ABOUT THE MEAN WIDTH OF SIMPLICES

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Abstract

We are interested in the maximal mean width of simplices in $\mathbf{E}^d$ having edge-length at most one. Probably the maximum is provided by the regular simplex with edge-length one. We prove it for $d \leq 5$ and support this conjecture with some additional arguments.

Keywords: finite packings, extremal properties.

Introduction

Let C be a convex, compact set in $\mathbf{E}^d$ where we always assume that $d \geq 2$. For a unit vector u, define $\Delta(C, u)$ as the length of the orthogonal projection of C onto a line parallel to u; i.e. the *width* of C in the direction of u. Moreover, denote by B^d the unit ball in $\mathbf{E}^d$ centered at the origin, by $\mathbf{S}^{d-1}$ the boundary of B^d, by κ_d the volume of B^d and by ω_{d-1} the surface-area of $\mathbf{S}^{d-1}$. Then the *mean width* of C is

$$M(C) = \frac{1}{\omega_{d-1}} \int_{\mathbf{S}^{d-1}} \Delta(C, u) du.$$

Observe that $M(C)$ is strictly monotonic, continuous and (positively) linear. It is useful to consider a renormalization of $M(C)$ which was introduced in [5]. The *first intrinsic volume* $V_1(C)$ of C is defined as

$$V_1(C) = \frac{d\kappa_d}{2\kappa_{d-1}} \cdot M(C).$$

This has the additional property that $V_1(C)$ does not depend on the dimension of the space containing C.

Assume that C is a d-dimensional polytope and denote by $\mathcal{E}$ the set of edges of C. Let p be any point of the relative interior of the edge e of C and $K(p)$ be the set of point x in $\mathbf{E}^d$ so that the closest point of C to x is

p. Then $K(p)$ is a polyhedral convex cone with vertex p, and for different choices of p from the relative interior of e, the resulted cones are congruent. Thus, we may define the *external angle* at e as

$$\alpha(e) = \frac{V(K(p) \cap (p + B^d))}{V(B^d)}.$$

As the length of the edge e is $V_1(e)$, the first intrinsic volume of C is (see [5])

$$V_1(C) = \sum_{e--} \alpha(e) \cdot V_1(e).$$

Denote by T^d the regular simplex with edge-length one and consider the family of simplices having edge-length at most one. Here we search for the simplex with the maximal mean width in this family, or in other words, the one with maximal first intrinsic volume. Thus, consider for $n \geq 2$ the family

$$\mathcal{F}_n^d = \{\mathrm{conv}\{x_0, \dots, x_n\} \,|\, x_0, \dots, x_n \in \mathbf{E}^d \quad \text{and} \quad d(x_i, x_j) \leq 1\}.$$

For $m < d$, we assume that $\mathbf{E}^m$, and hence also $\mathcal{F}_n^m$, is embedded into $\mathbf{E}^d$. Observe that $T^d \in \mathcal{F}_{d+1}^d$.

Conjecture 1. *Let $d \geq 2$ and $C \in \mathcal{F}_{d+1}^d$. Then $V_1(C) \leq V_1(T^d)$, with equality if and only if $C = T^d$.*

As in $\mathbf{E}^2$ the first intrinsic volume is half of the perimeter, the conjecture readily holds for $d = 2$. This paper proves the following results concerning the conjecture:

Theorem 2. *Let $P \in \mathcal{F}_{d+1}^d$ be so that $V_1(P) = max\{V_1(C) \,|\, C \in \mathcal{F}_{d+1}^d\}$. Then*

i) *$P = T^d$ if $\dim P \geq d - 1$,*
ii) *$P = T^d$ if $d = 3, 4, 5$ and*
iii) *$\dim P > 15 \ln d$ if d is large.*

The statements i), ii) and iii) are contained, respectively, in Theorem 7, Theorem 8 and Proposition 9.

Some General Observations

First we consider the general properties of $\mathcal{F}_n^d$ (see Lemma 3) and later the case $n = d + 1$ for any d (see Theorem 7).

Lemma 3. *Let* $n \geq 3$ *and* $P_n \in \mathcal{F}_n^d$ *be so that* $V_1(P_n) = max\{V_1(C) \,|\, C \in \mathcal{F}_n^d\}$.
Then

i) *dim* $P_n \geq 2$ *and* P_n *has* n *vertices,*
ii) $V_1(P_n) < V_1(P_{n+1})$,
iii) $V_1(P_n) < \frac{1}{2}V_1(B^d)$*and*
iv) $\lim_{n--} V_1(P_n) = \frac{1}{2}V_1(B^d)$.

Proof. If P_n is a segment, then $V_1(P_n) \leq 1$, and hence dim$P_n \geq 2$.

Let Q be a polytope having at most one diameter and at least two dimension and y be a point of the relative boundary of Q different from the vertices. Then $d(y,x) < 1$ for any $x \in Q$, and hence there exists a point y^- outside of Q so that the diameter of Q^-=conv$(Q \cup \{y^-\})$ is still at most one. This property yields i) and ii) by the strict monotony of the first intrinsic volume.

Finally, iii) follows as the first intrinsic volume is proportional to the mean width, and iv) holds because the unit ball can be approximated with inscribed polytopes.

Let dim $C \geq d-1$ for $C = \text{conv}\{x_0, \dots, x_d\}$, H=aff$\{x_1, \dots, x_d\}$ and g =aff$\{x_2, \dots, x_d\}$ have dimension $d-2$. In addition, assume that g does not contain x_0 and x_1 and if $C \subset H$, then g does not separate x_0 and x_1. Then we call g as an *axis* of C. Denote by H^+ the open halfspace of E^d determined by H and not containing x_0. By *rotating* x_1 *away from* x_0 we mean a rotation of x_1 around g into H^+. Observe that this rotation moves x_1 farther from x_0. The following lemma has a key role in the future considerations.

Lemma 4. *Let* C *=conv*$\{x_0, \dots, x_d\}$ *have dimension at least* $d-1$ *and* $g = aff\{x_2, \dots, x_d\}$ *be an axis of* C. *Then rotating* x_1 *away from* x_0 *strictly increases* $V_1(C)$.

Proof. Denote by y_1 the new position of x_1, by H the hyperplane perpendicularly bisecting the segment conv$\{x_1, y_1\}$, and let H^+ be the halfspace containing x_1. Observe that $g \subset H$, and that $x_0 \in$ intH^+ by $d(x_1, x_0) < d(y_1, x_0)$.

For any $x \in E^d$, let $\varphi(x)$ be the image of x by the reflection through H and let $y_0 = \varphi(x_0)$. The sets

$$C^- = \text{conv}\{y_0, y_1, x_2, \dots, x_d\},$$
$$M = \text{conv}\{x_0, y_1, x_2, \dots, x_d\}$$

and

$$M^- = \text{conv}\{y_0, x_1, x_2, \dots, x_d\}$$

satisfy $C^- = \varphi(C)$ and $M^- = \varphi(M)$, and the lemma states that $V_1(C) < V_1(M)$.

By the linearity of the intrinsic volumes, $V_1(M) = V_1(M_0)$ and $V_1(C) = V_1(C_0)$ for $M_0 = \frac{1}{2}(M + M^-)$ and $C_0 = \frac{1}{2}(C + C^-)$. We prove that C_0 is strictly contained in M_0, which in turn yields that $V_1(C) < V_1(M)$.

The points $u_0 = \frac{1}{2}(x_0 + x_1)$, $v_0 = \frac{1}{2}(y_0 + y_1)$, $u_1 = \frac{1}{2}(x_0 + y_1)$ and $v_1 = \frac{1}{2}(y_0 + x_1)$ satisfy $v_i = \varphi(u_i)$, $i = 0, 1$. These points occur in the sets

$$\sigma_C = \frac{1}{2}(\{x_0, x_1, x_2, \dots, x_d\} + \{y_0, y_1, x_2, \dots, x_d\})$$

and

$$\sigma_M = \frac{1}{2}(\{x_0, y_1, x_2, \dots, x_d\} + \{y_0, x_1, x_2, \dots, x_d\}).$$

We note that $C_0 = \operatorname{conv} \sigma_C$ and $M_0 = \operatorname{conv} \sigma_M$, and that $\sigma_M \backslash \sigma_C = \{u_0, v_0\}$ and $\sigma_C \backslash \sigma_M = \{u_1, v_1\}$.

As $y_i = \varphi(x_i)$ and H separates y_1 from x_0 and x_1, we have $u_1 \in \operatorname{conv}\{u_0, v_0\}$, and similarly $v_1 \in \operatorname{conv}\{u_0, v_0\}$. These yield $C_0 \subset M_0$ since u_1 and v_1 are the only points in $\sigma_C \backslash \sigma_M$.

In order to establish the strict inclusion, assume that H contains the origin and let w be the unit normal vector to H pointing into H^+. Define μ as

$$\mu = \max\{< w, x_0 >, < w, x_1 >\} = \max\{< w, z > | z \in C\}.$$

Any $z_0 \in C_0$ can be written in the form $z_0 = \frac{1}{2}(z + z^-)$ for some $z \in C$ and $z^- \in C^-$. Thus $< w, z^- > \leq 0$ and $< w, z > \leq \mu$ yield $< w, z_0 > \leq \frac{1}{2}\mu$. On the other hand, as $< w, x_0 >$ and $< w, x_1 >$ are positive and one of them is μ, we have $< w, u_0 > > \frac{1}{2}\mu$, which in turn yields that $u_0 \in M_0$ but $u_0 \notin C_0$. Therefore C_0 is strictly contained in M_0, and so $V_1(C) < V_1(M)$.

Remark: Note that $V_1(T^d)$ is a local maximum on $\mathcal{F}^d_{d+1}$ by Lemma 4.

Let σ be a finite subset of $\mathbf{E}^d$ containing at least $d + 1$ points. The points of σ are said to be *in general position* if no $d+1$ of them are contained in a hyperplane. In other words, if $x_0, \dots, x_d \in \sigma$ and coefficients $\alpha_0, \dots, \alpha_d$ satisfy

$$\alpha_0 \cdot x_0 + \dots + \alpha_d \cdot x_d = 0 \quad \text{and} \quad \alpha_0 + \dots + \alpha_d = 0,$$

then $\alpha_0 = \dots = \alpha_d = 0$. Now we modify slightly Radon's classical theorem (see [4]).

Lemma 5. *Let $x_0, \dots, x_{d+1}$ be points of $\mathbf{E}^d$ in general position. Then the points can be renumbered so that for certain m, $0 \leq m \leq d$, the intersection* *of conv*$\{x_0, \ldots, x_m\}$ *and* conv$\{x_{m+1}, \ldots, x_{d+1}\}$ *is a unique point. Moreover, for any pair of indices* i, j with $0 \le i \le m$ and $m+1 \le j \le d+1$, *the convex hull of the points* x_k *different from* x_i *and* x_j *is a facet of* $C = \text{conv}\{x_0, \ldots, x_{d+1}\}$.

Proof. For any $y = (y^1, \ldots, y^d) \in \mathbf{E}^d$ let $y^- = (y^1, \ldots, y^d, 1) \in \mathbf{E}^{d+1}$. The points $x_0^-, \ldots, x_{d+1}^-$ are dependent in $\mathbf{E}^{d+1}$, and hence there exist coefficients $\alpha_0, \ldots, \alpha_{d+1}$ so that not all of them are zero,

$$\alpha_0 \cdot x_0 + \ldots + \alpha_{d+1} \cdot x_{d+1} = 0 \quad \text{and} \quad \alpha_0 + \ldots + \alpha_{d+1} = 0. \tag{1}$$

Since $x_0, \ldots, x_{d+1}$ are in general position in $\mathbf{E}^d$, any $d+1$ out of the points $x_0^-, \ldots, x_{d+1}^-$ are independent in $\mathbf{E}^{d+1}$. This yields that none of the α_i's is zero and any other set of coefficients satisfying (1) is in the form $\{\lambda \cdot \alpha_0, \ldots, \lambda \cdot \alpha_{d+1}\}$ for some real number λ. We may assume that $\alpha_0, \ldots, \alpha_m$ are positive and $\alpha_{m+1}, \ldots, \alpha_{d+1}$ are negative for certain m, $0 \le m \le d$. The first statement follows from the fact that the point

$$\frac{\alpha_0 \cdot x_0 + \ldots + \alpha_m \cdot x_m}{\alpha_0 + \ldots + \alpha_m} = \frac{(-\alpha_{m+1}) \cdot x_{m+1} + \ldots + (-\alpha_{d+1}) \cdot x_{d+1}}{(-\alpha_{m+1}) + \ldots + (-\alpha_{d+1})}$$

is contained in both conv$\{x_0, \ldots, x_m\}$ and conv$\{x_{m+1}, \ldots, x_{d+1}\}$. This is the only point of the intersection because of the uniqueness condition on $\alpha_0, \ldots, \alpha_{d+1}$.

Now assume that aff$\{x_1, \ldots, x_d\}$ intersects conv$\{x_0, x_{d+1}\}$. Then

$$\beta_0 \cdot x_0 + \beta_{d+1} \cdot x_{d+1} = \beta_1 \cdot x_1 + \ldots + \beta_d \cdot x_d,$$

where β_0 and β_{d+1} are non-negative, $\beta_0 + \beta_{d+1} = 1$ and $\sum_{i=1}^d \beta_i = 1$. The uniqueness condition on $\alpha_0, \ldots, \alpha_{d+1}$ yields that α_0 and α_{d+1} have the same sign. This is absurd, hence, conv$\{x_1, \ldots, x_d\}$ is a facet of C (see [4]).

Note that if K is a convex body having at most one diameter then by Jung's theorem (see e.g. [2]),

$$R(K) \le R(T^d) = \sqrt{\frac{2d}{d+1}} < \sqrt{2} \tag{2}$$

holds for the circumradius $R(K)$ of K.

Lemma 6 *Let* $C = conv\{x_0, \ldots, x_{d+1}\}$*be a d-polytope with d+2 vertices so that* $d(x_i, x_j) \le 1$ *for any* i, j. *Then there exist two vertices of C, say* x_0 *and* x_1, *so that* $d(x_0, x_1) < 1$ *and conv* $\{x_2, \ldots, x_{d+1}\}$ *is a facet of C.*

Proof. First assume that the points $x_0, \ldots, x_{d+1}$ are in general position and that, contrary to our claim, there are no suitable pairs of vertices of C. By Lemma 5, we may assume that for certain index m, $M = \text{conv}\{x_0, \ldots, x_m\}$ and $N = \text{conv}\{x_{m+1}, \ldots, x_{d+1}\}$ intersect in a unique point y. Here $1 \leq m \leq d-1$ because each point out of $x_0, \ldots, x_{d+1}$ is a vertex of C. The indirect assumption and the second statement of Lemma 5 yield that $d(x_i, x_j) = 1$ for $i = 0, \ldots, m$ and $j = m+1, \ldots, d+1$. Thus, $d(y, x_0) = R(M)$, $d(y, x_{d+1}) = R(N)$ and aff M and aff N are orthogonal to each other. We deduce by (2) that $d(y, x_0)$ and $d(y, x_{d+1})$ are less than $\sqrt{2}$, hence, $d(x_0, x_{d+1}) < 1$ in the triangle $\text{conv}\{x_0, y, x_{d+1}\}$. This contradiction proves the lemma when the points $x_0, \ldots, x_{d+1}$ are in general position.

For the general case we proceed by induction on d. If $d = 2$, then C is a quadrilateral, and hence $x_0, \ldots, x_3$ are in general position. Let $d \geq 3$ and $x_0, \ldots, x_{d+1}$ be not in general position. Then we may assume that $x_0, \ldots, x_d$ span $\mathbf{E}^{d-1}$, and by induction that $d(x_0, x_1) < 1$ and $\text{conv}\{x_2, \ldots, x_d\}$ is a facet of $\text{conv}\{x_0, \ldots, x_d\}$ in $\mathbf{E}^{d-1}$. Now $\dim C = d$ yields that x_{d+1} is not contained in $\mathbf{E}^{d-1}$, and hence, $\text{conv}\{x_2, \ldots, x_{d+1}\}$ is a facet of C.

Theorem 7 *Let $P \in \mathcal{F}^d_{d+1}$ be so that $V_1(P) = max\{V_1(C) \,|\, C \in \mathcal{F}^d_{d+1}\}$. If $dimP \geq d-1$*, then $P = T^d$.

Proof. Assume that P is not congruent to T^d. Lemma 3 yields that P has $d+1$ vertices, and by Lemma 6 we may assume that $d(x_0, x_1) < 1$ and $g = \text{aff}\{x_2, \ldots, x_{d+1}\}$ is an axis of P. We conclude by Lemma 4 that $V_1(P)$ is not a local minimum on $\mathcal{F}^d_{d+1}$, and this contradiction proves the theorem.

Low and Large Dimensions

Simple calculations show that the external angle of T^3 at an edge is $\gamma = arccos(-\frac{1}{3})/2\pi$, and hence,

$$V_1(T^3) = 6 \cdot 1 \cdot \gamma = 1.8245.$$

Turning to T^4, let p be contained in the relative interior of the edge e of T^4 and $K(p)$ be the corresponding three-dimensional cone. Then $K(p)$ has three faces, and the angle of any two of these faces is γ. Let Δ be the spherical triangle on S^2 whose each angle is γ. As the surface-area of Δ and S^2 are $3\gamma - \pi$ and 4π, respectively, we deduce that

$$V_1(T^4) = 10 \cdot \alpha(e) = 10 \cdot \frac{3\gamma - \pi}{4\pi} = 2.0630.$$

Theorem 8. *Let $d = 3, 4, 5$ and $P \in \mathcal{F}^d_{d+1}$ be so that $V_1(P) = max\{V_1(C) \mid C \in \mathcal{F}^d_{d+1}\}$. Then $P = T^d$.*

Proof. If $d = 3$, then $P = T^d$ by i) of Lemma 3 and by Theorem 7. Let $d = 4, 5$ and observe that

$$\frac{1}{2}V_1(B^3) = 2 < V_1(T^4).$$

This yields that $\dim P \geq 4$ by Lemma 3, and hence, $P = T^d$ by Theorem 7.

In the proof of Proposition 9, we need the estimate (cf) [1]

$$\sqrt{\frac{2\pi}{d+1}} < \frac{\kappa_d}{\kappa_{d-1}} < \sqrt{\frac{2\pi}{d}}. \tag{3}$$

Proposition 9. *Let d be large and $P \in \mathcal{F}^d_{d+1}$ be so that $V_1(P) = max\{V_1(C) \mid C \in \mathcal{F}^d_{d+1}\}$. Then $dimP > 15 ln d$.*

Proof. *According to* [3], *we have* $V_1(T^d) \sim 2\sqrt{2\pi}\sqrt{\ln d}$ as d tends to infinity. Assume that d is large enough to ensure $V_1(T^d) > \frac{\sqrt{15}}{2}\sqrt{2\pi}\sqrt{\ln d}$.

Let $m \leq 15 \ln d$ and $C \in \mathcal{F}^m_{d+1}$. Then $V_1(C) < \frac{1}{2}V_1(B^m)$ by Lemma 3, and (3) yields that

$$V_1(C) < \frac{1}{2}\frac{m\kappa_m}{\kappa_{m-1}} < \frac{1}{2}\sqrt{2\pi}\sqrt{m} < V_1(T^d).$$

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