac{\mathrm{d}^3\theta}{\mathrm{d}t^3} + (\eta_1 M_1 s - J_1\eta_3)\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} - M_1 sg\frac{\mathrm{d}\theta}{\mathrm{d}t} + \eta_3 M_1 sg\theta = 0.$$

Since the first and third coefficients have opposite signs, the $\theta = 0$ solution is unstable whatever the choice of η_1 and η_3 is.

Despite of the discouraging results above, it is still possible to use a speed controller for stabilizing the inverted pendulum. For example, if we determine the speed as

$$\dot{q} = \int_0^t \Big(\kappa_1\theta(\tau) + \kappa_2\dot{\theta}(\tau) + \kappa_3 q(\tau) + \kappa_4\dot{q}(\tau)\Big)\mathrm{d}\tau,$$

or we use the equivalent constraint

$$\ddot{q} = \kappa_1\theta + \kappa_2\dot{\theta} + \kappa_3 q + \kappa_4\dot{q}, \tag{11}$$

the motion of the cart and the pendulum would be the same as if we applied a horizontal force F determined by (4) and (11). Namely,

$$F = (M_0 + M_1)\ddot{q} + M_1 s\cos(\theta)\ddot{\theta} - M_1 s\sin(\theta)\dot{\theta}^2 + D\dot{q},$$

where $\ddot{q}$ and $\ddot{\theta}$ could be expressed in terms of the generalized coordinates and velocities from (4b) and (11). If we compare the linear approximation

$$\begin{aligned} F(\theta,\dot{\theta},q,\dot{q}) \approx {} & \tfrac{(M_0J_1+M_1J_p)\kappa_1+M_1^2s^2g}{J_1}\theta + \tfrac{M_0J_1+M_1J_p)\kappa_2}{J_1}\dot{\theta} \\ & + \tfrac{(M_0J_1+M_1J_p)\kappa_3}{J_1}q + \tfrac{(M_0J_1+M_1J_p)\kappa_4+J_1D}{J_1}\dot{q}, \end{aligned} \tag{12}$$

with the linear feedback (6) in Section 2.1, a correspondence between the coefficients k_j and κ_j could be established:

$$\begin{aligned} k_1 = \tfrac{(M_0J_1+M_1J_p)\kappa_1+M_1^2s^2g}{J_1}; \qquad k_j = \tfrac{(M_0J_1+M_1J_p)\kappa_j}{J_1},\ j = 2,3; \\ k_4 = \tfrac{(M_0J_1+M_1J_p)\kappa_4+J_1D}{J_1}. \end{aligned} \tag{13}$$

With transformation (13), the stability conditions (*i*)–(*iv*) from Section 2.1 could be reformulated for the parameters κ_j and thus similar qualitative results could be obtained.

3. Experiments

We have used computer control combined with the speed controller to verify the theoretical results. In the experiments, only two of the four state variables were measured and sampled. These were θ and $\dot{q}$ with sampling time about 1.2 [ms]. The integration in the feedback was done by the computer (an IBM AT 486) and thus the estimated value for q contained some numeric error. This caused an error of about $\Delta q = 0.1$[m] in the final position of the cart. The time sequences of the measured θ and estimated q can be seen in *Fig. 2* and *3* bellow.

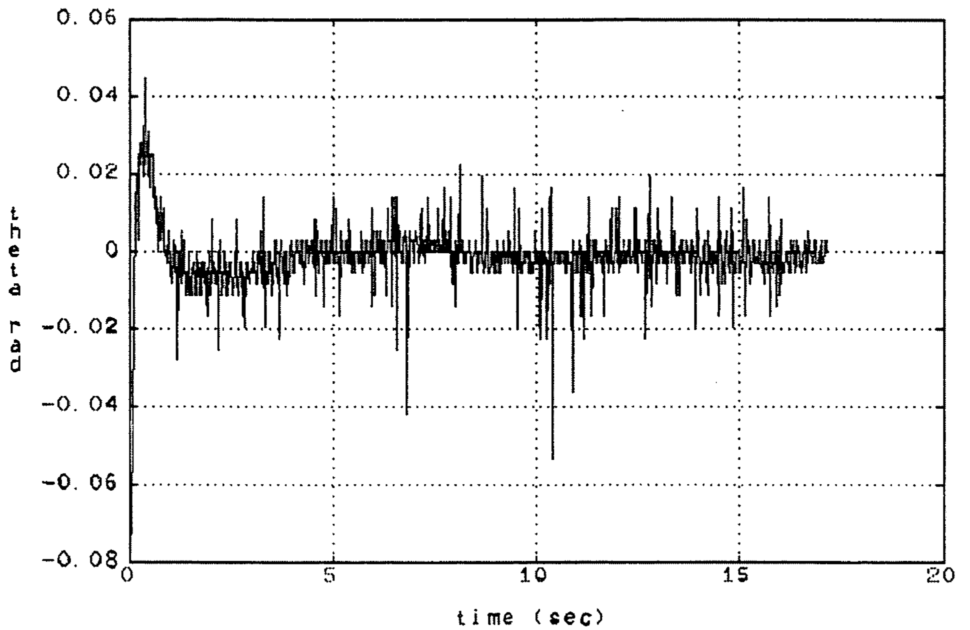

Fig. 2. Damped oscillation of the controlled inverted pendulum angle

The stochastic noise on the signals has been explained and identified as a slight chaotic behaviour of the system when linear and nonlinear digital effects, like sampling time and quantization, are also considered in the deterministic mathematical model [3].

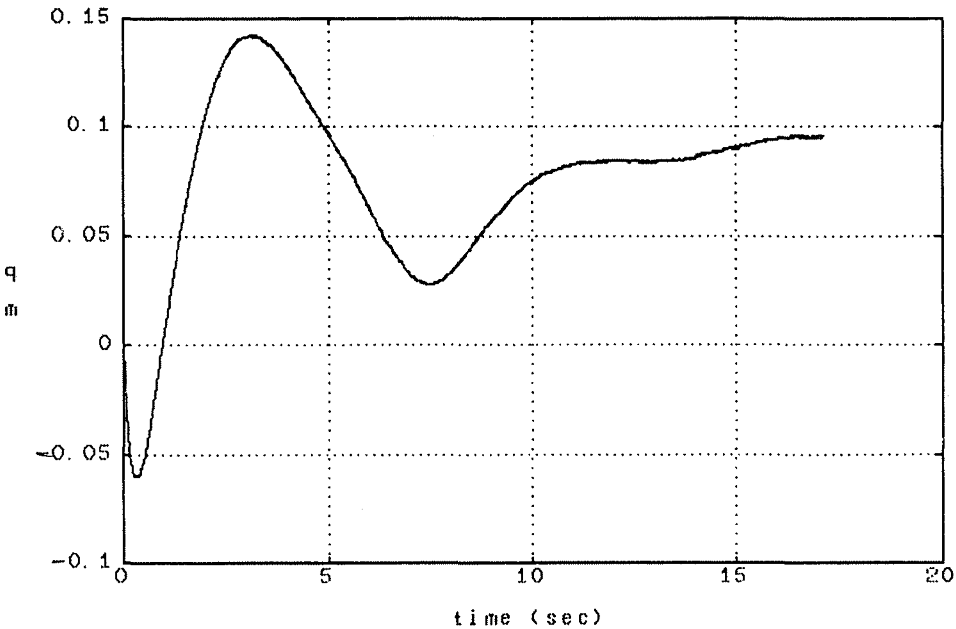

Fig. 3. Cart motion

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