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# PARABOLIC BLENDING SURFACES ALONG POLYHEDRON EDGES[1]

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## Abstract

In this paper parabolic blending surfaces are defined along a chain of polyhedron edges. The profile curve of each sweep surface generated for a given edge is a conic section, and every point of it moves on a conic section around a vertex. According to this, the patches at the corners are given in rational biquadratic form and they join to the cylindrical surfaces replacing the edges with 1st order continuity.

*Keywords:* blending surfaces, computer-aided geometric design, solid modelling, rounding.

## Introduction

In computer-aided design and manufacturing systems several methods have been proposed for generating free-form surfaces from polyhedra. Some of them construct surfaces interpolating the vertices, edges or faces of the initial polyhedron, others work by rounding off edges and vertices. Two types of smoothing methods may be distinguished. One group of methods generate a smooth or almost everywhere smooth surface preserving the global shape of the polyhedron, and others work locally. The ideas and methods differ depending on the purpose of the operation and on the type of applied surfaces.

Methods for generating free-form surfaces from a polyhedron were first proposed by DOO and SABIN [5,6] and CATMULL and CLARK [2]. Those are subdivision methods, where new vertices, edges and faces are constructed refining the polyhedron successively in such a way that a smooth surface arises as the limit of the process. Catmull and Clark generalize a recursive bicubic B-spline patch subdivision. The resulting surface is continuous almost everywhere, the vertices with $n$ edges, $n \neq 4$ become extraordinary points as the process evolves. The biquadratic subdivision method of Doo and Sabin shrinks every polyhedron face in every step, and the center points of faces are interpolated by the resulting surface. An analysis

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of the behaviour of the surface in the neighbourhood of the extraordinary points, arising from the center points of $n$-sided faces, $n \neq 4$, is given as well. In BRUNET's method [1] biquadratic recursive subdivision is used. The constructed surface interpolates the vertices of the initial polyhedron and its shape can be controlled during the process. STORRY and BALL [11] deal with the problem, how an $n$-sided hole can be filled out, that arises at a vertex point where $n$ edges meet, during the construction of a surface by recursive subdivision. Algebraic surfaces composed from cylinders and planes interpolating the edges of convex polyhedra are used for global rounding presented by LI JINGGONG et al. [9].

By local rounding methods only specified edges and vertices are smoothed down. CHIYOKURA and KIMURA [3,4] generate first a curve model by local modifications of specified edges, after this, Gregory patches of the curve meshes are generated. ROSSIGNAC and REQUICHA [10] construct a sequence of torus segments simulating the action of a 'rolling ball' along the edges to be rounded. Algebraic surfaces are used in the method of KOSTERS [8], where quadratic surfaces in implicit form are generated by the potential method for complex corners. In the author's previous rounding algorithm [12,13] rectangular bicubic patches are defined by their geometric data along the specified edges and around vertices of the polyhedron, where a range of rounding radii can be chosen, and the shape of the blending surfaces can be deformed by scalar parameters, making them flatter or sharper.

This paper presents a local rounding method based upon the idea of the rolling ball, but instead of a ball conic sections, tangential to the faces are pushed along polyhedron edges. The sweeped cylindrical surfaces are connected by corner patches represented in rational biquadratic form.

## Cylindrical Surface Replacing One Edge

Let a polyhedron be given by the segment edge system [13] and let a chain of edges be specified to be replaced by quadratic surfaces joining to the polyhedron faces and to each other with 1st order continuity. The proposed algorithm generates the smoothing blend surface as a sequence of cylindrical patches constructed along the edges and corner patches defined at the vertices of the edge chain. The edge blend surface is modelled as a sweep surface of conic section curve represented as a rational quadratic Bézier curve. The parametric equation of such a curve has the form

$$\mathbf{b}(t) = \frac{\omega_0(1-t)^2\mathbf{c}_0 + 2\omega_1(1-t)t\mathbf{c}_1 + \omega_2 t^2\mathbf{c}_2}{\omega_0(1-t)^2 + 2\omega_1(1-t)t + \omega_2 t^2}; \qquad 0 \le t \le 1, \tag{1}$$

where $\mathbf{c}_i$ are the position vectors of the control points $C_i$ and $\omega_i$ are the weights ($i = 0, 1, 2$) [7]. The endpoints of the curve segment are the control points $C_0$ and $C_2$ and the weights can be used in adjusting the shape of the curve. For the user it is more convenient to control the shape by changing one scalar parameter $\rho$, called 'fullness factor', in the range $(0, 1)$. For a given value of $\rho$ the weights are determined from

$$\omega_0 : \omega_1 : \omega_2 = 1 : \frac{\rho}{(1-\rho)} : 1.$$

As $\rho$ increases from 0 to 1, the conic section becomes 'fuller' nearing from the line $C_0C_2$ to the polygon $C_0C_1C_2$. If a circular arc is required, the fullness factor is set to

$$\rho = \frac{\cos\theta}{(1+\cos\theta)},$$

where $2\theta$ is the angle of the normals at the points $C_0$ and $C_2$. Substituting the weights 1, $\omega = \rho/(1-\rho)$, 1 into the *Eq.* (1) it becomes

$$\mathbf{b}(t) = \frac{(1-t)^2\mathbf{c}_0 + 2\omega(1-t)t\mathbf{c}_1 + t^2\mathbf{c}_2}{(1-t)^2 + 2\omega(1-t)t + t^2}; \qquad 0 \le t \le 1. \tag{2}$$

Let $e_1$ be the first edge of the edge chain and $\mathbf{n}_1$ and $\mathbf{n}_2$ the normals to the faces containing the edge (*Fig. 1*).

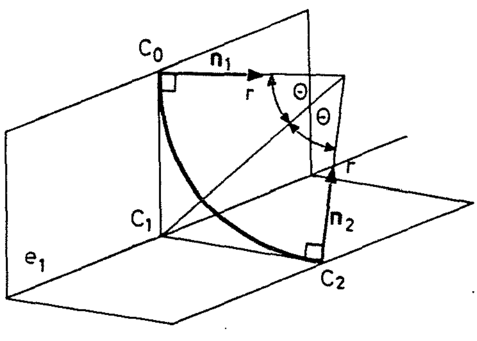


*Fig. 1.* Conic section defining the cylindrical surface for one edge

If the rounding radius $r$ is given for the starting edge $e_1$, the control points $C_0, C_1, C_2$ can be constructed in a plane orthogonal to the edge at the midpoint $C_1$ by measuring the segments $C_0C_1 = C_1C_2 = r\tan\theta$ in both faces. The equation of the circular arc is given by the *Eq.* (2) with the value $\omega = \cos\theta$. This construction has to be carried out only for the first edge. Namely, the parallel line to the edge $e_1$ passing through the point

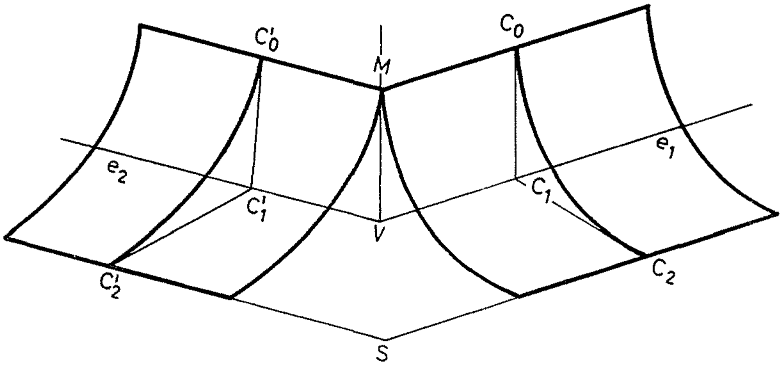


*Fig. 2.* Neighbouring cylindrical surfaces at a concave vertex

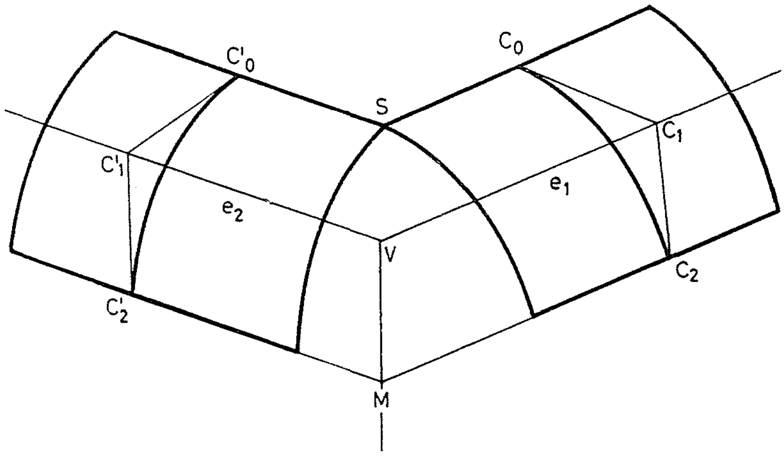


*Fig. 3.* Neighbouring cylindrical surfaces at a convex vertex

$C_0$ determines the point $M$ on the third edge at the vertex $V$ *(Fig.2)*, then the point $C_0'$ belonging to the neighbouring edge $e_2$ is determined by the line through $M$ parallel to $e_2$. After that $C_2'$ is also determined by $C_0'C_1' = C_1'C_2'$. In the case of a convex vertex the point $C_2'$ is first determined then $C_0'$ (*Fig. 3*).

This construction is to be repeated for all edges of the chain. The corner points of a cylindrical patch are constructed by measuring the segment $C_0M$ on the line $C_2S$ in the case of a concave vertex *(Fig. 2)*, or the segment $C_0S$ on the line $C_2M$, when the vertex is convex *(Fig. 3)*, and similarly in the other direction. The two opposite boundary curves $\mathbf{b}_1(t)$ and $\mathbf{b}_2(t)$ are congruent to the conic section determined by the control points $C_0$, $C_1$, $C_2$, assuming equal fullness parameters (*Fig. 4*).

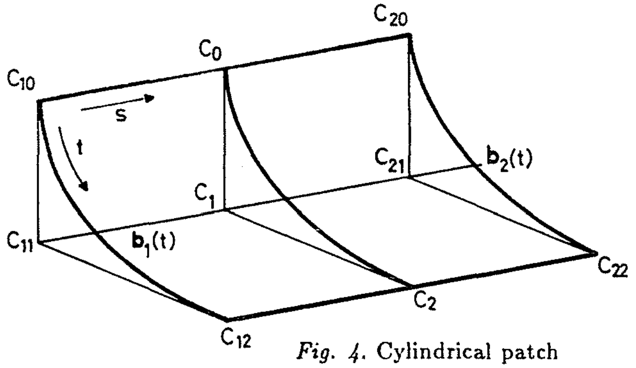


*Fig. 4.* Cylindrical patch

Let $C_{10}$, $C_{11}$, $C_{12}$ and $C_{20}$, $C_{21}$, $C_{22}$ denote the control points of these boundary curves, respectively. According to (2) the equations are

$$\begin{aligned} \mathbf{b}_1(t) &= R_0(t)\mathbf{c}_{10} + R_1(t)\mathbf{c}_{11} + R_2(t)\mathbf{c}_{12}, \\ \mathbf{b}_2(t) &= R_0(t)\mathbf{c}_{20} + R_1(t)\mathbf{c}_{21} + R_2(t)\mathbf{c}_{22}, \qquad 0 \le t \le 1. \end{aligned} \tag{3}$$

where

$$\begin{aligned} R_0(t) &= \frac{(1-t)^2}{N}, \quad R_1(t) = \frac{2\omega(1-t)t}{N}, \quad R_2(t) = \frac{t^2}{N}, \\ N &= (1-t)^2 + 2\omega(1-t)t + t^2, \end{aligned} \tag{4}$$

and $\omega$ is a common weight factor.

The parametric equation of the patch can be expressed as

$$\mathbf{p}(t,s) = (1-s)\mathbf{b}_1(t) + s\mathbf{b}_2(t), \qquad 0 \le t, s \le 1. \tag{5}$$

The $t$-parameter lines are congruent conic sections and the $s$-parameter lines are straight line segments parallel to the polyhedron edge.

Cylindrical patches of different rounding radii and different fullness parameters are shown in the *Figs 5* and *6*.

## Construction of Corner Blend Surfaces

As can be seen in the *Figs 2* and *3* , different patches are needed between two neighbouring cylindrical surfaces to fill out the hole around the vertex. The assumption of the construction is that a point moving along the $s$-parameter line of a cylindrical surface should describe a parabolic segment turning around the vertex, and keeps on moving continuously along an $s$-line of the next cylindrical patch.

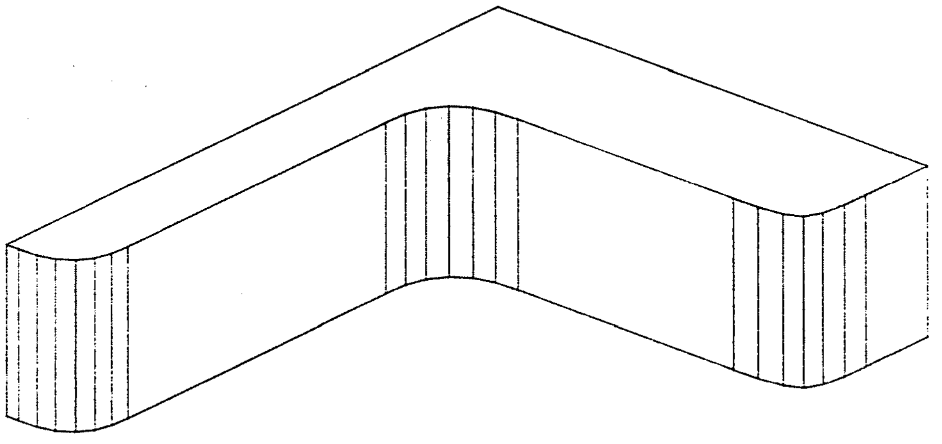

*Fig. 5.* Cylindrical patches of different fullness parameters

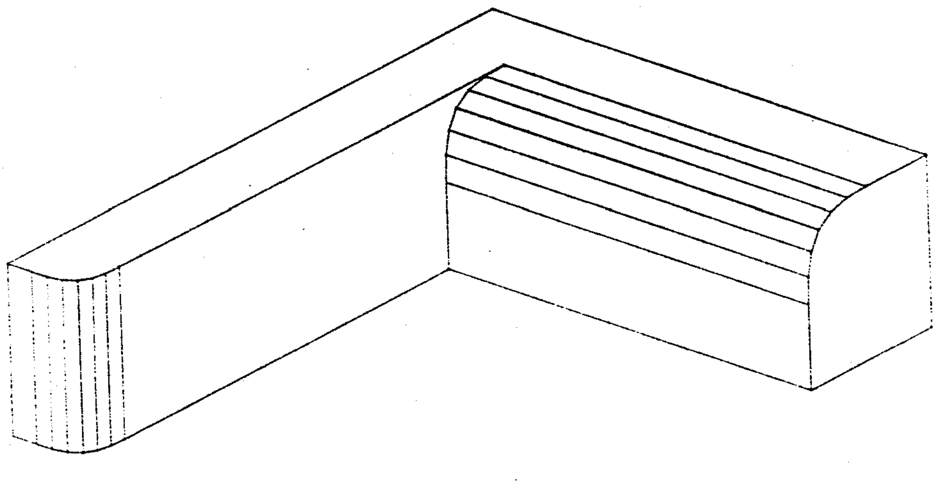

*Fig. 6.* Cylindrical patches of different rounding radii

*Patch Construction at a Concave Vertex*

The patch at a concave vertex is determined by three conic sections: the two boundary curves of the neighbouring patches denoted by $\mathbf{c}_0(t)$ and $\mathbf{c}_2(t)$, and the conic section $\mathbf{c}_1(t)$ determined by the control points $C_{10}$, $C_1$ and $C_2$ *(Fig. 7)*.

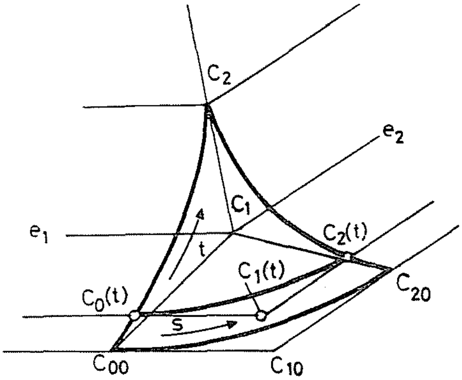


*Fig. 7.* Triangular patch at concave vertex

By taking $C_0(t)$, $C_1(t)$ and $C_2(t)$ with a fixed value of $t$ as the control points of an $s$-parameter line, the rational parametric equation of it has the form

$$\begin{aligned}\mathbf{p}(t,s) &= \frac{(1-s)^2\mathbf{c}_0(t) + 2\omega(1-s)s\mathbf{c}_1(t) + s^2\mathbf{c}_2(t)}{(1-s)^2 + 2\omega(1-s)s + s^2} \\ &= R_0(s)\mathbf{c}_0(t) + R_1(s)\mathbf{c}_1(t) + R_2(s)\mathbf{c}_2(t)\,, \qquad 0 \le s \le 1,\end{aligned} \tag{6}$$

where

$$R_0(s) = \frac{(1-s)^2}{N}, \qquad R_1(s) = \frac{2\omega(1-s)s}{N}, \qquad R_2(s) = \frac{s^2}{N}, \tag{7}$$
$$N = (1-s)^2 + 2\omega(1-s)s + s^2.$$

If $t$ is varying, the points $C_0(t)$, $C_1(t)$ and $C_2(t)$ are moving along the curves $\mathbf{c}_0(t)$, $\mathbf{c}_1(t)$ and $\mathbf{c}_2(t)$, respectively, given by the following equations:

$$\begin{aligned}\mathbf{c}_0(t) &= R_0(t,\omega_0)\mathbf{c}_{00} + R_1(t,\omega_0)\mathbf{c}_1 + R_2(t,\omega_0)\mathbf{c}_2, \\ \mathbf{c}_1(t) &= R_0(t,\omega_1)\mathbf{c}_{10} + R_1(t,\omega_1)\mathbf{c}_1 + R_2(t,\omega_1)\mathbf{c}_2, \\ \mathbf{c}_2(t) &= R_0(t,\omega_2)\mathbf{c}_{20} + R_1(t,\omega_2)\mathbf{c}_1 + R_2(t,\omega_2)\mathbf{c}_2\end{aligned} \tag{8}$$

where $0 \le t \le 1$, and the rational scalar functions $R_i(t)$ $(i = 0, 1, 2)$ are given by the equations (4), but with different weight factors $\omega_i$ for the three curves. In this way the function $\mathbf{p}(t, s)$ represents the triangular patch shown in *Fig. 7*.

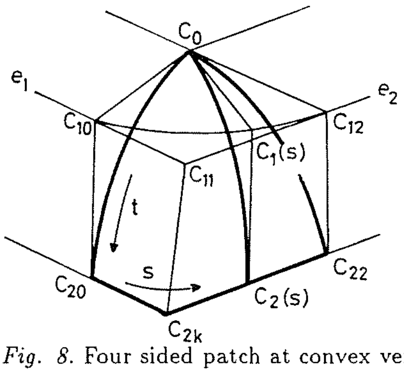


*Fig. 8.* Four sided patch at convex vertex

### *Patch Construction at a Convex Vertex*

The patch at a convex vertex has two straight line segments $C_{20}C_{2k}$ and $C_{2k}C_{22}$ as boundaries lying on polyhedron faces *(Fig. 8)*. The other two boundary curves are conic segments, the boundaries of neighbouring cylindrical patches. Such a patch can be generated by the $t$-parameter lines as conic segments with the control points $C_0, C_1(s)$ and $C_2(s)$ with a fixed value of $s$. According to this the equation of the patch can be expressed as

$$\mathbf{p}(t,s) = R_0(t)\mathbf{c}_0 + R_1(t)\mathbf{c}_1(s) + R_2(t)\mathbf{c}_2(s), \qquad 0 \le t, s \le 1, \qquad (9)$$

where the rational scalar functions $R_0(t)$, $R_1(t)$ and $R_2(t)$ are given by the *Eq.* (4). As the value of $s$ varies from 0 to 1 the point $C_1(s)$ is moving along the curve $\mathbf{c}_1(s)$ given as conic segment by the equation

$$\mathbf{c}_1(s) = R_0(s)\mathbf{c}_{10} + R_1(s)\mathbf{c}_{11} + R_2(s)\mathbf{c}_{12}, \qquad 0 \le s \le 1,$$

where the functions $R_0(s)$, $R_1(s)$ and $R_2(s)$ are given by the *Eqs.* (7). The points $C_2(s)$ lie on the boundary line segments

$$\mathbf{c}_2(s) = \begin{cases} (1-s)\mathbf{c}_{20} + s\mathbf{c}_{2k}, & 0 \le s \le 0.5, \\ (1-s)\mathbf{c}_{2k} + s\mathbf{c}_{22}, & 0.5 \le s \le 1. \end{cases}$$

The $s$-parameter lines are quadratic curves for each $t$, $0 < t < 1$. The point $C_0$ is a singular point at $t = 0$. For $t = 1$ the boundary lines $\mathbf{c}_2(s)$ are obtained.

## Derivatives at Patch Boundaries

The continuity of the constructed blend surface composed from cylindrical surfaces and corner patches can be investigated by the derivatives of the parametric functions $\mathbf{p}(t, s)$ computed at boundary points. Determining the derivatives of the scalar functions $R_0(u)$, $R_1(u)$ and $R_2(u)$ at $u = 0$ and $u = 1$ separately, we have

$$\begin{aligned} R_0'(0) &= -2\omega, & R_0'(1) &= 0 \\ R_1'(0) &= 2\omega, & R_1'(1) &= -2\omega \\ R_2'(0) &= 0 & R_2'(1) &= 2\omega. \end{aligned} \tag{10}$$

The derivatives of the $t$-parameter lines of the cylindrical surface given by the equations (3,4,5) are

$$\frac{\partial \mathbf{p}(t, s)}{\partial t} = (1 - s)\dot{\mathbf{b}}_1(t) + s\dot{\mathbf{b}}_2(t).$$

The tangent vectors of the boundary curves $\mathbf{b}_1(t)$ and $\mathbf{b}_2(t)$ at the corner points can be computed by using the values written in (10). At the upper corner points we have

$$\begin{aligned} t = 0,\ s = 0: &\qquad 2\omega(\mathbf{c}_{11} - \mathbf{c}_{10}), \\ t = 0,\ s = 0: &\qquad 2\omega(\mathbf{c}_{21} - \mathbf{c}_{20}), \end{aligned}$$

that lie in the upper face containing the edge (Fig. 4). At the lower corner points the values of the $t$-derivatives are

$$\begin{aligned} t = 1,\ s = 0: &\qquad 2\omega(\mathbf{c}_{12} - \mathbf{c}_{11}), \\ t = 1,\ s = 1: &\qquad 2\omega(\mathbf{c}_{22} - \mathbf{c}_{21}), \end{aligned}$$

that lie in the horizontal polyhedron face.

The derivatives of the $s$-parameter lines are

$$\frac{\partial \mathbf{p}(t, s)}{\partial s} = \mathbf{c}_2(t) - \mathbf{c}_1(t),$$

which is the generating vector of the cylindrical patch, parallel to the edge. According to this, the tangent planes along the boundaries $C_{10}C_{20}$ and $C_{12}C_{22}$ are the polyhedron faces.

A triangular patch given by (6) joining to a cylindrical patch has the following cross derivatives at the points of the common boundary curve

$$\begin{aligned} \left.\frac{\partial \mathbf{p}(t, s)}{\partial s}\right|_{s=0} &= 2\omega(\mathbf{c}_1(t) - \mathbf{c}_0(t)), \\ \left.\frac{\partial \mathbf{p}(t, s)}{\partial s}\right|_{s=1} &= 2\omega(\mathbf{c}_2(t) - \mathbf{c}_1(t)). \end{aligned}$$

These vectors are parallel to the edges $e_1$ and $e_2$, respectively, if the weights $\omega_i$ $(i = 0, 1, 2)$ in the equations of the curves $\mathbf{c}_0(t)$, $\mathbf{c}_1(t)$ and $\mathbf{c}_2(t)$ are equal. In this case the two patches have common tangent planes at the common boundary points, in other words the surfaces join with 1st order continuity.

The cross derivative along the boundary curve $t = 0$ is

$$\left.\frac{\partial \mathbf{p}(t,s)}{\partial t}\right|_{t=0} = 2\omega[R_0(s)(\mathbf{c}_1 - \mathbf{c}_{00}) + R_1(s)(\mathbf{c}_1 - \mathbf{c}_{10}) + R_2(s)(\mathbf{c}_1 - \mathbf{c}_{20})],$$

which is a vector in the polyhedron face containing the curve. Therefore the tangent plane of the patch at the points of this boundary curve is the containing polyhedron face itself. The point $C_2$ is a singular point of the patch, where the tangent plane does not exist.

Investigating the derivatives of the patch given by (9) at a convex vertex, the following is obtained: the tangents of the $t$-parameter lines at the corner points

$$\begin{aligned} t = 0,\ s = 0: &\qquad 2\omega(\mathbf{c}_{10} - \mathbf{c}_0), \\ t = 0,\ s = 1: &\qquad 2\omega(\mathbf{c}_{12} - \mathbf{c}_0) \end{aligned}$$

lie in the upper horizontal polyhedron face *(Fig. 8)*. The tangents at the corner points

$$\begin{aligned} t = 1,\ s = 0: &\qquad 2\omega(\mathbf{c}_{20} - \mathbf{c}_{10}) \\ t = 1,\ s = 1: &\qquad 2\omega(\mathbf{c}_{22} - \mathbf{c}_{12}) \end{aligned}$$

lie in the left and right vertical faces, respectively.

However, the cross derivatives along the boundary lines $t = 1$ at the points $0 < s < 1$ do not lie in the polyhedron faces with the exception when the fullness parameter $\rho$ for the curve $\mathbf{c}_1(t)$ is equal to 1. But in the case of $\rho = 1$ the derivatives of the $t$-parameter lines are not continuous at the points belonging to $s = 0.5$. A compromise is, when the value of $\rho$ is chosen for almost 1, that means a big value for the weight $\omega$, then the tangent planes at the inner points of the boundary lines $C_{20}C_{2k}$ and $C_{2k}C_{22}$ are almost the corresponding polyhedron faces. The patch joins to the cylindrical patches with 1st order continuity, because the cross derivatives at the common boundary points are parallel to the edges $e_1$ and $e_2$, respectively. Specially the values of the derivatives of the $s$-parameter lines at the corner points are

$$\begin{aligned} t = 1,\ s = 0: &\qquad \mathbf{c}_{2k} - \mathbf{c}_{20} \\ t = 1,\ s = 1: &\qquad \mathbf{c}_{22} - \mathbf{c}y_{2k} \end{aligned}$$

Therefore the path of a point moving along the $s$-parameter lines of the constructed surfaces is a composed curve, where the straight line and quadratic segments join with 1st order continuity.

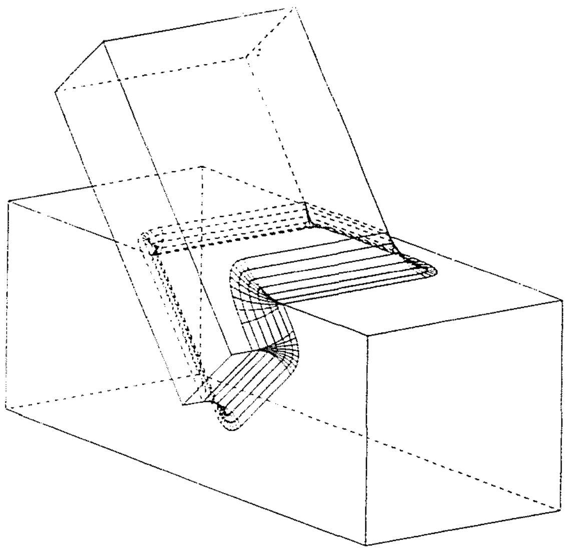

*Fig. 9.* Blend surface along a closed edge chain

## Conclusions

A method of constructing edge and corner blend surfaces for a given polyhedron is presented in this paper. The algorithm works along a chain of edges and generates a surface composed of cylindrical and corner patches given in rational quadratic parametric form. By assumption at every vertex of the chain three polyhedron edges meet and only the two belonging to the chain are rounded off. In the *Fig. 9* the resulting surface can be seen constructed along the line of intersection of two polyhedra. Further studies are required in order to construct blend surfaces in more general cases.

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