PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 36, NOS. 3–4, PP. 265–271 (1992)

CONIC SECTIONS – AN INTRODUCTION TO THE RATIONAL SPLINES

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Received: November 17, 1992

Abstract

The rational splines have been included in the **IGES** (International Graphics Exchange Specification) standard for about ten years, but they have been subjects of interest since 1967 [1]. The current kind of rational splines of our days are the **NURBS** (Non-Uniform Rational *B-splines*), which are generalizations of *B-splines* and the rational Bézier curves and surfaces at the same time. The popular scientific articles and the manuals (e.g. [2]) as well frequently mention as an advantage that these spline curves are accurate for conic sections. For this reason, in this article we propose the rational representation of conic sections as an introduction to the **NURBS**.

Keywords: conics, rational Bézier curve, nonuniform rational B-spline.

Introduction

The general equation of a conic section in the affine plane:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

The homogeneous equation in the projective plane is:

$$a_{00}x^0x^0 + 2a_{01}x^0x^1 + 2a_{02}x^0x^2 + a_{11}x^1x^1 + 2a_{12}x^1x^2 + a_{22}x^2x^2 = 0, \quad (1)$$

where the correspondence between the affine and the homogeneous projective coordinates is:

$$x = \frac{x^1}{x^0}, \quad y = \frac{x^2}{x^0}, \quad \text{if} \quad x^0 \neq 0 \quad \text{and} \quad \begin{matrix} f = a_{00} & d = 2a_{01} & e = 2a_{02}, \\ a = a_{11} & b = 2a_{12} & c = a_{22}. \end{matrix}$$

Introducing the bilinear form

$$\langle \mathbf{x}; \mathbf{y} \rangle := x^i a_{ij} y^j$$

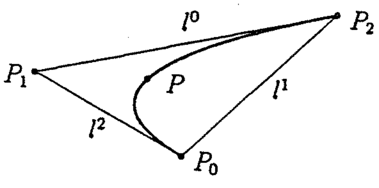
[1]Supported by Hungarian Nat. Found. for Sci. Research (OTKA) No. 1615 (1991).

with $\mathbf{x} = \mathbf{e}_i x^i$, $\mathbf{y} = \mathbf{e}_j y^j$ and the symmetric coefficients $a_{ij} = a_{ji}$ from (1), we can see that $\langle \mathbf{e}_i ; \mathbf{e}_j \rangle = a_{ij}$ gives a geometric relation between the conic section (by its coefficients) and the coordinate base vectors

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Representation in Projective Coordinate System

Let the conic section be given by its two tangents l^0, l^2 that have point P_1 in common. The points of contact are denoted by P_2 and P_0, and a third point of the curve not collinear with any two points of the triangle $P_0 P_1 P_2$ is denoted by P in *Fig. 1*.



$$P_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Fig. 1. Projective coordinate system fitted to the conics

Let us introduce the projective reference system by the base points $P_0(\mathbf{e}_0), P_1(\mathbf{e}_1), P_2(\mathbf{e}_2)$ and $P(\mathbf{e})$, where $\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2$. Then the conic section has the equation

$$z^0 z^2 - z^1 z^1 = 0 \tag{2}$$

with the coordinates (z^0, z^1, z^2) of the running point. Both parametric representations below satisfy the *Eq.* (2):

$$z^0 = 1, \qquad z^1 = t, \qquad z^2 = t^2, \tag{3}$$

$$z^0 = (1-t)^2, \qquad z^1 = t(1-t), \qquad z^2 = t^2. \tag{4}$$

The *Figs 2a* and *2b* show the correspondence between particular values of the parameter and the given points in the case of (3) and (4), respectively. Both parametrisations have their own benefits to be discussed.

The reference system used was a very special one. To get the parametric representation in general form we consider the points P_0, P_1, P_2 given by their pointing vectors $\mathbf{p}_i = \mathbf{e}_j p_i^j$ and for the fourth point $P(\mathbf{p})$ $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. Let the pointing vector of the running point be denoted by $\mathbf{z}$,

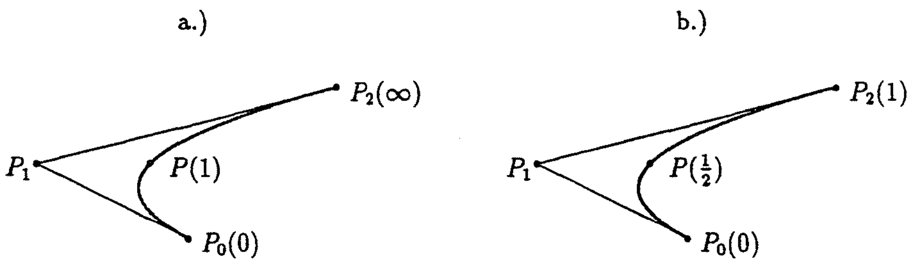


Fig. 2. Parametrisations

this means $\mathbf{z} = \mathbf{p}_i z^i$. Of course the base transformation $\mathbf{p}_i = \mathbf{e}_j p_i^j$ serves also the coordinate transformation.

$$\mathbf{x} = \mathbf{e}_j x^j = \mathbf{z} = \mathbf{p}_i z^i = \mathbf{e}_j p_i^j z^i,$$

i.e.

$$x^j = p_i^j z^i.$$

Here we used the Einstein summation convention for the same lower and upper indices. In details we get the

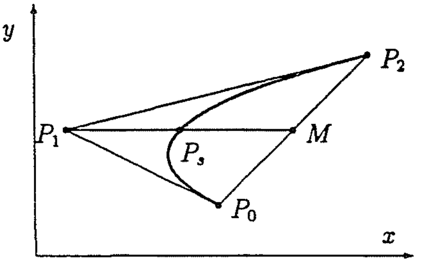
$$\begin{aligned} x^0 &= p_0^0 + p_1^0 t + p_2^0 t^2, \\ x^1 &= p_0^1 + p_1^1 t + p_2^1 t^2, \\ x^2 &= p_0^2 + p_1^2 t + p_2^2 t^2. \end{aligned} \tag{5}$$

$$\begin{aligned} x^0 &= p_0^0(1-t)^2 + p_1^0(1-t)t + p_2^0 t^2, \\ x^1 &= p_0^1(1-t)^2 + p_1^1(1-t)t + p_2^1 t^2, \\ x^2 &= p_0^2(1-t)^2 + p_1^2(1-t)t + p_2^2 t^2. \end{aligned} \tag{6}$$

from (3) and (4), respectively. The representation (6) is the form we are going to deal with.

Return to the Cartesian Coordinates

The points P_0, P_1, P_2 are supposed to be given on the screen or in the **UCS** (User Coordinate System) by their pointing vectors and Cartesian coordinates.



$$P_0(\mathbf{r}_0(x_0 = p_0^1/p_0^0\ ,\ y_0 = p_0^2/p_0^0))$$
$$P_1(\mathbf{r}_1(x_1 = p_1^1/p_1^0\ ,\ y_1 = p_1^2/p_1^0))$$
$$P_2(\mathbf{r}_2(x_2 = p_2^1/p_2^0\ ,\ y_2 = p_2^2/p_2^0))$$

Fig. 3. Cartesian coordinate system

The fourth point P_s ('s' comes from the word 'shoulder') not collinear with any two points of P_0, P_1, P_2 is introduced into the role of P.

$$P_s(\mathbf{r}_s) = \frac{1}{h_0 + 2h_1 + h_2}(h_0\mathbf{r}_0 + 2h_1\mathbf{r}_1 + h_2\mathbf{r}_2) \tag{7}$$

with the $h_0, 2h_1, h_2$ as weights in P_0, P_1, P_2. By using the notation of *Fig. 3* we can demonstrate the geometrical meaning of the weights, e. g. the ratio

$$\frac{P_0M}{MP_2} = \frac{h_2}{h_0} \text{ and } \frac{MP_s}{P_sP_1} = \frac{2h_1}{h_0 + h_2}.$$

In *Fig. 3* we have chosen $h_0 = h_1 = h_2 = 1$. The $(h_0, 2h_1, h_2)$ are proportional coordinates of the point P_s. It takes some explaining why the weight of P_1 was denoted by $2h_1$ (instead of h_1). Anticipation must be made that in the special **NURBS**, which with conics coincide, appear the second degree Bernstein polynomials: $(1-t)^2, 2t(1-t), t^2$; that is why we have introduced the coefficient 2. The Cartesian coordinates of the point P_s as from (7) follows

$$P_s(\mathbf{r}_s(x_s = \frac{h_0x_0 + 2h_1x_1 + h_2x_2}{h_0 + 2h_1 + h_2}\ ,\ y_s = \frac{h_0y_0 + 2h_1y_1 + h_2y_2}{h_0 + 2h_1 + h_2})).$$

The pointing vectors of the base points by their homogeneous Cartesian coordinates are

$$\begin{aligned}
&\mathbf{p}_0(h_0, h_0x_0, h_0y_0),\\
&\mathbf{p}_1(2h_1, 2h_1x_1, 2h_1y_1),\\
&\mathbf{p}_2(h_2, h_2x_2, h_2y_2),\\
&\mathbf{p}_s(h_0 + 2h_1 + h_2, h_0x_0 + 2h_1x_1 + h_2x_2, h_0y_0 + 2h_1y_1 + h_2y_2),
\end{aligned}$$

which can be substituted into the equation system (6):

$$\begin{aligned}
x^0(t) &= h_0(1-t)^2 + 2h_1(1-t)t + h_2t^2,\\
x^1(t) &= h_0x_0(1-t)^2 + 2h_1x_1(1-t)t + h_2x_2t^2,\\
x^2(t) &= h_0y_0(1-t)^2 + 2h_1y_1(1-t)t + h_2y_2t^2.
\end{aligned} \tag{8}$$

Finally the $\mathbf{r}(t)$ vector-valued function describing the conic section is:

$$\begin{aligned}
\mathbf{r}(t) = &\frac{h_0(1-t)^2}{h_0(1-t)^2 + 2h_1t(1-t) + h_2t^2}\mathbf{r}_0 +\\
&+ \frac{2h_1t(1-t)}{h_0(1-t)^2 + 2h_1t(1-t) + h_2t^2}\mathbf{r}_1 +\\
&+ \frac{h_2t^2}{h_0(1-t)^2 + 2h_1t(1-t) + h_2t^2}\mathbf{r}_2
\end{aligned} \tag{9}$$

This simple and short method emphasizes the benefit of using homogeneous coordinates.

The Correspondence between NURBS and Conic Sections

The **NURBS** can be introduced as the formal generalisation of *B-splines:*

$$\mathbf{r}(t) = \sum_{i=0}^{k-1} R_{i,m}(t)\mathbf{r}_i,$$

where $\mathbf{r}_i$, $i = 0, 1, \ldots, k-1$ are the position vectors of the defining polygon and $R_{i,m}(t)$ are the corresponding rational *B-spline* basic functions:

$$R_{i,m}(t) = \frac{h_iN_{i,m}(t)}{\sum_{i=0}^{k-1} h_iN_{i,m}(t)}, \tag{10}$$

where

- k : the number of the control points
- h_i : weights at the points $P_0, P_1, \ldots, P_{k-1}$, for all $h_i : 0 \leq h_i$
- m : polynomial degree
- $N_{i,m}(t)$: *B-spline* basic functions
- $[t_0, t_1, \ldots, t_{k+m}]$: the knot vector of the *B-splines*:

recalling that the recursive definition of the basic functions:

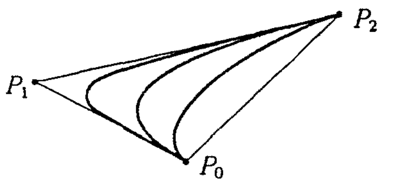
$$N_{i,0}(t) = \begin{cases} 1 & \text{if } \; t_i \leq t \leq t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,m}(t) = \frac{t - t_i}{t_{i+m} - t_i} N_{i,m-1}(t) + \frac{t_{i+m+1} - t}{t_{i+m+1} - t_{i+1}} N_{i+1,m-1}(t) \qquad t_i \leq t \leq t_{i+m+1}$$

Let us take the following restrictions of parameters into considerations:

- $k = 3$
- $h_0, \; h_1, \; h_2 \geq 0$
- $m = 2$ second degree curve
- $[0, 0, 0, 1, 1, 1]$ the knot vector of the *B-splines*

By substituting the above values into the formula (10) of the **NURBS**-curve we can obtain a special rational curve in the form of the function (9). The character of the curve depends on h_0, h_1, h_2.



$$\begin{aligned} h_1 &= 0 : P_0P_2 \text{ segment} \\ 0 \leq h_1 &< 1 : \text{ellipse} \\ h_1 &= 1 : \text{parabola} \\ h_1 &> 1 : \text{hyperbola} \end{aligned}$$

Fig. 4. Dependence of conics on h_1 with $h_0 = h_2 = 1$

That means *'the well-known'* conic sections can be regarded as special **NURBS**, or in another point of view, the **NURBS** are accurate for the conics. It is really a perception of a familiar geometrical concept (without quotation mark), if the theory of rational parametric representation of conic sections takes precedence of the **NURBS**-curve theory. The kinematic surfaces, especially the kinematic Bézier-patches are important in technical sciences. O. RÖSCHEL has developed the theory of the kinematic Bézier patches (see the articles [3] and [4]), where the conics appear as paths of points. In this sense, the rational representation of conic sections can serve as an introduction to the kinematic rational Bézier patches too.

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