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EXTREMA OF SOME THERMOELASTIC PROPERTIES OF SYMMETRIC LAMINATES

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Abstract

Extrema of coefficients of thermal expansion of a symmetric and balanced angle ply laminate and extrema of lowest eigenfrequency of two types of simply supported angle ply plates are studied. It is shown that for two of the three cases considered explicit expressions for the angle leading to extremum can be derived.

Keywords: laminate, extrema, CTA, eigenfrequency.

Introduction

Composite materials have found increasing use in many technical applications during the last three decades. The first reason for their wide use was the high strength to weight and stiffness to weight ratios. Applications employing these features can be found e.g. in aerospace-, aeroplane-, ship-, and car industries.

Another advantage giving still more potentials is the ability to tailor the properties of the structure by varying ply thicknesses and ply orientations. When the thickness of an isotropic plate made of given material is fixed, the in–plane stiffness, bending stiffness, coefficients of thermal expansion (CTE), strength margin due to given loads etc. are also fixed. On the other hand, by using one ply type (see *Table 1* for typical thermomechanical properties for commercially used plies [1]) and fixed number of plies, one has still opportunities to adjust the above mentioned properties according to the needs - of course within certain limits, depending on the ply type used.

The above mentioned properties depend quite complicately on lamination angles, and each one in different way. Therefore, trial and error design is almost hopeless. If one has a specification, say for a symmetric laminate consisting of 16 plies, the number of ply angle combinations to be analyzed is 18^8, if the search is restricted to laminates where the lamination angles are accepted to be changed only in steps of 10 degrees. Many of

Table 1
Thermoelastic properties of some laminae [1]

No	Fiber/Matrix	E_{11} GPa	E_{22} GPa	G_{12} GPa	ν_{12}	α_{11} $10^{-6}/°C$	α_{22} $10^{-6}/°C$
1	T300/N5208	181	10.30	7.17	0.28	0.02	22.5
2	B(4)/N5505	204	18.50	5.59	0.23	6.1	30.3
3	AS/3501	138	8.96	7.10	0.3	-0.3	28.1
4	E-glass/epoxy	38.6	8.27	4.14	0.26	8.6	22.1
5	Kev.49/epoxy	76.0	5.50	2.30	0.34	-4.0	79.0

the properties are very sensitive with respect to lamination angle. Hence, search with step 5 degrees would be in many cases necessary. This would be impracticable even for a nation. If we assume that in beforehand an own angle combination is now given to every Hungarian (10 million people) and suppose that analyzing one combination takes 5 minutes, all the combinations are analyzed after about 14 years. This is why synthesis methods are needed to search laminate lay–ups satisfying the specification.

The present paper has grown up from the results obtained for testing and developing such a method, [2] and [3]. By using the algorithms developed one can minimize, maximize or search a target value for some properties keeping at the same time others within certain limits. During the development and testing of the algorithms, cases corresponding to minimum and maximum values of certain properties of $[\Theta/-\Theta]_s$ laminates have been considered. It is not necessary nor possible to describe all the explicit results obtained. For brevity two typical cases are dealt with in what follows. The first one deals with a local property, coefficient of thermal expansion (CTE) of a symmetric and balanced angle ply laminate. The second case deals with the first eigenfrequency of a simply supported laminated plate.

Extrema of CTE

It has been known for a long time that by using certain laminate lay up, laminates having negative CTE in longitudinal direction can be created although the laminae used have positive CTEs both in fiber direction and in direction perpendicular to it. The theoretical explanation, [4] and [5] , based on classical lamination theory is not very old. The expressions obtained are, however, in the form allowing only numerical treatment, i. e. calculation of CTEs for given angles.

The calculations performed for material 1 (*Table 2*) by the present author for 8 and 12 layered symmetric and balanced angle ply laminates indicated that the absolute minimum of CTE will be obtained when the magnitudes of the angles are equal, i.e. the laminate is of type $[\Theta/-\Theta]$. Even for this kind of simple laminate, the calculations for explicit expressions of CTEs are tedious if done manually. Therefore a symbolic manipulation program DERIVE [6], was employed.

Table 2
$\alpha^0_{xx}/(10^{-6}/°C)$ of a laminate $[\Theta_1/Theta_1/\Theta_2/-\Theta_2]_s$ made of material No 1 in *Table 1*.

		$\Theta_1/1°$					
		0	10	20	30	40	50
	0	0.02	-0.24	-0.80	-1.03	-0.56	0.19
	10	-0.24	-0.50	-1.07	-1.28	-0.72	0.14
$\Theta_2/1°$	20	-0.80	-1.07	-1.70	-1.94	-1.14	0.07
	30	-1.03	-1.28	-1.95	-2.37	-1.46	0.25
	40	-0.56	-0.72	-1.14	-1.46	-0.64	1.52
	50	0.19	0.14	0.07	0.25	1.52	4.46

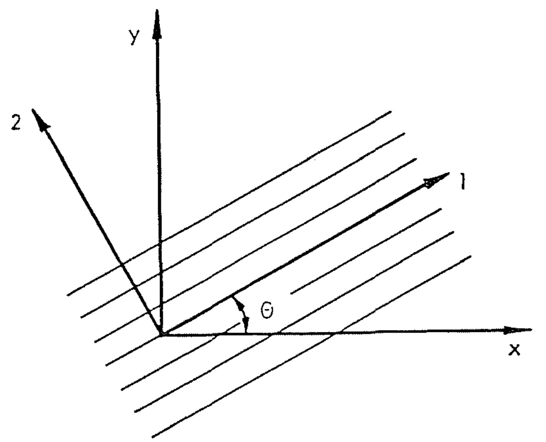

Fig. 1. Co-ordinate systems used, notations

The co-ordinate systems used are shown in *Fig. 1.* x,y-co-ordinates are the laminate coordinates whereas 1,2-coordinates are ply coordinates, such that axis 1 is along the fiber direction. Using the assumptions of linear elasticity, transformations of components of stress and strain tensor one obtains first the following expressions for the CTEs

$$\alpha^0_{xx} = \frac{A_{22}S_1 - A_{12}S_2}{A_{11}A_{22} - A^2_{12}} \tag{1}$$

and

$$\alpha^0_{yy} = \frac{A_{11}S_2 - A_{12}S_1}{A_{11}A_{22} - A^2_{12}}, \tag{2}$$

where A_{ij} and S_i are elements of matrix ($\mathbf{h}_k$ is the ply thickness)

$$[A] = \sum_{k=1}^{2n}[\bar{Q}]\mathbf{h}_k \tag{3}$$

and vector

$$\{S\} = \sum_{k=1}^{2n}\{\bar{R}\}_k\mathbf{h}_k. \tag{4}$$

The expressions for the elements of the transformed material stiffness matrix $[\bar{Q}]$ are

$$\begin{aligned}
\bar{Q}_{11} &= Q_{11}\cos^4\Theta + 2(Q_{12} + 2Q_{66})\sin^2\Theta\cos^2\Theta + Q_{22}\sin^4\Theta,\\
\bar{Q}_{12} &= Q_{12}(\sin^4\Theta + \cos^4\Theta) + (Q_{11} + Q_{22} - 4Q_{66})\sin^2\Theta\cos^2\Theta,\\
\bar{Q}_{22} &= Q_{11}\sin^4\Theta + 2(Q_{12} + 2Q_{66})\sin^2\Theta\cos^2\Theta + Q_{22}\cos^4\Theta,\\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})\sin\Theta\cos^3\Theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\Theta\cos\Theta,\\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\Theta\cos\Theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\Theta\cos^3\Theta,\\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\Theta\cos^2\Theta + Q_{66}(\sin^4\Theta + \cos^4\Theta)
\end{aligned} \tag{5}$$

where

$$\begin{aligned}
Q_{11} &= \frac{E_{11}}{1 - \nu_{12}\nu_{21}},\\
Q_{22} &= \frac{E_{22}}{1 - \nu_{12}\nu_{21}},\\
Q_{12} &= \frac{\nu_{12}E_{22}}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_{11}}{1 - \nu_{12}\nu_{21}},\\
Q_{66} &= G_{12}.
\end{aligned} \tag{5'}$$

By using symbolic manipulation program one finally obtains (for more details, see [7]) explicit expressions for CTEs

$$\alpha^0_{xx} = \frac{(\alpha_{11} - \alpha_{22})\sin 2\Theta(C_1\sin 4\Theta + C_2\sin 2\Theta)}{2(C_1\cos 4\Theta + C_3)} \tag{6}$$

$$+\frac{(\alpha_{11}-\alpha_{22})\cos 2\Theta}{2}+\frac{(\alpha_{11}+\alpha_{22})}{2}.$$

$$\alpha_{yy}^{0}=\frac{(\alpha_{22}-\alpha_{11})\sin 2\Theta(C_1\sin 4\Theta-C_2\sin 2\Theta)}{2(C_1\cos 4\Theta+C_3)}+\frac{(\alpha_{22}-\alpha_{11})\cos 2\Theta}{2}+\frac{(\alpha_{11}+\alpha_{22})}{2} \tag{7}$$

where

$$C_1=E_{11}(E_{22}-G_{12})-E_{22}G_{12}(2\nu_{12}+1), \tag{8}$$

$$C_2=2G_{12}(E_{11}-E_{22}) \tag{9}$$

and

$$C_3=E_{11}(E_{22}+G_{12})+E_{22}G_{12}(2\nu_{12}+1). \tag{10}$$

As an application of the above results the angle dependence of the longitudinal CTE is plotted in *Fig. 2* for the materials given in *Table 1.*

From *Fig. 2* we see that for some laminates there is a third extremum in addition to the – trivial – extrema at 0 and 90 degrees. From *Eqs* 6 and 7 we see that all the terms containing Θ have the same coefficient, namely $(\alpha_{11}-\alpha_{22})$. Thus the angles giving extrema to α_{xx}^{0} and α_{yy}^{0} depend solely on elastic properties.

When the right hand side of *Eq.* (6) is differentiated with respect to Θ, one finally obtains an expression the nominator of which is

$$g(\Theta)=\sin\Theta\cos\Theta(b_2\cos^4\Theta+b_1\cos^2\Theta+b_0), \tag{11}$$

where

$$b_2=4[E_{11}(E_{22}-G_{12})-E_{22}G_{12}(2\nu_{12}+1)], \tag{12}$$

$$b_1=-4[E_{11}(E_{22}-2G_{12})-2E_{22}G_{12}\nu_{12}], \tag{13}$$

and

$$b_0=E_{11}(E_{22}-4G_{12})-4E_{22}G_{12}\nu_{12}. \tag{14}$$

By further study of expression (11) (for more details, see [7]) one can see that α_{xx}^{0} has in the open interval $(0^\circ, 90^\circ)$ - i. e. in addition to the trivial ones at 0° and 90° – two, one or

$$\begin{array}{ll} 0<G_{12}<\frac{E_{11}}{4(E_{11}/E_{22}+\nu_{12})}, & \text{2 extrema} \\ \frac{E_{11}}{4(E_{11}/E_{22}+\nu_{12})}\le G_{12}<\frac{E_{11}}{4(1+\nu_{12})}, & \text{1 extremum} \\ G_{12}\ge\frac{E_{11}}{4(1+\nu_{12})}, & \text{none extrema} \end{array} \tag{15}$$

By using the values of *Table 1* we see that all those materials belong to the 'middle' class. However, there can be materials belonging to the first and third category.

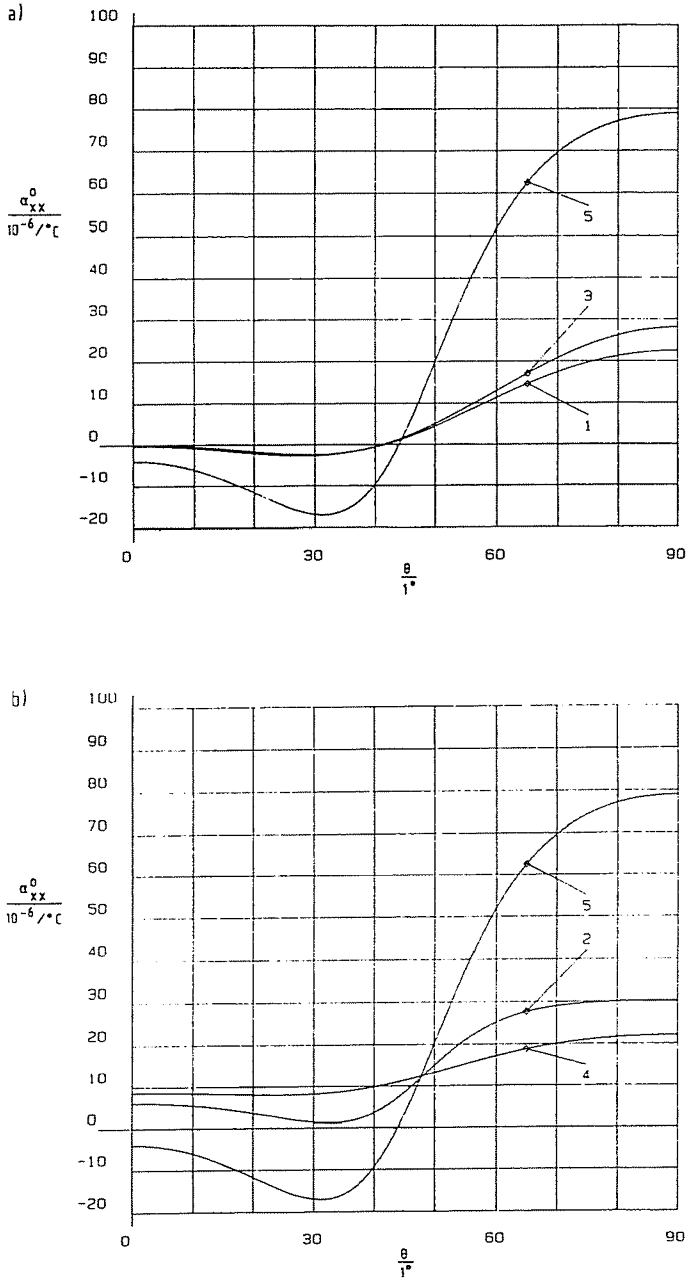

Fig. 2. Lamination angle dependence of the longitudinal CTE of laminates made of materials given in *Table 1*.

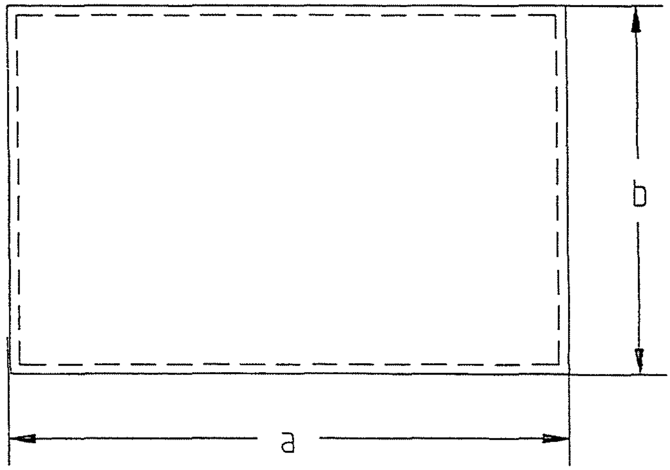

Fig. 3. Rectangular plate, notations

Extrema of Lowest Eigenfrequency

Let us consider a rectangular simply supported plate having symmetric laminate lay up, *Fig. 3.* The equation of motion reads

$$D_{11}w_{,xxxx} + 4D_{16}w_{,xxxy} + 2(D_{12} + 2D_{66})w_{,xxyy}$$

$$+4D_{26}w_{,xyyy} + D_{22}w_{,yyyy} + \rho\ddot{w} = 0, \qquad (16)$$

where w is the transverse deflection, ρ is the mass per unit area, a dot indicates differentiation with respect to time, w_{xxxx} fourth derivative with respect to x etc., and

$$D_{ij} = \int_{-h/2}^{h/2} \bar{Q}_{ij}z^2 \mathrm{d}z \qquad (17)$$

Substitution of the trial $w = We^{i\omega t}$ leads to an eigenvalue problem

$$D_{11}W_{,xxxx} + 4D_{16}W_{,xxxy} + 2(D_{12} + 2D_{66})W_{,xxyy}$$

$$+4D_{26}W_{,xyyy} + D_{22}W_{,yyyy} - \omega^2\rho W = 0. \qquad (18)$$

The solution would be easy if D_{16} and D_{26} would be equal to zero. For a general $[\Theta/-\Theta]$s laminate they, however, are nonzero.

From *Eq.* (17) we easily obtain, that for a special $[\Theta/-\Theta]$ ort laminate, where the thickness of the middle layers is equal to $h/\sqrt[3]{16}$ (h is

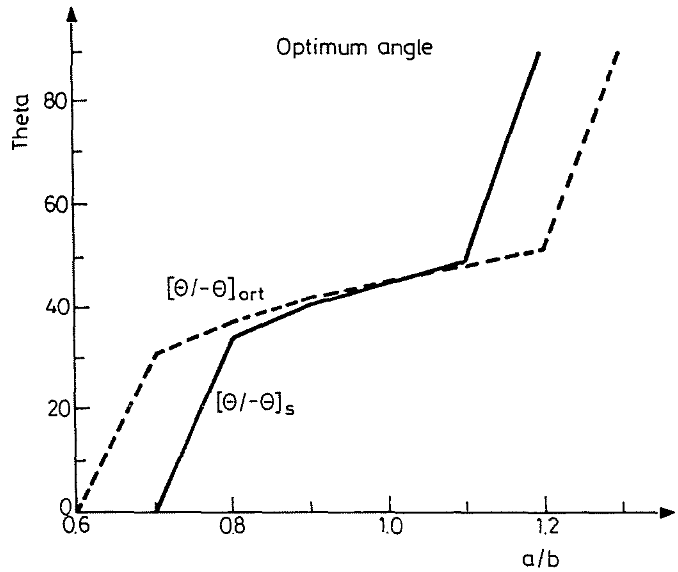

Fig. 4. Optimum angle as a function of aspect ratio.

the total thickness of the laminate) these difficult terms disappear. The well-known orthotropic solution

$$W = A \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{19}$$

gives for the square of the lowest circular eigenfrequency following equation

$$\omega^2 = \frac{\pi^4}{\rho}[D_{11}\frac{1}{a^4} + 2(D_{12} + 2D_{66}\frac{1}{a^2}\frac{1}{b^2} = D_{22}\frac{1}{b^4}] \tag{20}$$

By using *Eqs* (5), (5') and (17) we can write this expression solely in terms of lamination angle and the elastic coefficients of the ply used. The expression is lengthy and lends itself not for further manual manipulation. However, by using symbolic manipulation by a computer, one obtains that the derivative of the right hand side of (20) with respect to Θ is zero at 0 and 90 degrees and, moreover, when

$$\cos\Theta = \sqrt{\frac{2\alpha^4 a_1 - 3\alpha^2 a_2 - 2a_3}{2(\alpha^4 a_1 - 3\alpha^2(21) + 1)a^2}} \tag{21}$$

$$
\begin{aligned}
a_1 &= E_{11}^2 - E_{11}(E_{22}\nu_{12} + 2G_{12}) + 2E_{22}G_{12}\nu_{12}^2 \\
a_2 &= E_{11}^2 - E_{11}[E_{22}(2\nu_{12} - 1) + 4G_{12}] + 4E_{22}G_{12}\nu_{12}^2 \\
a_3 &= E_{11}[E_{22}(\nu_1 2 - 1) + 2G_1 2] - 2E_{22}G_{12}\nu_{12}^2
\end{aligned}
\tag{22}
$$

Eq. (21) gives an angle for extremum only when the right hand side belongs to interval [0,1]. This in turn depends on the ply elastic constants and on the aspect ratio (= a/b).

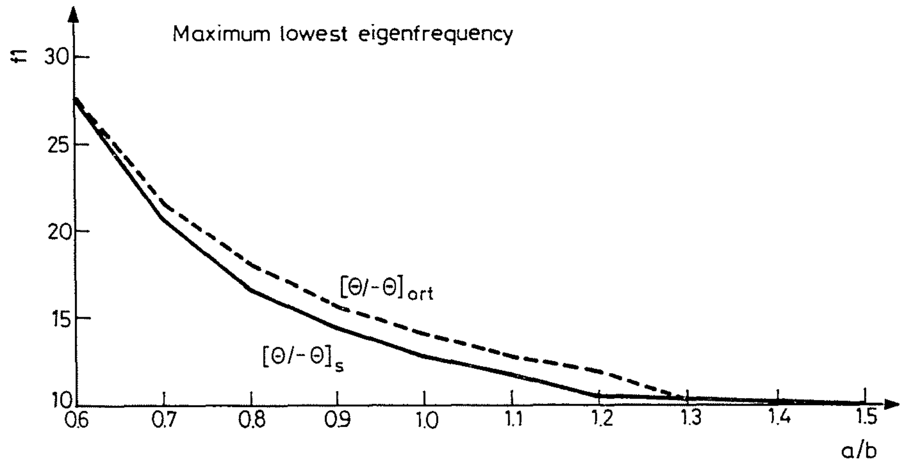

Fig. 5. Maximum lowest eigenfrequency as a function of aspect ratio.

When the specially orthotropic solution is not possible to use, i.e. the plies are of equal thickness, Rayleigh-Ritz method must be used in extracting the eigenvalues. By using the trial

$$W = \sum_{j=1}^{n} a_j \chi_j(x, y) \tag{23}$$

We are led to the familiar matrix eigenvalue problem

$$[[K] - \omega^2[M]]\{a\} = \{0\} \tag{24}$$

In search for maximum lowest eigenvalue, numerical algorithms must be used. The algorithms implemented into the program system described in reference 2 are the Flexible Tolerance Method and the SQP-method.

As an example let us consider a simply supported angle ply plate made of material 1 in *Table I.* Moreover, assume that the thickness of the plate is 2 mm and the density of the material is 1600 kg/m3. The optimum angle - giving maximum lowest eigenfrequency - is plotted in *Fig. 4* as a fuction of the aspect ratio for both the abovementioned laminate types. As can be seen, they do not differ considerably from each other. Moreover, of course, they are equal when aspect ratio is small (optimum angle 0°) and when it is large (optimum angle 90°). *Fig. 5* shows the maximum natural frequency (in HZ) for both laminate types as a function of the aspect ratio. In this figure the length b was equal to 1 m.

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