PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 36, NOS. 3-4, PP. 273-278 (1992)

ON SOME SYMPLECTIC GROUP ACTIONS WHERE ALL THE ORBITS ARE EQUIVARIANTLY ISOMORPHIC AND DIFFEOMORPHIC TO A FIXED ORBIT OF THE COADJOINT ACTION

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Received: November 16, 1992.

Abstract

In conformity with the 'Foundations of Mechanics' given by R. ABRAHAM and J. E. MARSDEN [1] let (P, ω) be a symplectic manifold and

$$\Phi : G \times P \to P$$

a Hamiltonian action of a compact, connected Lie group G on the manifold P.

Considering this setting J. SZENTHE [2] found the following result:

If the isotropy subgroups of the action Φ are of maximal rank then all the orbits of Φ are equivariantly isomorphic.

Consequently, P is the total space of a differentiable fibre bundle, where the base manifold is the orbit space of the action Φ and the fibres are diffeomorphic to a fixed orbit of the coadjoint action.

The aim of the present paper is to develop further characterizations of the above situation as it was suggested by J. J. DUISTERMAAT.

Keywords: symplectic manifold, Lie group, Hamiltonian action, momentum map.

Introduction

For sake of conciseness a detailed introduction of some basic facts and notations contained in J. SZENTHE's paper [2] will be omitted here.

Let $\mathfrak{J}$ be an almost complex structure and $\langle , \rangle_P$ a Kählerian metric on the given manifold P. (They exist since P is supposed to be paracompact). Moreover

$$\omega(X, Y) = \langle \mathfrak{J}X, Y \rangle_P$$

holds for all $X, Y \in \tau(P)$ and both $\mathfrak{J}$ and $\langle , \rangle_P$ are invariant with respect to the action Φ.

Let further

$$\mu : P \to \mathfrak{g}^*$$

[1]Supported by Hungarian Nat. Found. for Sci. Research (OTKA) No. 1615 (1991).

be an equivariant momentum map, where g^* is the dual of the Lie algebra g of G. The momentum map μ is supposed to be equivariant with respect to the given Hamiltonian action

$$\Phi : G \times P \to P \qquad \text{and the coadjoint action}$$
$$\Psi : G \times g^* \to g^* \qquad \text{defined by}$$
$$\Psi(g,\xi) = \mathrm{Ad}^*(g^{-1})\xi \qquad \text{for } g \in G \text{ and } \xi \in g^*.$$

Consider now for any fixed $z \in P$ the isotropy subgroup G_z, the orbit $G(z) \cong G/G_z$ of the action Φ and the orthogonal decomposition of the tangent space

$$T_zP = T_zG(z) \oplus N_zG(z)$$

taken with respect to the Kählerian metric $\langle , \rangle_P$.

Let $\xi = \mu(z) \in g^*$. Then the tangent linear map

$$T_z\mu : T_zP \to T_\xi g^* \cong g^*$$

of the momentum map μ has the following basic properties:

$$\mathrm{Ker}\, T_z\mu = \mathrm{J}_z N_zG(z),$$
$$\mathrm{Im}\, T_z\mu = m_z^* \subset g^*,$$

where $m_z^*\{\xi \mid \xi(g_z) = 0, \xi \in g^*\}$ is the annihilator space of the Lie subalgebra g_z belonging to G_z.

By the equivariance of μ the restricted map

$$T_z\mu \mid T_zG(z) : T_zG(z) \to T_\xi G(\xi)$$

is surjective and so

$$\dim T_zG(z) \geq \dim T_\xi G(\xi) \qquad \text{holds in general.}$$

More precisely, consider the decomposition

$$T_zP = \mathrm{J}_zT_zG(z) \oplus \mathrm{J}_zN_zG(z).$$

Since the almost complex structure J preserves orthogonality the restricted map

$$T_z\mu \mid \mathrm{J}_zT_zG(z) : \mathrm{J}_zT_zG(z) \to m_z^*$$

is an isomorphism. Using the notation $W = \mathrm{J}_z T_z G(z) \cap N_z G(z)$, the subspace $\mathrm{J}_z W$ is the kernel of the map $T_z\mu \mid T_z G(z)$ and so

$$\dim T_z G(z) = \dim T_\xi G(\xi) + \dim W \qquad \text{is valid.}$$

Proposition 1
The orbit $G(z)$ is a symplectic submanifold of P if and only if $\dim W = 0$, i. e. $Im\ T_z\mu = T_\xi G(\xi)$ holds.
Proof It is enough to show that the restricted symplectic form ω is non-degenerated on $G(z)$ if and only if for any non-zero tangent vector $\overline{Y} \in T_z G(z)$ $\quad T_z\mu(\overline{Y}) \neq 0$ holds.
As it is well known, the momentum map μ has the following basic property:

$$\iota_{\overline{X}}\omega = d\langle \mu(z), X\rangle$$

holds on P for any $X \in \mathrm{g}$, where the vectorfield $\overline{X}$ is the infinitesimal generator of the action Φ belonging to the element $X \in \mathrm{g}$.

Thus

$$\omega(\overline{X}, \overline{Y}) = \langle T_z\mu(\overline{Y}), X\rangle$$

is valid at $z \in P$ implying the statement of proposition 1.
Remark The above notation $\overline{Y} \in T_z G(z)$ is justified by the fact that having a reductive decomposition

$$\mathrm{g} = \mathrm{g}_z \oplus \mathrm{m}_z$$

of the Lie algebra g there is a canonical isomorphism between $T_z G(z)$ and m_z.
Definition *Let $R_z \subset T_z P$ be the subspace defined by*

$$R_z = \{X \mid T_z\Phi_g(X) = X \text{ for } g \in G_z, \quad X \in T_z P\},$$

where $T_z\Phi_g : T_z P \to T_z P$ is the induced action belonging to the element $g \in G_z$. A non-zero element $X \in R_z \cap T_z G(z)$ is called an isotropy fixed vector of the action Φ.
Proposition 2
Let $G(z)$ be a principal orbit of the action Φ and suppose that $T_z G(z)$ does not contain isotropy fixed vector. Then $G(z)$ is a symplectic submanifold of P.

Conversely, let $G(z)$ be a symplectic submanifold of P. Then $T_z G(z)$ does not contain isotropy fixed vector.

Proof:

a) In the case of a principal orbit $N_zG(z) \subset R_z$ is valid. The absence of isotropy fixed vectors implies then that $N_zG(z) = R_z$. Using also the fact that $J_zR_z = R_z$ holds

$$\operatorname{Ker} T_z\mu = N_zG(z) \qquad \text{is obtained.}$$

Thus

$$W = J_zT_zG(z) \cap N_zG(z) = \{0\} \qquad \text{is valid.}$$

b) If the orbit $G(z)$ is a symplectic submanifold of P then the map

$$T_z\mu \mid T_zG(z) : T_zG(z) \to T_\xi G(\xi) \qquad (\xi = \mu(z))$$

is an isomorphism and so $\dim G_z = \dim G_\xi$ holds.
On the other hand the equivariance

$$\mu(\Phi_g z) = \Psi_g(\xi) \text{ for all } g \in G_z \text{ implies that } G_z \subset G_\xi.$$

Consequently $G_z = G_\xi$ is valid since $G(\xi)$ is simply connected and so G_ξ must be connected.

Suppose now indirectly that $\overline{Y} \in T_zG(z)$ is an isotropy fixed vector, where $Y \in \mathrm{m}_z$.

Then

$$T_z\Phi_g\overline{Y} = \overline{Ad(g)Y} = \overline{Y} \text{ holds for all } g \in G_z \text{ or equivalently}$$
$$[Z, Y] = 0 \text{ for all } Z \in \mathrm{g}_\xi = \mathrm{g}_z.$$

But now $Y \notin \mathrm{g}_\xi$ yields that

$$\operatorname{rank} \mathrm{g}_\xi < \operatorname{rank} \mathrm{g}$$

which contradicts the fact that all the isotropy subgroups G_ξ have maximal rank at the coadjoint action Ψ.

Remark If $G(z)$ is a symplectic submanifold of P and it is a principal orbit of Φ then

$$\operatorname{Ker} T_z\mu = N_zG(z) = R_z \text{ is valid.}$$

The preceding propositions allow to formulate a consequence of J. Szenthe's result as follows:

Theorem A

Let (P, ω) be a symplectic manifold and $\Phi : G \times P \to P$ a Hamiltonian action of a compact, connected Lie group G. If the orbits of Φ are symplectic submanifolds of P then they are all equivariantly isomorphic, i. e.

principal orbits. Moreover the given equivariant momentum map μ maps the manifold P onto a single orbit $G(\xi)$ of the coadjoint action Ψ.

In order to complete this statement with its converse part the following theorem is given:

Theorem A

If an equivariant momentum map μ maps the symplectic manifold P onto a single orbit $G(\xi)$ of the coadjoint action Ψ then all the orbits of the action Φ are symplectic submanifolds of P and they are equivariantly isomorphic to each other, i. e. the manifold P is a disjoint union of principal orbits.

Proof of the theorems

The proof of theorem A can be carried out analogously to that given in J. Szenthe's paper.

Theorem B can be proved in the following simple way:

Consider the isomorphism

$$T_z\mu \mid \mathrm{J}_zT_zG(z) : \mathrm{J}_zT_zG(z) \to \mathrm{m}_z^*.$$

Since in our case $\mathrm{m}_z^* = T_\xi G(\xi)$ is valid, the map

$$T_z\mu \mid T_zG(z) : T_zG(z) \to T_\xi G(\xi) \qquad (\xi = \mu(z))$$

is again an isomorphism and so the subspace

$$\mathrm{J}_zW = T_zG(z) \cap \mathrm{J}_zN_zG(z) \text{ is 0-dimensional.}$$

Thus $G(z)$ is a symplectic submanifold of P by proposition 1. Different orbits of the action Φ are equivariantly isomorphic to each other then through the equivariant momentum map μ.

Remark The converse part of J. Szenthe's originally given result would be the following:

If an equivariant momentum map μ maps the symplectic manifold P onto a single orbit $G(\xi)$ of the coadjoint action Ψ then all the isotropy subgroups $G_z (z \in P)$ of the action Φ are of maximal rank.

The proof of this statement is known however only in the case where the Lie group G is supposed to be semisimple. In this case a result of B. P. KOMRAKOV [3] implies that if the orbit $G(z)$ is a symplectic submanifold then the isotropy subgroup G_z is of maximal rank.

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