PERIODICA POLYTECHNICA SER. MECH. ENG. VOL. 36, NOS. 3-4, PP. 279-290 (1992)

QUASIPLANAR MAPS BETWEEN THREE-DIMENSIONAL MANIFOLDS

L. VERHÓCZKI[1]

Department of Geometry
Faculty of Mechanical Engineering
Technical University of Budapest

Received: November 16, 1992

Abstract

The problem of quasiplanar maps between spaces with affine connection was set by N. S. SZINYUKOV in [4]. Quasiplanar maps can be regarded as generalizations of maps of affine spaces which preserve plane curves. In this paper we study a special class of quasiplanar maps between three-dimensional manifolds. Among others we give a sufficient and necessary condition for a map to be quasiplanar. Our main results are analogous to the results of SZINYUKOV and J. MIKES who studied in [2] quasiplanar maps in spaces with more than three dimensions.

Keywords: manifold, affine connection, torsion, quasiplanar map, deformation tensor.

1. Introduction

In the seventies N. S. SZINYUKOV studied almost geodesic curves in spaces with torsion free affine connection. Almost geodesic curves can be considered as generalizations of plane curves in affine spaces. Investigating curves of this type, SZINYUKOV has found the so-called φ-planar and F-planar curves which form two distinct classes of almost geodesic curves. He has shown that if we fix a tensor field of type (1,1) satisfying some conditions, then we can determine a subset of almost geodesic curves of the considered space (see [4]).

Later SZINYUKOV and MIKES in their paper [2] defined F-planar curves in spaces endowed with torsion free affine connection and with tensor field F. These F-planar curves are connected with special curves in Kählerian spaces (see [3]).

They called a diffeomorphism between two spaces F-planar (or in other words quasiplanar) if it maps each F-planar curve of the first space into an F-planar curve of the second one. In the paper written in common the authors considered that case when the dimension of the space is greater

[1]Supported by Hungarian Nat. Found. for Sci. Research (OTKA) No. 1615 (1991).

than three and gave a necessary and sufficient condition for a map to be F-planar.

In this paper we study quasiplanar maps between 3-dimensional manifolds with affine connection. Our main purpose is to give a necessary and sufficient condition for a map to be quasiplanar. We strive to make our investigations in modern form, that is, we try to avoid computations with tensor components. The obtained results are analogous to the equations concerning higher dimensional case which were given by SZINYUKOV and MIKES in [2]. In the second chapter we deal with the tensor algebra of a real 3-dimensional vector space and prove some auxiliary facts which we use later. In the third chapter we give the correct definitions and characterize the quasiplanar maps.

The notions used in this paper, which are not defined here, can be found in the book [1]. Manifolds, tensor fields, curves are always supposed to be of class C^∞. Throughout the paper we use the Einstein summation convention, that is, the repeated index means summation over its complete range. As usual, components of tensors will be denoted by the same letters with indices which indicate the types of tensors.

2. Auxiliary Results in Tensor Algebras

Let V be an n-dimensional vector space over the set of real numbers $\mathcal{R}$ ($n \geq 3$). Let V^* denote the dual space and $\mathbf{T}^r_s(V)$ denote the tensor space of contravariant degree r and covariant degree s over V. It is well known that each element Q of $\mathbf{T}^r_s(V)$ can be regarded as a multilinear map of $V^s \times V^{*r}$ into $\mathcal{R}$. Therefore we say that Q has got s contravariant and r covariant variables. The product operation in the mixed tensor algebra will be denoted by the sign $\otimes$. The canonical bilinear form mapping $V \times V^*$ into $\mathcal{R}$ will be denoted by $<\ ,\ >$. Let us take a fixed basis $e_1, \ldots, e_n$ in V and its dual basis $\varepsilon^1, \ldots, \varepsilon^n$ in V^*.

For example, let us consider a tensor Q of type $(2,3)$. Using the Einstein summation convention, Q can be written in the form

$$Q = Q^{hi}_{jkl}\, \varepsilon^j \otimes \varepsilon^k \otimes \varepsilon^l \otimes e_h \otimes e_i,$$

where the real numbers Q^{hi}_{jkl} $(h, i, j, k, l = 1, \ldots, n)$ are called the components of Q with respect to the basis $e_1, \ldots, e_n$ in V.

We introduce some operations on tensors which will be used later. Let P, D, F be tensors of type $(1, s_1)$, $(1, s_2)$, $(1, s_3)$ over V, respectively. Their tensor product $P \otimes D \otimes F$ of type $(3, s)$ can be regarded as a 3-linear map of $V^* \times V^* \times V^*$ into the tensor space $\mathbf{T}_s(V)$ of covariant degree s, where $s = s_1 + s_2 + s_3$.

Definition 1. The contravariant exterior product $P\wedge D\wedge F$ of the tensors given above is the tensor of type (3,s) having the equality

$$P\wedge D\wedge F(\omega^1,\omega^2,\omega^3)=\sum_{\pi} sign(\pi)\ (P\otimes D\otimes F)(\omega^{\pi(1)},\omega^{\pi(2)},\omega^{\pi(3)})$$

for any dual vectors ω^1, ω^2, ω^3 in V^*, where the summation is taken over all permutations π of (1,2,3).

Remark 1. Obviously, the exterior product of three vectors is antisymmetric with respect to the covariant variables. Let P be a tensor of type (1,2) and let D, F be two tensors of type (1,1). By the above definition the components of their exterior product can be expressed as

$$(P\wedge Q\wedge F)^{hij}_{klrs}=\sum_{\pi} sign(\pi)P^{\pi(h)}_{kl}D^{\pi(i)}_{r}F^{\pi(j)}_{s}$$

where the summation is taken over all permutations π of (h,i,j).

Remark 2. Considering a vector φ in V as one of the three tensors, we have

$$P\wedge D\wedge\varphi=-P\wedge\varphi\wedge D=\varphi\wedge P\wedge D.$$

Later we use the following simple lemma which we give without proof.

Lemma 1. Let P be a tensor of type $(1,s)$ and let η,φ be two linearly independent vectors in V. If the exterior product tensor $P\wedge\eta\wedge\varphi$ vanishes, then there exist uniquely two tensors A, B of covariant degree s such that the following equality

$$P=A\otimes\eta+B\otimes\varphi$$

holds. Furthermore, if P is symmetric with respect to contravariant variables, then A and B are also symmetric.

Definition 2. Let us regard a tensor Q of type (r,s) as an s-linear map of V^s into $\mathbf{T}^r(V)$. The covariant symmetrization of Q is a tensor σQ of type (r,s) having the equality

$$\sigma Q(v_1,\dots,v_s)=\sum_{\pi} Q(v_{\pi(1)},\dots,v_{\pi(s)})$$

for any vectors $v_1,\dots,v_s$ in V, where the summation is taken over all permutations π of $(1,\dots,s)$.

Remark 3. If the tensor Q is symmetric with respect to the contravariant variables, then we have $\sigma Q=s!Q$.

Concerning the notion given above, we can make the following assertion the easy proof of which is left out.

Lemma 2. Let Q be a tensor of type (r, s) which is symmetric with respect to contravariant variables and where $s = 3$ or $s = 4$. If taking any vector v in V s times as variables, the equality $Q(v, \dots, v) = 0$ holds, then Q vanishes.

Definition 3. Let Q be a tensor of type (r, s) and let be given m vectors $v_1, \dots, v_m$ in V $(m \leq s)$ as last m variables of Q. Consider Q as an m-linear map of V^m into the tensor space $\mathbf{T}^r_{s-m}(V)$ of type $(r, s-m)$. Taking this map assigned to Q, the image of the m-tuple of vectors $(v_1, \dots, v_m)$ is said to be the contraction of Q with the vectors $v_1, \dots, v_m$. The contracted tensor will be denoted by $Q[v_1, \dots, v_m]$.

Remark 4. Let Q be a tensor of type (2,3) and let be given two vectors $\eta = \eta^i e_i$, $\varphi = \varphi^j e_j$ in V. The components of the contracted tensor $C = Q[\eta, \varphi]$ are

$$C^{hi}_j = Q^{hi}_{jkl}\eta^k \varphi^l.$$

Let us consider a tensor $F = F^h_i \varepsilon^i \otimes e_h$ of type (1,1). Obviously, F can be regarded as an endomorphism of V. The image of a vector η by F will be denoted by $F(\eta)$. Let I denote the identity map of V onto itself. The components of this special tensor of type (1,1) coincide with the Kronecker symbols δ^h_i.

Lemma 3. Let η and φ be two linearly independent vectors in V and A be a symmetric tensor of type (0,2). If the covariant symmetrization of the tensor $Q = A \otimes (I \wedge \eta \wedge \varphi)$ vanishes, then $A = 0$ holds.

Proof. Suppose that $\sigma Q = 0$ is valid. Take a vector ζ such that the vectors ζ, η, φ are linearly independent, that is, $\zeta \wedge \eta \wedge \varphi \neq 0$. Let us contract σQ with (ζ, ζ, ζ). Therefore we get $6A(\zeta, \zeta)(\zeta \wedge \eta \wedge \varphi) = 0$ which implies $A(\zeta, \zeta) = 0$. Contracting σQ with (ζ, ζ), we obtain the equality

$$2A(\zeta, \zeta)(I \wedge \eta \wedge \varphi) + 4A[\zeta] \otimes (\zeta \wedge \eta \wedge \varphi) = 0,$$

which means $A[\zeta] = 0$. At last if we contract Q with ζ, we get

$$2A[\zeta] \otimes (I \wedge \eta \wedge \varphi) + 2(I \wedge \eta \wedge \varphi) \otimes A[\zeta] + 2A \otimes (\zeta \wedge \eta \wedge \varphi) = 0.$$

The equality obtained above proves our assertion.

Lemma 4. Let P be a symmetric tensor of type (1,2) and let F be a tensor of type (1,1) such that there exists a vector η which is not an eigenvector. If the equality

$$\sigma(P \wedge I \wedge F) = 0 \tag{1}$$

holds, then there exist dual vectors ψ, ξ and symmetric tensors A, B of type (0,2) such that P can be expressed in the following form

$$P = \psi \otimes I + I \otimes \psi + \xi \otimes F + F \otimes \xi + A \otimes \eta + B \otimes F(\eta). \tag{2}$$

Furthermore, the tensor fields A and B in (2) satisfy the equalities

$$A[\eta] = 0, \qquad B[\eta] = 0. \tag{3}$$

Proof. Suppose that $\sigma(P \wedge I \wedge F)$ vanishes. Contracting this tensor with (η, η, η, η), we get

$$24P(\eta,\eta) \wedge \eta \wedge F(\eta) = 0.$$

Since η and $F(\eta)$ are linearly independent, this implies that $P(\eta,\eta)$ can be expressed as a linear combination of the two other vectors. Hence with suitable real numbers a, b

$$P(\eta,\eta) = 2a\eta + 2bF(\eta) \tag{4}$$

holds. Let us contract $\sigma(P \wedge I \wedge F)$ with the 3-tuple of vectors (η, η, η). Then we obtain

$$6P(\eta,\eta) \wedge \eta \wedge F + 6P(\eta,\eta) \wedge I \wedge F(\eta) + 12P[\eta] \wedge \eta \wedge F(\eta) = 0.$$

Using the equality (4), we have

$$(P[\eta] - aI - bF) \wedge \eta \wedge F(\eta) = 0.$$

By Lemma 1, from this follows that there exist two dual vectors ψ, ξ in V^* satisfying the equality

$$P[\eta] - aI - bF = \psi \otimes \eta + \xi \otimes F(\eta). \tag{5}$$

Contracting (5) with η, we get

$$P(\eta,\eta) = (a + <\psi, \eta>)\eta + (b + <\xi, \eta>)F(\eta),$$

where $< \, , \, >$ denotes the canonical bilinear form on $V^* \times V$. Therefore by the equality (4) this implies that $a = <\psi, \eta>$, $b = <\xi, \eta>$ are valid. Last we contract the tensor $\sigma(P \wedge I \wedge F)$ with (η, η). Using the equality (5), it can be seen that this operation results

$$(P - \psi \otimes I - I \otimes \psi - \xi \otimes F - F \otimes \xi) \wedge \eta \wedge F(\eta) = 0.$$

By Lemma 1 there exist symmetric tensors A, B of type (0,2) such that the equality (2) holds.

In order to verify (3) we contract (2) with η and subtract the obtained equality from (5). Therefore we get

$$0 = A[\eta] \otimes \eta + B[\eta] \otimes F(\eta),$$

which implies that the covariant vectors $A[\eta]$ and $B[\eta]$ vanish.

Henceforth we suppose that the ground vector field V is 3-dimensional. Let F be an endomorphism of V. The image subspace of F will be denoted by L and the kernel subspace of F will be denoted by K, that is,

$$L = \{F(v) \mid v \in V\}, \quad K = \{v \mid v \in V,\ F(v) = 0\}.$$

Suppose that *dimension* $L = 2$ and take two vectors $\varphi_1,\ \varphi_2$ which span the image subspace L. Then there exist uniquely two dual vectors ω^1 and ω^2 so that

$$F = \omega^1 \otimes \varphi_1 + \omega^2 \otimes \varphi_2 \tag{6}$$

holds. If we take two other vectors $\hat{\varphi}_1,\ \hat{\varphi}_2$ from L having expressions $\hat{\varphi}_\alpha = c_{\alpha 1}\varphi_1 + c_{\alpha 2}\varphi_2 \quad (\alpha = 1, 2)$, with suitable numbers $c_{\alpha\beta}$, then the corresponding dual vectors are $\hat{\omega}^\alpha = d^{1\alpha}\omega^1 + d^{2\alpha}\omega^2$, where the matrix formed with elements $d^{\alpha\beta}$ coincides with the inverse of the matrix formed with numbers $c_{\alpha\beta}$.

Conditions for the endomorphism F.
In the following lemma we consider an endomorphism F which satisfies the conditions given below:

a) The dimension of the image subspace L is not greater than 2.
b) If L is 2-dimensional, then there exists a vector η in the subspace L such that η is not an eigenvector of F.

Lemma 5. Let V be a 3-dimensional vector space, P a symmetric tensor of type (1,2) over V, and F an endomorphism of V satisfying the conditions given above. If for any vector λ in V

$$P(\lambda, \lambda) = a\lambda + bF(\lambda) \tag{7}$$

holds with suitable numbers $a,\ b$, then there exist dual vectors $\psi,\ \xi$ such that the following equality holds

$$P = \psi \otimes I + I \otimes \psi + \xi \otimes F + F \otimes \xi \tag{8}$$

Proof. Suppose that for any λ in V the vector $P(\lambda,\lambda)$ can be expressed as a linear combination of λ and $F(\lambda)$. From this follows that

$$P(\lambda,\lambda)\wedge\lambda\wedge F(\lambda)=0$$

which is equivalent to

$$(P\wedge I\wedge F)(\lambda,\lambda,\lambda,\lambda)=0.$$

Hence, by Lemma 2 we obtain that $\sigma(P\wedge I\wedge F)$ vanishes and so Lemma 4 can be applied. Therefore we have to show that the tensors A and B in the equality (2) are equal to 0. In order to prove this fact, we shall consider three different cases depending on the product expression of F. In all the cases first we shall show that $A=0$ holds.

Let us introduce the following tensors of type (3,4)

$$G=A\otimes(\eta\wedge I\wedge F),\quad H=B\otimes(F(\eta)\wedge I\wedge F).$$

Substituting the expression (2) of P into (1), we obtain

$$\sigma(G+H)=0. \tag{9}$$

Therefore contracting $\sigma(G+H)$ with the vector η, by the equalities (3) we get

$$\sigma(\hat{G}+\hat{H})=0, \tag{10}$$

where $\hat{G}=A\otimes(\eta\wedge I\wedge F(\eta))$ and $\hat{H}=B\otimes(F(\eta)\wedge\eta\wedge F)$).

Case I. rang F = 1. Let us take a vector φ and a dual vector ω so that $F=\omega\otimes\varphi$ is valid. It is obvious that in this case $\hat{H}$ vanishes and

$$\hat{G}=<\omega,\eta>A\otimes(\eta\wedge I\wedge\varphi)$$

holds. It can be easily seen that this implies

$$\sigma(A\otimes(\eta\wedge\varphi\wedge I))=0.$$

Therefore by Lemma 3 the tensor A vanishes.

Let λ be a vector so that $<\omega,\lambda>=0$ and λ, φ are independent. By the equality (7) we obtain $P(\lambda,\lambda)=a\lambda$. However, contracting (2) with (λ,λ), we get

$$P(\lambda,\lambda)=2<\psi,\lambda>\lambda+B(\lambda,\lambda)<\omega,\eta>\varphi.$$

From this follows that $B(\lambda,\lambda)=0$ holds. Let us consider two vectors λ, ζ in K so that neither of λ, ζ, $\lambda+\zeta$ is parallel to φ. Using the above results and (3), we can see that $B(v_1,v_2)=0$ is valid if the vectors v_1, v_2 are chosen from λ, ζ, η, which form a basis in V.

Case II. rang $(F)=2$, *K is contained in L.* Consider an expression (6) of F and introduce the notations $K_\alpha=\{v\in V \mid <\omega^\alpha, v>=0\}$ $(\alpha=1,2)$. Since V is 3-dimensional and K is contained in L, by the suitable choice of φ_1, φ_2 we can reach that the subspaces K_1 and L coincide. Regarding the requirements for F, we can fix a vector η in $L=K_1$ so that η and $F(\eta)=<\omega^2,\eta>\varphi_2$ are not parallel. Since $\varphi_2\wedge\eta\wedge\varphi_\alpha=0$ is valid $(\alpha=1,2)$, we obtain that the tensor $\hat{H}$ vanishes. Therefore from (9) follows that $\sigma\hat{G}=0$ holds. Hence, like in the preceding case, Lemma 3 implies that A vanishes.

Using the vectors chosen above, we have

$$H=<\omega^2,\eta>B\otimes(\varphi_2\wedge I\wedge(\omega^1\otimes\varphi_1)).$$

Take a vector λ which is not contained in the plane $L=K_1$. Contracting the equality $\sigma H=0$ with $(\lambda,\lambda,\lambda,\lambda)$, we get

$$24<\omega^2,\eta><\omega^1,\lambda>B(\lambda,\lambda)(\varphi_2\wedge\lambda\wedge\varphi_1)=0,$$

which implies that B also vanishes.

Case III. rang $(F)=2$, *K is not contained in L.* It is trivial that for any two vectors φ_1, φ_2 spanning L, the subspaces $K_1\cap L$ and $K_2\cap L$ are 1-dimensional. By the requirements for F we can take the vectors φ_1 and φ_2 so that $K_1\cap L$ is not invariant with respect to F. Fix a vector η in $K_1\cap L$ different from 0. Then using the same procedure as in the above case, we can show that $A=0$, $B=0$ hold.

3. Quasiplanar Maps between 3-dimensional Manifolds with Affine Connection

Let M be a 3-dimensional connected manifold with torsion free affine connection ∇. The tangent bundle of M will be denoted by TM, the tangent space at a point p will be denoted by T_pM. Furthermore, the ring of differentiable real-valued functions and the space of smooth vector fields on M will be denoted by $\mathcal{F}(M)$ and $\mathcal{V}(M)$, respectively. Let (U,x) be a local coordinate system in M around p. Hence the vector fields $\frac{\partial}{\partial x^i}$ $(i=1,2,3)$ and the 1-forms dx^1, dx^2, dx^3 form a basis for the tangent bundle TU and for the dual bundle TU^*, respectively. For simplicity's sake, the coordinate vector fields $\frac{\partial}{\partial x^i}$ will be denoted by X_i. Using the Einstein summation convention, we have

$$\nabla_{X_i} X_j = \Gamma_{ij}^h X_h,$$

where Γ_{ij}^h $(h, i, j = 1, 2, 3)$ are the Christoffel symbols with respect to the coordinate system (U, x). Let F be a tensor field of type (1,1) over M. Clearly, the restriction of F on U can be expressed in the form $F = F_i^h dx^i \otimes X_h$, and F determines at each point p of M an endomorphism of the tangent space T_pM. Henceforth, a manifold M endowed with a torsion free affine connection ∇ and with a differentiable tensor field F will be denoted by $M(\nabla, F)$.

Let $\gamma : J \to M$ be a differentiable curve where J is an open interval in $\mathcal{R}$. The tangent vector of γ at a parameter t will be denoted by $\dot{\gamma}(t)$.

Definition 4. A differentiable curve γ in $M(\nabla, F)$ is said to be F-planar if the parallel displacement of each tangent vector of the curve along γ is contained in the subspace (plane) spanned by the vectors $\dot{\gamma}(t)$ and $F(\dot{\gamma}(t))$.

Considering the above definition, it can be easily seen that γ is an F-planar curve if and only if for some functions $a,\ b : J \to \mathcal{R}$ the following equality holds

$$\nabla_{\dot{\gamma}} \dot{\gamma} = a\dot{\gamma} + bF(\dot{\gamma}). \tag{11}$$

Remark 5. It is trivial that geodesic curves in $M(\nabla)$ are always F-planar irrecpective of the tensor field F. Let I denote the tensor field of type (1,1) over M which presents the identity map of T_pM at each point p. Obviously, I has the local expression

$$I = \delta_i^h dx^i \otimes X_h,$$

where δ_i^h denote the Kronecker symbols $(h, i = 1, 2, 3)$. Using the equality (11), it can be seen that if $F = cI$ holds for a function c on M, then the F-planar curves are only the geodesics. Therefore the tensor fields of this type will be excluded from our investigations. We suppose that the endomorphism determined by the considered tensor field F is different from cI at each point. Hence, we can raise the question whether there is a tensor field satisfying this condition on a given manifold M. Obviously, the answer is positive if M is diffeomorph with an open subset of $\mathcal{R}^3$.

Remark 6. Let F and $\bar{F}$ be two tensor fields of type (1,1) over a manifold M with torsion free affine connection ∇. It can be shown that the F-planar curves coincide with the $\bar{F}$-planar ones if and only if the equality $\bar{F} - F = cI$ is valid for a real function c over M. This fact suggests us to introduce the following notion.

Definition 5. Let be given two tensor fields F and $\bar{F}$ of type (1,1) on a 3-dimensional manifold M. F and $\bar{F}$ are called equivalent if $\bar{F} - F = cI$ holds for a function c. In this case we say that F and $\bar{F}$ determine the same planar structure on M.

Let $\widetilde{M}$ be another 3-dimensional manifold with a torsion free affine connection $\widetilde{\nabla}$ and with a tensor field $\widetilde{F}$. Suppose that the map $m : M \to \widetilde{M}$ is a diffeomorphism. By the tangent linear map $Tm : TM \to T\widetilde{M}$ we obtain over M the so called induced connection $\bar{\nabla} = m^*\widetilde{\nabla}$ and the induced tensor field $\bar{F} = m^*\widetilde{F}$ (for details see [1]). Let us consider the local coordinate system $(m(U), x \circ m^{-1})$ on $\widetilde{M}$. Furthermore, let us denote by $\widetilde{\Gamma}^h_{ij}$ and by $\widetilde{F}^h_i$ the Christoffel symbols of $\widetilde{\nabla}$ and the component functions of the tensor field $\widetilde{F}$ with respect to this coordinate system, respectively. Obviously, for the components of $\bar{\nabla}$ and $\bar{F}$ in (U, x) the following equalities hold

$$\bar{\Gamma}^h_{ij} = \widetilde{\Gamma}^h_{ij} \circ m, \qquad \bar{F}^h_i = \widetilde{F}^h_i \circ m.$$

Definition 6. A diffeomorphism $m : M(\nabla, F) \to \widetilde{M}(\widetilde{\nabla}, \widetilde{F})$ is said to be a quasiplanar map if m maps each F-planar curve of $M(\nabla, F)$ into an $\widetilde{F}$-planar curve of $\widetilde{M}(\widetilde{\nabla}, \widetilde{F})$.

Definition 7. A diffeomorphism $m : M(F) \to \widetilde{M}(\widetilde{F})$ is said to be preserving the planar structure if F and the induced tensor field $\bar{F}$ over M are equivalent to each other.

Considering the equality (11) describing F-planar curves, it can be easily proved that the following assertion is true.

Proposition 1. Let $m : M(\nabla, F) \to \widetilde{M}(\widetilde{\nabla}, \widetilde{F})$ be a diffeomorphism. Then m is a quasiplanar map if and only if the identity map $id : M(\nabla, F) \to M(\bar{\nabla}, \bar{F})$ is quasiplanar.

Let be given two torsion free affine connections ∇ and $\bar{\nabla}$ on a manifold. It is known that their difference P is a tensor field of type (1,2). More precisely, considering P as an $\mathcal{F}(M)$-linear map of $\mathcal{V}(M) \times \mathcal{V}(M)$ into $\mathcal{V}(M)$, the following equality holds

$$P(X, Y) = \bar{\nabla}_X Y - \nabla_X Y$$

for any two vector fields X, Y over M. This tensor field is said to be the deformation tensor field between $\bar{\nabla}$ and ∇. Since the affine connections are torsion free, P is symmetric.

Theorem 1. Let M be a 3-dimensional manifold and let F be a tensor field of type (1,1) defining a planar structure over M. Let ∇ and $\bar{\nabla}$ be two different torsion free affine connections on M. Then the identity map

$id : M(\nabla, F) \to M(\bar{\nabla}, F)$ is quasiplanar if and only if there exist two covariant vector fields ψ and ξ such that for the deformation tensor field P the following equality holds

$$P = \psi \otimes I + I \otimes \psi + \xi \otimes F + F \otimes \xi. \tag{12}$$

Proof. By definition we get that if the condition (12) is satisfied, then id is a quasiplanar map.

Conversely, suppose that the diffeomorphism $id : M(\nabla, F) \to M(\bar{\nabla}, F)$ is quasiplanar. Let $\gamma : J \to M$ be an F-planar curve in $M(\nabla, F)$. Since γ is also F-planar in $M(\bar{\nabla}, F)$, for suitable functions $\bar{a}$, $\bar{b}$ we have

$$\bar{\nabla}_{\dot{\gamma}}\dot{\gamma} = \bar{a}\dot{\gamma} + \bar{b}F(\dot{\gamma}). \tag{$\bar{11}$}$$

Subtracting (11) from the equality ($\bar{11}$), we obtain

$$P(\dot{\gamma}, \dot{\gamma}) = (\bar{a} - a)\dot{\gamma} + (\bar{b} - b)F(\dot{\gamma}). \tag{13}$$

Since the geodesics are always F-planar, for any tangent vector λ in TM we can find an F-planar curve where λ is a tangent vector. Hence, (13) implies that there exist real valued functions α and β on TM so that

$$P(\lambda, \lambda) = \alpha(\lambda)\lambda + \beta(\lambda)F(\lambda) \tag{14}$$

holds for any element λ of TM.

We show pointwise that the equality (12) is valid. Let us fix an arbitrary point p in M and take the tensors P, F over the tangent space T_pM. The equality (14) shows that for any λ the vector $P(\lambda, \lambda)$ can be expressed as a linear combination of λ and $F(\lambda)$. In order to apply Lemma 5 we have to point out that we can find a tensor field which is equivalent to F and satisfy at p the conditions given in Lemma 5. Since T_pM is 3-dimensional, the endomorphism F has got eigenvectors. Therefore if F does not satisfies these conditions, then using one of its characteristic values, we can take an equivalent tensor field $\hat{F} = F - cI$ so that Lemma 5 can be applied for $\hat{F}$. This fact completes the proof of our theorem.

Theorem 2. Let ∇, $\bar{\nabla}$ be two different torsion free affine connections and let F, $\bar{F}$ be two tensor fields of type (1,1) over a manifold M. If $id : M(\nabla, F) \to M(\bar{\nabla}, \bar{F})$ is a quasiplanar map, then the tensor fields F and $\bar{F}$ are equivalent to each other.

Proof. Suppose that the identity map is quasiplanar. Since all the geodesic curves in $M(\nabla)$ are $\bar{F}$-planar in $M(\bar{\nabla}, \bar{F})$, we obtain that for suitable functions α, β on TM the equality

$$P(\lambda, \lambda) = \alpha(\lambda)\lambda + \beta(\lambda)\bar{F}(\lambda)$$

holds for any element λ in TM. Regarding the proof of Theorem 1, this implies that there exist covariant vector fields ψ, ξ so that we have

$$P = \psi \otimes I + I \otimes \psi + \xi \otimes \bar{F} + \bar{F} \otimes \xi. \tag{15}$$

Let us consider an arbitrary F-planar curve γ in $M(\nabla, F)$. Using the equalities (11) and (15) we obtain

$$\begin{aligned}\bar{\nabla}_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\dot\gamma + P(\dot\gamma, \dot\gamma)\\ &= (a + 2 < \psi, \dot\gamma >)\dot\gamma + (bF + 2 < \xi, \dot\gamma > \bar{F})(\dot\gamma).\end{aligned}$$

Since γ is an $\bar{F}$-planar curve, from this follows that for any element λ in TM the vector $F(\lambda)$ can be expressed as a linear combination of $\bar{F}(\lambda)$ and λ. Therefore the tensor fields F and $\bar{F}$ determine the same planar structure on M which completes the proof.

Regarding Proposition 1 and the preceding theorems, we can state the following assertions.

Corollary 1. Let be given a diffeomorphism $m : M(\nabla, F) \to \widetilde{M}(\widetilde{\nabla}, \widetilde{F})$ between two 3-dimensional manifolds. Then m is a quasiplanar map if and only if m preserves the planar structure and there exist covariant vector fields ψ, ξ so that for the tensor field $P = m^*\widetilde{\nabla} - \nabla$ the equality (12) holds.

Corollary 2. If a diffeomorphism $m : M(\nabla, F) \to \widetilde{M}(\widetilde{\nabla}, \widetilde{F})$ is quasiplanar, then the inverse of m is also a quasiplanar map.

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Address:

László VERHÓCZKI
Department of Geometry
Faculty of Mechanical Engineering
Technical University of Budapest
H-1521 Budapest, Hungary